J. Austral. Math. Soc. (*Series A*) **42** (1987), 409–420

EXACT COVERINGS OF TRIPLES WITH SPECIFIED LONGEST BLOCK LENGTH

R. G. STANTON and J. L. ALLSTON

(Received 1 July 1985)

Communicated by W. D. Wallis

Abstract

A minimal $(1, 3; v)$ covering occurs when we have a family of proper subsets selected from v elements with the property that every triple occurs exactly once in the family and no family of smaller cardinality possesses this property. Woodall developed a lower bound W for the quantity $g^{(k)}(1, 3; v)$ which represents the cardinality of a minimal family with longest block of length k. The Woodall bound is only accurate in the region when $k \geqslant v/2$. We develop an expression for the excess of the true value over the Woodall bound and apply this to show that, when $k \geqslant v/2$, the value of $g(1, 3; v) = W + 1$ when k is even and $W + 1 + \binom{v-k}{2}$ when k is odd.

1980 *Mathematics subject classification* (*Amer. Math. Soc.*): 05 B 30.

1. Introduction

A minimal $(\lambda, t; v)$ covering occurs when we have a family of proper subsets selected from v elements with the property that every t-set occurs exactly λ times in the family and no family of smaller cardinality possesses this property. Occasionally, this covering can be achieved by using a family of k-sets; in this case, the covering is called a Steiner System $S_\lambda(t, k, v)$. However, Steiner systems are rare, and the sets of a $(\lambda, t; v)$ covering are usually of unequal sizes; we use $g(\lambda, t; v)$ to denote the cardinality of a minimal covering.

In [8], we introduced $g^{(k)}(\lambda, t; v)$; this was the covering number under the restriction that there was a block of size k but no block of size greater than k. Clearly,

$$g(\lambda, t; v) = \min_{t \leqslant k \leqslant v-1} g^{(k)}(\lambda, t; v).$$

https://doi.org/10.1017/S1446788700028688 Published online by Cambridge University Press

It thus appears that the behaviour of $g^{(k)}(\lambda, t; v)$ is more fundamental than that of $g(\lambda, t; v)$. In [8], [4], and [15], we have studied $g^{(k)}(1,2; v)$; this function decreases as k goes from 2 to a value in the vicinity of $v^{1/2}$, and then increases to a maximum in the neighbourhood of $k = 2v/3$. Thereafter the function decreases to v for $k = v - 1$ and becomes unity for $k = v$.

We thus see that the value of $g(1,2; v)$ is almost an accident; it depends on whether the minimum in the neighbourhood of $k = v^{1/2}$ is less than the functional value for $k = v - 1$. For a complete discussion, including a diagram, we refer to [6]. Of course, the Erdös-de Bruijn Theorem $g(1,2; v) \geqslant v$, proved in [1], can be easily deduced from the behaviour of $g^{(k)}(1,2; v)$; cf. [10], [11].

2. The behaviour of $g(1,3; v)$ for large k

Three general bounds for $g(1,3; v)$ are known (see [11] and [6]). These are as follows (in any case that a bound is non-integral, we must take the next integer above).

The Combinatorial Bound is

$$C = \frac{v(v-1)(v-2)}{k(k-1)(k-2)}. \tag{2.1}$$

The Stanton–Kalbfleisch Bound (cf. [13] and [11]) is

$$SK = 1 + \frac{k-1}{v-2}\binom{k}{2}(v-k). \tag{2.2}$$

The Woodall Bound (cf. [17] and [11]) is

$$W = 1 + (v-k)\binom{k}{2}\left(1 - \frac{v-k-1}{2(k-1)}\right). \tag{2.3}$$

It is useful to write W in the form

$$W = 1 + \frac{(v-k)k(3k-v-1)}{4}. \tag{2.4}$$

Just as in the case $t = 2$, the bound C predominates for small k. Then the SK bound takes over, and finally the W bound predominates. We give a table for the case $v = 16$ (this is a value of v large enough to be typical).

