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NASH EQUILIBRIUM PAYOFFS FOR STOCHASTIC DIFFERENTIAL GAMES WITH TWO REFLECTING BARRIERS

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Abstract

In this paper we study Nash equilibrium payoffs for nonzero-sum stochastic differential games with two reflecting barriers. We obtain an existence and a characterization of Nash equilibrium payoffs for nonzero-sum stochastic differential games with nonlinear cost functionals defined by doubly controlled reflected backward stochastic differential equations with two reflecting barriers.

Keywords: Nash equilibrium payoff; stochastic differential game; backward stochastic differential equation

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1. Introduction

Duffie and Epstein [7] introduced a special kind of backward stochastic differential equation (BSDE) in order to investigate a stochastic differential recursive utility which dependents not only on the instantaneous consumption rate, but also on the future utility. Thus, it generalizes the standard additive utility. Pardoux and Peng [13] introduced nonlinear BSDEs. The theory of BSDEs has many applications in mathematical finance and mathematical economics, e.g. Knightian uncertainty problems in economics, asset pricing, and hedging of contingent claims. See El Karoui *et al.* [8], and the references therein for more applications.

Fleming and Souganidis [9] were the first to study zero-sum stochastic differential games in a rigorous way. Since this pioneering work, stochastic differential games have been investigated by many authors. We refer the reader to Buckdahn *et al.* [2], Buckdahn and Li [4], and the references therein.

Recently, Buckdahn *et al.* [3] studied Nash equilibrium payoffs for stochastic differential games with linear cost functionals. Lin [11], [12] generalizes the earlier result in [3]. In Lin [11], [12], the admissible control processes can depend on events occurring before the beginning of the stochastic differential game; thus, the cost functionals are not necessarily deterministic. Moreover, the cost functionals are defined with the help of BSDEs, and, thus, they are nonlinear.

The objective of this paper is to investigate Nash equilibrium payoffs for nonzero-sum stochastic differential games with two reflecting barriers whose cost functionals are defined by doubly controlled reflected backward stochastic differential equations (RBSDEs) with two reflecting barriers. Cvitanic and Karatzas [6] first studied RBSDEs with two reflecting barriers. This kind of RBSDE has many applications in economics, e.g. in Dynkin games and mixed zero-sum games. We shall study Nash equilibrium payoffs for nonzero-sum stochastic different

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games, which are different from the earlier ones [3], [11], and [12]. We shall also study Nash equilibrium payoffs for nonzero-sum stochastic differential games with more general running cost functionals, which are defined with the help of RBSDEs with two reflecting barriers. In comparison with [5] we shall study nonzero-sum stochastic differential games of the *strategy against strategy* type, while [5] considered the games of the *strategy against control* type.

In comparison with [11] and [12], this paper has the following improvements and advantages. First, the cost functionals of both players are defined by BSDEs without reflecting barriers as in [11] and BSDEs with one reflecting barriers as in [12]. In this paper, the cost functionals of both players are defined by BSDEs with two reflecting barriers. Thus, our results are more general. Second, for the proof of our results in this paper, we make use of elementary mathematical analysis techniques and the properties of BSDEs with two reflecting barriers. Finally, the presence of two reflecting barriers in this paper brings with it much difficulty and adds a level of supplementary complexity.

The paper is organized as follows. In Section 2 we introduce some notation and present some preliminary results concerning reflected RBSDEs with two reflecting barriers, which are useful in what follows. In Section 3 we introduce nonzero-sum stochastic differential games with reflection and obtain the associated dynamic programming principle. In Section 4 we give a probabilistic interpretation of systems of Isaacs' equations with two reflecting barriers. In Section 5 we obtain the main results of this paper, i.e. an existence and a characterization of Nash equilibrium payoffs for nonzero-sum stochastic differential games with two reflecting barriers.

2. Preliminaries

In this section we provide some notation and some results about BSDEs, which are useful in what follows. In this paper we shall work on the classical Wiener space $(\Omega, \mathcal{F}, \mathbb{P})$. For an arbitrarily fixed time horizon $T > 0$, Ω is the set of continuous functions from $[0, T]$ to $\mathbb{R}^d$, with the initial value 0, and $\mathcal{F}$ is the Borel σ-algebra over Ω, completed by the Wiener measure on $\mathbb{P}$. With respect to $\mathbb{P}$, the coordinate process $B_s(\omega) = \omega_s$, $s \in [0, T]$, $\omega \in \Omega$, is a d-dimensional Brownian motion. The filtration $\mathbb{F} = \{\mathcal{F}_t,\ 0 \le t \le T\}$ is generated by B and augmented by all $\mathbb{P}$-null sets, i.e.

