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ON A CLASS OF RIGHT HEREDITARY SEMIGROUPS

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1. **Introduction.** Throughout this paper *all semigroups will have identity*, and *all S-systems* (operands) *will be right unitary S-systems.* All homomorphisms will be S-homomorphisms unless specified.

An S-system P is *projective* iff for every epimorphism $g: M \to R$, and every homomorphism $h: P \to R$, there exists a homomorphism $k: P \to M$, such that $gk = h$, where M and R are S-systems. A semigroup S is called right hereditary provided every right ideal is projective.

This note provides a structure for right hereditary, principal right ideal semigroups with central idempotents as a union of left cancellative, principal right ideal semigroups.

1.1 Definition. An S-system F is *free* provided there exists a subset X of F such that each element y of F has a unique representation $y = xs$, $x \in X$, $s \in S$. X is called a *basis* for F.

1.2 Definition. An S-system N is called a *retract* of an S-system M provided there exists a diagram,

$$M \underset{g}{\overset{f}{\rightleftarrows}} N$$

such that $fg = 1_N$. Here f is an epimorphism, and g is a monomorphism.

The proofs of the following statements follow the usual diagrammatic procedures.

1.3 Statement. Every free S-system is projective, and every retract of a projective S-system is projective.

1.4 Statement. Every S-system is the epimorphic image of a free S-system.

1.5 Statement. Every projective S-system, which is the epimorphic image of an S-system M, is a retract of M.

1.6 Statement. An S-system is projective iff it is the retract of a free S-system.

1.7 Statement. A semigroup S is right hereditary iff every subsystem of a projective S-system is projective.

1.8 Statement. Let S be a semigroup where every right ideal is principal, and the left cancellation law holds, then every subsystem of a free S-system is free.

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2. **Properties of right hereditary semigroups.** Note if e is any idempotent in S, then the mapping $s \to es$ combined with the identity mapping on eS implies that eS is a retract of S. Since S is free, then eS is projective by 1.3. Thus we have

2.1 PROPOSITION. *If every right ideal of S is generated by an idempotent, then S is right hereditary.*

By 2.2 of [2, p. 16] we can write

2.2 PROPOSITION. *Every completely right injective semigroup is right hereditary.*

2.3 NOTATION. If $x \in S$ and xS is projective, we shall denote by $f_x : S \to xS$, where $f_x(s) = xs$. Since xS is a retract of S, there exists $g_x : xS \to S$ such that $f_x g_x$ is the identity on xS.

2.4 PROPOSITION. *If S is right hereditary, then every principal right ideal xS is isomorphic to a right ideal eS, where $e^2 = e$, and $Sx \subset Se$.*

Proof. From 2.3, $xs = f_x g_x(xs) = x g_x(xs)$. For $s = 1$, $x = x g_x(x)$. Applying g_x, we have $g_x(x) = g_x(x) g_x(x)$. Hence xS is isomorphic to $g_x(x)S$, and $Sx \subset S g_x(x)$.

2.5 PROPOSITION. *If S is right hereditary with only one idempotent, then S is left cancellative.*

Proof. If $xa = xb$, then $g_x(x)a = g_x(x)b$. Since $g_x(x) = 1$, then $a = b$.

From the preceding discussion we can say,

2.6 PROPOSITION. *Let S be a principal right ideal semigroup. Then S is right hereditary iff each right ideal is isomorphic to a right ideal generated by an idempotent.*

3. **Principal right ideal semigroups with central idempotents.** In this section, let *S always denote a right hereditary, principal right ideal semigroup with central idempotents*. Since the idempotents of S are central, by 2.4 we have $xS \subset g_x(x)S$, which is a two sided ideal.

3.1 PROPOSITION. *If $xS \subset eS \subset g_x(x)S$ where $e^2 = e$, then $e = g_x(x)$.*

Proof. Since $xe = x$, then $g_x(x)e = g_x(x)$. Since $eS \subset g_x(x)S$, then $g_x(x)e = e = g_x(x)$.

In this way we can associate with each right ideal xS the two sided ideal $g_x(x)S$, which is minimal with respect to the properties of containing xS and being generated by an idempotent.

Since the right ideals of S are dually well ordered, we can write the ideals generated by idempotents as a chain $S = e_0 S \supset e_1 S \supset e_2 S \supset \cdots \supset e_\alpha S \supset \cdots$ where the subscripts belong to the set M_γ of all ordinals less than an ordinal γ. Let $T_\alpha = e_\alpha S \backslash e_{\alpha+1} S$.

3.2 LEMMA. *If $x \in T_\alpha$, $y \in T_\beta$ where $\beta \leq \alpha$ ($e_\beta S \supset e_\alpha S$), then xy and yx belong to T_α, and ye_α is a unit of T_α.*

Proof. Since the right ideals are principal they are dually well ordered by inclusion ($aR \cup bR = aR$ or bR), we can write by 3.1,

$$xS \subset e_\alpha S \subset yS \subset e_\beta S.$$

Then $g_x g_y(yx) = g_x(g_y(y)x) = g_x(e_\beta x) = g_x(x) = e_\alpha$. Suppose $yxS \subset eS \subset e_\alpha S$, where $e^2 = e$. Then $g_x g_y(yx) = g_x g_y(yxe) = g_x g_y(yx)e$, or $e_\alpha = e_\alpha e$. Thus $e_\alpha = ee_\alpha$, since the idempotents are central. Hence $e_\alpha S = eS$. Thus it is impossible for yx to be contained in eS if eS is properly contained in $e_\alpha S$. Hence $yx \in T_\alpha$ by 3.1. Now $g_y g_x(xy) = g_y((g_x(x)y) = g_y(yg_x(x)) = e_\beta g_x(x) = g_x(x) = e_\alpha$. Suppose as before $yxS \subset eS \subset e_\alpha S$. Then $g_y g_x(xy) = g_y g_x(xye) = g_y(xy)e$. Thus $e_\alpha = e_\alpha e$ and $e_\alpha S = eS$. Then, as before, by 3.1 we have $xy \in T_\alpha$. Thus T_α is a semigroup with identity e_α.