In addition, there is a bound due to D. R. Stinson which improves (2.3). For this bound, one needs to determine $s = [(v-2)/(k-1)]$. The bound then takes the form (cf. [16])

$$S = 1 + \frac{(v-k)}{s(s+1)}\binom{k}{2}\left(2s + 1 - \frac{v-2}{k-1}\right). \tag{2.5}$$

https://doi.org/10.1017/S1446788700028688 Published online by Cambridge University Press

TABLE I: Lower bounds for $g^{(k)}(1,3;16)$

k	C	SK	W	S
3	560			
4	140			
5	56	27		28
6	28	55	16	56
7		82	64	85
8		113	113	113
9			159	
10			196	
11			221	
12			229	
13			216	
14			176	
15			106	
16			1	

It is easy to deduce from (2.3) and (2.4) that $W \geqslant SK$ so long as

$$v/2 \leqslant k \leqslant v - 1$$

(the equality occurs if and only if $v/2 = k$ or $k = v - 1$). In this paper, we show that, with the exception of small perturbations, $g^{(k)}(1, 3; v)$ is equal to the bound W in this range; a more precise statement will be given later.

3. An improvement on the bound W

We first note that there are three trivial cases in which the bound W is exact.

(a) Clearly, if $k = v$, then $W = 1$ and the bound is exact (usually we exclude $k = v$ as a possibility).

(b) If $k = v - 1$, then

$$W = 1 + \frac{(v-1)(v-2)}{2}.$$

But, if $k = v - 1$, then we need this single long block plus all triples made up of the remaining element taken with every pair from the long block. So the value is

$$g^{(k-1)}(1,3;v) = 1 + \binom{v-1}{2} = W.$$

(c) If $k = v - 2$, *and if v is even*, then

$$W = 1 + \frac{2(v-2)(2v-7)}{4}.$$

https://doi.org/10.1017/S1446788700028688 Published online by Cambridge University Press

We need to take the single long block and make quadruples consisting of the two elements not in the long block together with a set of disjoint pairs from the long block; we also need triples consisting of an element not from the long block together with all pairs from the long block not previously used. Thus we have

$$g^{(k-1)}(1,3;v) = 1 + \frac{v-2}{2} + 2\frac{v-2}{2}(v-4),$$

where we employ the well known fact that the elements of the long block have $(v-3)$ 1-factors. Simplifying, we find that, in this case,

$$g^{(k-2)}(1,3;v) = 1 + \frac{v-2}{2}(2v-7) = W.$$

Henceforth, we exclude cases (a), (b), and (c). We now refer to [11] and quote the result

$$\sum_j \binom{j}{x} \sum_{A(j)} \binom{k_i - j}{t-x} = \lambda \binom{k}{x}\binom{v-k}{t-x}$$

proved there in Theorem 1 (the k_i are the various block lengths). By placing $\lambda = 1$, writing $x = t-1$ and $x = t-2$, and combining the equations, it was shown there that

$$(t-1)\sum_{A(t-1)} \frac{(k_i - t + 1)(k_i - t - 2)}{2} + \sum_{A(t-2)} \binom{k_i - t + 2}{2}$$
$$+(t-1)(v-k)\binom{k}{t-1}\left(1 - \frac{v-k-1}{2(k-t+2)}\right) = 0.$$

Here $\Sigma_{A(n)}$ denotes the summation over all blocks which meet the longest block in n elements. This equation can be written as

$$(3.1)\qquad (t-1)\sum_{A(t-1)} \frac{(k_i - t + 1)(k_i - t - 2)}{2}$$
$$+ \sum_{A(t-2)} \binom{k_i - t + 2}{2} + (t-1)(W-1) = 0.$$

Now put $t = 3$ to give

$$(3.2)\qquad 2\sum_{A(2)} \frac{(k_i - 2)(k_i - 5)}{2} + \sum_{A(1)} \binom{k_i - 1}{2} + 2(W-1) = 0.$$

The first term can be written as

$$(3.3)\qquad 2\left\{\sum_{A(2)} \binom{k_i - 3}{2} - \sum_{A(2)} 1\right\} = 2\sum_{A(2)} \binom{k_i - 3}{2} - 2\alpha_2$$

where we write α_i to denote the number of blocks which meet the long block in i elements.

