Canad. Math. Bull. Vol. 19 (3), 1976

A NOTE ON BEST SIMULTANEOUS APPROXIMATION IN NORMED LINEAR SPACES

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The purpose of the present note is to point out that the results of D. S. Goel, A. S. B. Holland, C. Nasim and B. N. Sahney [1] on best simultaneous approximation are easy consequences of simple facts about convex functions. Given a normed linear space X, a convex subset K of X, and points x_1, x_2 in X, [1] discusses existence and uniqueness of $k^* \in K$ such that

$$\max(\|x_1 - k^*\|, \|x_2 - k^*\|) = \inf_{k \in K} \max(\|x_1 - k\|, \|x_2 - k\|).$$

If we let $\varphi_i(k) := \|x_i - k\|$ for $i = 1, 2$, let $\varphi := \varphi_1 \vee \varphi_2$ and let

$$A := \{k \in K \mid \varphi(k) = \inf \varphi(K)\},$$

then existence of k^* means that A is non-empty, and uniqueness of k^* means that A is at most one-pointed. Now, φ_1 and φ_2 are norm-continuous and convex functions on K, and therefore φ is norm-continuous and convex on K. In particular, the level sets

$$L(\alpha) := \{k \in K \mid \varphi(k) \leqq \alpha\}$$

are norm-closed (relatively to K) and convex. Furthermore, they are bounded. Note that

$$A = \bigcap\{L(\alpha) \mid \inf \varphi(K) < \alpha\}.$$

From this we may conclude the following:

The set A is convex; cf. [1, Lemma 2.3]. In fact, A is a level set.

When K is (a closed subset of) a finite dimensional subspace of X, then $A \neq \varnothing$; cf. [1, Lemma 2.2]. In fact, under the conditons stated the level sets are compact. (Here convexity is not involved.)

When X is a reflexive Banach space (e.g. when X is a uniformly convex Banach space), and K is closed, then $A \neq \varnothing$; cf. the existence statement of [1, Proposition 4.1]. In fact, being norm-closed and convex, the level sets are weakly closed. By the boundedness and the reflexivity of X it next follows that the level sets are weakly compact.

In order to obtain uniqueness statements we need the following easy result:

LEMMA. *If X is a strictly convex space, then φ is not constant on any segment.*

https://doi.org/10.4153/CMB-1976-053-8 Published online by Cambridge University Press

Proof. Suppose that $\varphi = a$ on a segment $[k_0, k_1]$. By the strict convexity of X, φ_1 cannot be constant on $[k_0, k_1]$. Therefore, we must have $\varphi_1(k) < a$ for some $k \in [k_0, k_1]$. By the continuity of φ_1 on $[k_0, k_1]$ it next follows that we have $\varphi_1 < a$ on a whole subsegment of $[k_0, k_1]$. But then we must have $\varphi_2 = a$ on this subsegment, which is contradicted by the strict convexity of X.

Now, it follows from the lemma that if X is strictly convex, then A is at most one-pointed. This proves [1, Proposition 3.1] and the uniqueness statement of [1, Proposition 4.1].

Reference

1. D. S. Goel, A. S. B. Holland, C. Nasim and B. N. Sahney, *On best simultaneous approximation in normed linear spaces,* Canad. Math. Bull. **17** (1974), pp. 523–527.

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