Since $e_\alpha \in yS$, then $e_\alpha = ys$. Thus $e_\alpha = (ye_\alpha)(se_\alpha)$ for $se_\alpha \in T_\alpha$. [For $se_\alpha \in e_\alpha S$, and se_α cannot be in a twosided ideal smaller than $e_\alpha S$ since $e_\alpha = (ye_\alpha)(se_\alpha)$].

We have shown ye_α is a right unit of T_α. Now T_α is left cancellative as will be shown in the next proposition. Hence ye_α is a unit of T_α.

3.3 Proposition *The sets T_α of S are principal right ideal, left cancellative semigroups.*

Proof. We have shown in 3.2 that T_α is a semigroup with identity e_α. If $xy = xz$ for $x, y, z \in T_\alpha$, then $g_x(x)y = g_x(x)z$ and $e_\alpha y = e_\alpha z$. Thus $y = z$.

Let H be a right ideal of T_α. Then $K = H \cup e_{\alpha+1}S$ is a right ideal of S. For if $z \in e_{\alpha+1}S$, then $zS \subseteq e_{\alpha+1}S$. If $h \in H$ and $s \in e_\beta S$, $\beta > \alpha$ then $hs \in e_\beta S \subseteq e_{\alpha+1}S$. If $h \in H$ and $s \in e_\beta S$, $\beta \leq \alpha$, then $hs = (he_\alpha)s = h(e_\alpha s) \in H$. Hence $K = xS$ for $x \in T_\alpha$. By 3.2, $xS = xT_\alpha \cup e_{\alpha+1}S$ since if $e_\alpha S \subset yS$, then $xy = x(e_\alpha y) \in xT_\alpha$. Hence $H = xT_\alpha$.

Define the mapping $f_{\alpha\beta}: T_\alpha \to T_\beta$ for $\alpha \leq \beta$ by $f_{\alpha\beta}(a_\alpha) = a_\alpha e_\beta$ with $a_\alpha \in T_\alpha$. The mappings $f_{\alpha\beta}$ are semigroup homomorphisms, where $f_{\alpha\alpha}$ is the identity on T_α, and $f_{\beta\gamma} f_{\alpha\beta} = f_{\alpha\gamma}$ for $\gamma > \beta > \alpha$. In addition, the image of $f_{\alpha\beta}$ is contained in the group of units of T_β.

We have proved the converse part of

3.4 Structure Theorem.[1] *Let M be a well ordered set such that for each $\alpha \in M$, there corresponds a left cancellative, principal right ideal semigroup T_α with identity e_α. For each α, β of M with $\alpha < \beta$, let there correspond a homomorphism $f_{\alpha\beta}: T_\alpha \to T_\beta$ such that $f_{\beta\gamma} f_{\alpha\beta} = f_{\alpha\gamma}$ for $\gamma > \beta > \alpha$, and where the image of $f_{\alpha\beta}$ is contained in the group of units of T_β. Let $f_{\alpha\alpha}$ denote the identity mapping on T_α, and S be the union of the T_α. Define the product $a_\alpha b_\beta = f_{\alpha\gamma}(a_\alpha) f_{\beta\gamma}(b_\beta)$ where $\gamma = \alpha \vee \beta$ for $a_\alpha \in T_\alpha$, $b_\beta \in T_\beta$. Then S is a right hereditary, principal right ideal semigroup whose idempotents are in the center. Conversely, each such semigroup is of this form.*

[1] Statement suggested by A. H. Clifford.

Proof. The associativity of $S=\cup T_\alpha$ follows directly as in [1, p. 128]. Now $f_{\alpha\beta}(e_\alpha)=e_\beta$, since $f_{\alpha\beta}(e_\alpha)$ is an idempotent unit of T_β. Hence $e_\beta e_\alpha=f_{\beta\beta}(e_\beta)f_{\alpha\beta}(e_\alpha)=e_\beta e_\beta=e_\beta$ and $e_\alpha e_\beta=f_{\alpha\beta}(e_\alpha)f_{\beta\beta}(e_\beta)=e_\beta e_\beta=e_\beta$ for $\alpha<\beta$. Thus $e_\alpha S\supset e_\beta S$ for $\alpha<\beta$, and $T_\alpha=e_\alpha S\backslash e_{\alpha+1}S$. By the definition of $f_{\alpha\beta}$ the idempotents are in the center.

Let H be a right ideal of S. If $x\in H$ and $x\in T_\alpha$, then for $\alpha<\beta$, H contains $xe_\beta=f_{\alpha\beta}(x)e_\beta$, which is a unit of T_β by hypothesis. Thus H contains $e_\beta S$ for $\beta>\alpha$. Let α be the least index with respect to the condition that $x\in H$, $x\in T_\alpha$. Then $H=K\cup e_{\alpha+1}S$ where K is a right ideal of T_α. Thus $K=yT_\alpha$, $y\in T_\alpha$. Then $H=yS$, which follows directly by using the definition of multiplication.

The mapping $xs\to e_\alpha s$, for all $s\in S$ is an S-isomorphism of xS onto $e_\alpha S$. By 2.6, then S is right hereditary.

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