Also, since W is a bound, we can write

$$(3.4) \qquad g^{(k)}(1,3;v) = 1 + \alpha_0 + \alpha_1 + \alpha_2 = W + \varepsilon,$$

where ε denotes the excess over the bound W. When we substitute (3.3) and (3.4) into (3.2), we obtain

$$(3.5) \qquad 2\sum_{A(2)}\binom{k_i-3}{2} - 2\alpha_2 + \sum_{A(1)}\binom{k_i-1}{2} + 2(\alpha_0+\alpha_1+\alpha_2-\varepsilon) = 0.$$

Divide by 2 and simplify to obtain

$$(3.6) \qquad \varepsilon = \alpha_0 + \alpha_1 + \sum_{A(2)}\binom{k_i-3}{2} + \tfrac{1}{2}\sum_{A(1)}\binom{k_i-1}{2}.$$

We might remark that analogous formulae hold for $t = 2$ and $t = 4$. For reference, we record these as

$$(3.7) \qquad \varepsilon = \alpha_0 + \sum_{A(0)}\binom{k_i}{2} + \sum_{A(1)}\binom{k_i-2}{2}$$

and

$$(3.8) \qquad \varepsilon = \alpha_0 + \alpha_1 + \alpha_2 + \sum_{A(3)}\binom{k_i-4}{2} + \tfrac{1}{3}\sum_{A(2)}\binom{k_i-2}{2}.$$

Now, suppose that there are 3 or more elements not in the long block; they must occur in a block, and it will meet the long block in 0, 1, or 2 elements. If it meets the long block in 0 elements, then $\alpha_0 > 0$; if it meets the long block in 1 element, then $\alpha_1 > 0$; if it meets the long block 2 elements, then $k_i = 5$ and $\sum_{A(2)}\binom{k_i-3}{2} > 0$. In any case, we have $\varepsilon > 0$.

If there are 2 elements not in the long block and if $k = v - 2$ is odd, then there must be a triple which meets the long block in exactly one element; again $\alpha_1 > 0$, and so $\varepsilon > 0$.

On conclusions can be stated as

THEOREM 1. *For $g^{(k)}(1,3;v)$, we have*

$$\varepsilon = \alpha_0 + \alpha_1 + \sum_{A(2)}\binom{k_i-3}{2} + \tfrac{1}{2}\sum_{A(1)}\binom{k_i-1}{2}.$$

Furthermore, if $k = v$ or $k = v - 1$ or $k = v - 2$ (v even), then

$$g^{(k)}(1,3;v) = W.$$

In all other cases, we have $\varepsilon > 0$ and $g^{(k)}(1,3;v) \geqslant W + 1$.

We shall see that this result cannot be sharpened, since the bound $W + 1$ is attained in many cases.

4. The case of a long block of even length

We first recall the well known fact that a graph K_{2a} possesses $(2a - 1)$ disjoint 1-factors (cf., for example, [12]). Thus the pairs from K_6 can be split into K_2's as follows.

12, 34, 56	13, 25, 46
16, 23, 45	15, 24, 36
14, 26, 35	

This splitting is called a 1-factorization. It is useful to consider 1-factorizations of K_{2a-1} as well. In this case, a 1-factor consists of K_2's and a single K_1; no K_1 can be repeated. Thus, K_{2a-1} has $(2a - 1)$ 1-factors (again, cf. [12]); for example, the splitting for K_5 is simply

12, 34, 5	13, 25, 4	23, 45, 1
15, 24, 3	14, 35, 2	

These results on 1-factors will be useful in our next constructions.

First consider the case that k is even. The remaining points form a set of $v - k$ elements. Suppose first that $v - k$ is also even. Form a block of length $v - k$ which is disjoint from the long block (clearly, $v - k \leqslant k$, that is, $k \leqslant v/2$ for this to be possible). We also take $v - k > 2$, by virtue of the result of Theorem 1 when $v - k = 2$.

Form quadruples by taking the Cartesian product of all one-factors from the $(v - k)$ points with $(v - k - 1)$ 1-factors from the k points. The number of these is

$$\frac{k}{2}\frac{v-k}{2}(v-k-1),$$

Now form triples by taking the elements from the set of $(v - k)$ points with the remaining $(k - 1) - (v - k - 1)$ 1-factors from the k points. The number of these is

$$(v-k)\frac{k}{2}(2k-v).$$

All triples have now been accounted for, and the number of blocks is

$$2 + \frac{k}{4}(v-k)(v-k-1+4k-2v) = 2 + \frac{k(v-k)(3k-v-1)}{4} = W + 1.$$

https://doi.org/10.1017/S1446788700028688 Published online by Cambridge University Press

Since, by Theorem 1, we cannot do better, we obtain

THEOREM 2. *If $v/2 \leqslant k \leqslant v-2$, and if k and $v-k$ are even, then*

$$g^{(k)}(1,3;v) = W + 1 = 2 + \frac{k(v-k)(3k-v-1)}{4}. \tag{4.1}$$

COROLLARY 2.1. *The bound $W+1$ can only be achieved in the way indicated ($v-k$ elements in a single disjoint block).*

PROOF. This is immediate from (2.10), since α_1 must be zero (otherwise $\alpha_1 + \frac{1}{2}\Sigma_{A(1)}\binom{k_i-1}{2} > 1$), and α_2 must be zero (otherwise, since $v-k \geqslant 4$, $k_i \geqslant 6$ and we would have $\Sigma_{A(2)}\binom{k_i-3}{2} > 1$). Then $\alpha_0 = 1$, and we have our result.

We now consider the case that k is even and $v-k$ is odd, and we employ a similar construction. The number of quadruples formed by taking all 1-factors from the $(v-k)$ points with $(v-k)$ 1-factors from the k points is

$$\frac{k}{2}\,\frac{v-k+1}{2}(v-k).$$

The number of triples formed by taking elements from the $(v-k)$ points with the remaining $(k-1)-(v-k) = 2k-v-1$ 1-factors is

$$\frac{k}{2}(v-k)(2k-v-1).$$

So the total number of blocks is

$$2 + \frac{k}{4}(v-k)(v-k+1+2(2k-v-1)) = 2 + \frac{k(v-k)(3k-v-1)}{4}$$
$$= W + 1.$$

This gives us

THEOREM 3. *If $v/2 \leqslant k \leqslant v-2$, and if k is even and $v-k$ is odd, then*

$$g^{(k)}(1,3;v) = W + 1.$$

COROLLARY 3.1. *The bound $W+1$ can only be achieved by placing all $v-k$ elements not in the long block in a single disjoint block.*

PROOF. This follows as for Corollary 2.1.

5. The case of a long block of odd length

The situation when k is odd is somewhat different in that, whereas $\varepsilon = 1$ for k even, we find that $\varepsilon > 1$ for k odd. This basically stems from the result of the following lemma.

LEMMA 5.1. *If AB represents any pair of points from the $v - k$ points not in the long block and if k is odd, then there is at least one block containing AB that intersects the long block in a single point.*

PROOF. AB must occur with each element from the long block; the intersections of blocks containing AB with the long block can contain only 1 element or 2 elements; and the intersections cannot all contain 2 elements, since k is odd.

Now let us illustrate what happens in a couple of cases. Suppose that $v - k = 3$. If the pairs AB, AC, BC, all appear in separate blocks (triples), then they contribute $\varepsilon = 3(1.5) = 4.5$. On the other hand, if there is a single block ABC meeting the long block in a point, then $\varepsilon = 1 + 1.5 = 2.5$.

As a more complicated illustration, let $v - k = 10$ and suppose that the blocks *ABCD*, *AEFG*, *AHKL*, *BEH*, *CFK*, *DGL*, *DEK*, *BFL*, *CGH*, *CEL*, *DFH*, *BGH*, all meet the long block in single points. Their contribution to ε is

$$12 + \frac{3}{2}(6) + \frac{9}{2}(3) = \frac{69}{2},$$

as opposed to $45 + 45/2 = 135/2$ if the pairs had all been in separate blocks. However, one block *ABCDEFGHKL* only contributes $1 + 45/2 = 47/2$ to the excess. We are thus led to

LEMMA 5.2. *The minimal contribution to the excess from the fact that every pair of points not in the long block must occur in a block meeting the long bock in a single point is* $1 + \frac{1}{2}\binom{v-k}{2}$.

PROOF. As in the last example, let the $v - k$ points be pair-covered by a set of blocks of lengths $m_1, m_2, \ldots, m_r$. Then

$$\sum \binom{m_i}{2} = \binom{v-k}{2}.$$

Each block of length m_i extends to a block of length $m_i + 1$ by meeting the long block in a single point; so the total contribution to the excess is

$$r + \frac{1}{2}\sum \binom{m_i}{2} = r + \frac{1}{2}\binom{v-k}{2}.$$

On the other hand, if all $v - k$ points are put in a block of length $(v - k + 1)$, then the contribution to the excess is only

$$1 + \frac{1}{2}\binom{v-k}{2}.$$

https://doi.org/10.1017/S1446788700028688 Published online by Cambridge University Press

Clearly, this is the best we can do. Also, we need $v - k + 1 \leqslant k$, that is, $k \geqslant (v+1)/2$. Lemma 5.2 immediately gives us

THEOREM 4. *If* $(v+1)/2 \leqslant k \leqslant v - 2$, *and if* k *is odd, then*

$$g^{(k)}(1,3;v) \leqslant W + 1 + \frac{1}{2}\binom{v-k}{2}.$$

COROLLARY 4.1. *Under these conditions,*

$$g^{(k)}(1,3;v) \leqslant 2 + \frac{(v-k)(k-1)(3k-v+1)}{4}.$$

We shall now show that this bound is actually attained for k odd in the range $(v+1)/2 \leqslant k \leqslant v-2$.

First, let k be odd and let $v-k$ be even. In addition to the two blocks of lengths k and $v-k+1$, we require the following.

(a) $(k-1)(v-k)$ triples containing the point A which lies on both blocks and also containing one point from each block.

(b) $\frac{1}{2}(v-k)\frac{1}{2}(k-1)(v-k-1)$ quadruples formed by taking the Cartesian product of all one-factors from the $(v-k)$ elements with $(v-k-1)$ 1-factors from the $(k-1)$ points (less A) in the long block.

(c) $(v-k)\frac{1}{2}(k-1)(2k-v-1)$ triples formed by combining the $v-k$ points not in the long block with the remaining 1-factors of the $(k-1)$ points (less A) in the long block.

The total number of these blocks, which cover all triples on the v points, is

$$2 + \frac{(v-k)(k-1)}{4}\{4 + (v-k-1) + (4k-2v-2)\}$$
$$= 2 + \frac{(v-k)(k-1)(3k-v+1)}{4}.$$

Since this is the bound in Corollary 4.1, we can do no better and thus have shown that the bound is attained.

The construction for k odd and $v-k$ odd is similar, although the counts differ. We have two blocks intersecting in (A), together with the following blocks:

(a) $(k-1)(v-k)$ triples as before.

(b) $\frac{1}{2}(v-k-1)\frac{1}{2}(k-1)(v-k)$ blocks (some are quadruples and some are triples) formed by taking the Cartesian product of 1-factors.

(c) $(v-k)\left(\frac{k-1}{2}\right)(2k-v-2)$ triples formed by taking single elements with 1-factors from the $k-1$ points different from A on the long block.

https://doi.org/10.1017/S1446788700028688 Published online by Cambridge University Press

The total number of blocks then is given by

$$2 + \frac{(k-1)(v-k)}{4}\{4 + (v-k+1) + 2(2k-v-2)\}$$
$$= 2 + \frac{(v-k)(k-1)(3k-v+1)}{4},$$

as before. These two calculations establish

THEOREM 5. *If* $(v+1)/2 \leqslant k \leqslant v-2$, *and if* k *is odd, then*

(5.1) $$g^{(k)}(1,3;v) = 2 + \frac{(v-k)(k-1)(3k-v+1)}{4}.$$

COROLLARY 5.1. *For the minimal configuration giving*
$$g^{(k)}(1,3;v) = 2 + \frac{(v-k)(k-1)(3k-v+1)}{4},$$
we must have two blocks of lengths k *and* $v-k+1$ *intersecting in a single point, the other blocks are triples or quadruples.*

PROOF. Any other configuration would give (by Lemma 5.2) a contribution to the excess that would push the value over the stated lower bound.

6. Table for small values of v

In this section, we make use of the results obtained to tabulate $g^{(k)}(1,3;v)$ for v up to 12. In forming Table II, we have used the obvious fact that, for $k = 4$, we take $D(3,4,v)$ quadruples plus as many triples as are needed. Since the packing number $D(3,4,v)$ is known for all v in our table (cf. [14], [2], [5]), the second row is merely

$$D(3,4,v) + \left\{\binom{v}{3} - 4D(3,4,v)\right\}.$$

TABLE II. $g^{(k)}(1,3;v)$ for $3 \leqslant v \leqslant 12$.

k \ v	3	4	5	6	7	8	9	10	11	12
3	1	4	10	20	35	56	84	120	165	220
4		1	7	11	14	14	30	30	35	51
5			1	11	20	26	30	42*	*	*
6				1	16	28	38	44	47	47
7					1	22	41	56	68	77
8						1	29	53	74	90
9							1	37	87	98
10								1	46	86
11									1	56
12										1

This fact, with the results of the earlier sections, gives all entries except the three marked with an asterisk.

LEMMA 6.1. $g^{(5)}(1,3;10) = 42$.

PROOF. Clearly 42 is a lower bound since we can take two disjoint blocks of length 5. Each has five 1-factors, and the Cartesian product of the 1-factors contains $3(3) - 1 = 8$ blocks (drop the block of length 2). Thus

$$g^{(5)}(1,3;10) \leqslant 2 + 5(8) = 42.$$

Now let the long block be 12345 and the other points be A, B, C, D, E. We must cover A, B, C, D, E, by 10 blocks, 7 blocks, or 1 block (see the table). We have already dealt with one block $ABCDE$ (it must be disjoint).

If the cover is 10 triples of the form ABC, they meet the long block in 0, 1, or 2 elements. An intersection of 0 or 2 contributes 1 to the value of E, whereas an intersection of 1 contributes 2.5. However, Lemma 5.1 and the fact that a pair-covering of 5 elements contains at least 4 triples (such as ABC, ADE, BDE, CDE) guarantee that E is at least $6(1) + 4(2.5) = 16$. Hence, since $W = 26$, we cannot obtain a value less than 42 in this way.

If the cover is $ABCD$, EAB, EAC, EAD, EBC, EBD, ECD, and if the pair-covering is made up of the six triples, then these contribute a minimum of $6(2.5)$, and $ABCD$ contributes a minimum of 1. Again, we cannot get a value less than 42.

Finally, let the cover be $ABCD$, EAB, EAC, EAD, EBC, EBD, ECD, and suppose the pair covering is $ABCD$, EAB, ECD. Then these blocks contribute to E an amount at least $4 + 2(2.5) + 4(1) = 13$. However, Lemma 5.1 guarantees that AB and CD meet the long block in an odd number of unit intersections; hence there are two triples ABX and CDX at least, and they contribute another $2(1.5) = 3$ units to E. Hence, again, in this case, we can do no better than 42. This completes the demonstration of the lemma.

We leave the values of $g^{(5)}(1,3;11)$ and $g^{(5)}(1,3;12)$ to another paper, since they are longer and are closely connected with a difficult problem (cf. [13], [10], [7], [9], [3]). However, the same sort of arguments apply.

References

[1] N. G. de Bruijn and P. Erdös, 'On a combinatorial problem', *Nederl. Akad. Wetensch. Indag. Math.* **10** (1948), 421–423.

[2] M. R. Best, A. E. Brouwer, F. J. McWilliams, A. M. Odlyzko, and N. J. A. Sloane, 'Bounds for binary codes of length less than 25', *IEEE Trans. Information Theory* **24** (1978), 81–93.

[3] A. Hartman, R. C. Mullin, and D. R. Stinson, 'Exact covering configurations and Steiner systems', *J. London Math. Soc.* (2) **25** (1982), 193–200.
[4] R. C. Mullin, R. G. Stanton, and D. R. Stinson, 'Perfect pair-coverings and an algorithm for certain (1-2) factorizations of the complete graph K_{2s+1}', *Ars Combinatoria* **12** (1981), 73–80.
[5] N. J. A. Sloan and F. J. McWilliams, *The Theory of Error-Correcting Codes* (North-Holland, Amsterdam, 1977).
[6] R. G. Stanton, 'Old and new results on perfect coverings (Combinatorial Mathematics IX, Springer-Verlag, Berlin, Heidelberg, New York, 1981), 142–149.
[7] R. G. Stanton, J. L. Allston, and D. D. Cowan, 'Determination of an exact covering by triples', *Congr. Numer.* **31** (1981), 253–258.
[8] R. G. Stanton, J. L. Allston, and D. D. Cowan, 'Pair-coverings with restricted largest block length', *Ars Combinatoria* **11** (1981), 85–98.
[9] R. G. Stanton, J. L. Allston, P. D. Eades, and D. D. Cowan, 'Computation of the $g(1,3;20)$ cover', *J. of Combinatorics, Information, Syst. Sci.* **6** (1980), 173–177.
[10] R. G. Stanton and P. H. Dirksen, 'Computation of $g(1,3;12)$' (Combinatorial Mathematics IV, Springer-Verlag, Berlin, Heidelberg, New York, 1976), 232–234.
[11] R. G. Stanton, P. D. Eades, G. H. J. van Rees, and D. D. Cowan, 'Computation of some exact g-coverings', *Utilitas Math.* **18** (1980), 269–282.
[12] R. G. Stanton and I. P. Goulden, 'Graph factorization, general triple systems, and cyclic triple systems', *Aequationes Math.* **22** (1981), 1–28.
[13] R. G. Stanton and J. G. Kalbfleisch, 'The $\lambda - \mu$ problem: $\lambda = 1$ and $\mu = 3$' (Proc. Second Chapel Hill Conf. on Combinatorics, Univ. of North Carolina, 1972), 451–462.
[14] R. G. Stanton and J. G. Kalbfleisch, 'Maximal and minimal coverings of $(k-1)$-tuples by k-tuples', *Pacific J. Math.* **26** (1968), 131–140.
[15] R. G. Stanton and D. R. Stinson, 'Perfect pair-coverings with block sizes two, three, and four', *J. of Combinatorics, Information, Syst. Sci.* **8** (1983), 21–25.
[16] D. R. Stinson, 'Applications and generalizations of the variance method in combinatorial designs', *Utilitas Math.* **22** (1982), 323–333.
[17] D. R. Woodall, 'The λ-μ problem', *J. London Math. Soc.* **1** (1969), 505–519.

Department of Computer Science
University of Manitoba
Winnipeg, R3T 2N2
Canada