$$\mathcal{F}_t = \sigma\{B_r, 0 \le r \le t\} \vee \mathcal{N}_{\mathbb{P}},$$

where $\mathcal{N}_{\mathbb{P}}$ is the set of all $\mathbb{P}$-null sets. Let us introduce some spaces:

$$L^2(\Omega, \mathcal{F}_T, \mathbb{P}; \mathbb{R}^n) = \{\xi \mid \xi\colon \Omega \to \mathbb{R}^n \text{ is an } \mathcal{F}_T\text{-measurable random variable such that } \mathbb{E}[|\xi|^2] < +\infty\},$$

$$S^2(0, T; \mathbb{R}) = \Bigl\{\varphi \Bigm| \varphi\colon \Omega \times [0, T] \to \mathbb{R} \text{ is an adapted continuous process such that } \mathbb{E}\Bigl[\sup_{0\le t\le T} |\varphi_t|^2\Bigr] < +\infty\Bigr\},$$

$$\mathcal{H}^2(0, T; \mathbb{R}^d) = \Bigl\{\varphi \Bigm| \varphi\colon \Omega \times [0, T] \to \mathbb{R}^d \text{ is a progressively measurable process such that } \mathbb{E}\int_0^T |\varphi_t|^2\,\mathrm{d}t < +\infty\Bigr\}.$$

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We consider the following two barriers reflected BSDEs with data (f, ξ, L, U):

$$
\begin{aligned}
Y_t = \xi + \int_t^T f(s, Y_s, Z_s)\,\mathrm{d}s + K_T^+ - K_t^+ - K_T^- + K_t^- \\
- \int_t^T Z_s\,\mathrm{d}B_s, \qquad L_t \le Y_t \le U_t, \quad t \in [0, T],
\end{aligned} \tag{1a}
$$

$$
\int_0^T (Y_t - L_t)\,\mathrm{d}K_t^+ = \int_0^T (Y_t - U_t)\,\mathrm{d}K_t^- = 0, \tag{1b}
$$

where $\{K_t^+\}$ and $\{K_t^-\}$ are adapted, continuous, and increasing processes such that $K_0^+ = 0$ and $K_0^- = 0$, $f\colon \Omega \times [0, T] \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ and we make the following assumptions:

(H2.1) $f(\cdot, 0, 0) \in \mathcal{H}^2(0, T; \mathbb{R})$,

(H2.2) there exists some constant $L > 0$ such that for all $y, y' \in \mathbb{R}$ and $z, z' \in \mathbb{R}^d$,

$$
|f(t, y, z) - f(t, y', z')| \le L(|y - y'| + |z - z'|),
$$

(H2.3) L and U are continuous processes such that $L, U \in S^2(0, T; \mathbb{R})$.

We have the existence and uniqueness theorem for solutions of (1). For its proof, see [10].

Lemma 1. *Under assumptions (H2.1)–(H2.3), if* $\xi \in L^2(\Omega, \mathcal{F}_T, \mathbb{P}; \mathbb{R})$, $L_T \le \xi \le U_T$, *and* $L_t < U_t, 0 \le t \le T$ *almost surely (a.s.), then (1) has a unique solution* (Y, Z, K^+, K^-).

We now provide the following estimate of the solutions of BSDEs with two reflecting barriers, which plays an important role in this paper. Since some of the proof technique is derived from Pham and Zhang [15], we omit the proof here.

Lemma 2. *We suppose that* (ξ^1, f^1, L^1, U^1) *and* (ξ^2, f^2, L^2, U^2) *satisfy the assumptions in Lemma 1. Let* $(Y^1, Z^1, K^{+,1}, K^{-,1})$ *and* $(Y^2, Z^2, K^{+,2}, K^{-,2})$ *be the solutions of the reflected BSDEs (1) with data* (ξ^1, f^1, L^1, U^1) *and* (ξ^2, f^2, L^2, U^2)*, respectively. Write*

$$
\begin{gathered}
\Delta\xi = \xi^1 - \xi^2, \qquad \Delta f = f^1 - f^2, \qquad \Delta L = L^1 - L^2, \\
\Delta U = U^1 - U^2, \qquad \Delta Y = Y^1 - Y^2, \qquad \Delta Z = Z^1 - Z^2, \\
\Delta K^+ = K^{+,1} - K^{+,2}, \qquad \Delta K^- = K^{-,1} - K^{-,2}.
\end{gathered}
$$

Then there exists a constant C such that

$$
\begin{aligned}
&\mathbb{E}\bigg[\sup_{t \le s \le T} |\Delta Y_s|^2 + \int_t^T |\Delta Z_s|^2\,\mathrm{d}s + |\Delta K_T^+ - \Delta K_t^+ - \Delta K_T^- + \Delta K_t^-|^2 \mid \mathcal{F}_t\bigg] \\
&\quad \le C\mathbb{E}\bigg[|\Delta\xi|^2 + \bigg(\int_t^T |\Delta f(s, Y_s^1, Z_s^1)|\,\mathrm{d}s\bigg)^2 \mid \mathcal{F}_t\bigg] \\
&\qquad + C\bigg(\mathbb{E}\bigg[\sup_{t \le s \le T} (|\Delta L_s|^2 + |\Delta U_s|^2) \mid \mathcal{F}_t\bigg]\bigg)^{1/2} (A_{t,T} + B_{t,T}),
\end{aligned}
$$

where

$$
A_{t,T} = \mathbb{E}\bigg[|\xi^1|^2 + \bigg(\int_t^T |f^1(s, 0, 0)|\,\mathrm{d}s\bigg)^2 + |\xi^2|^2 + \bigg(\int_t^T |f^2(s, 0, 0)|\,\mathrm{d}s\bigg)^2 \,\bigg|\, \mathcal{F}_t\bigg]^{1/2},
$$

and

$$B_{t,T} = \bigg(\mathbb{E}\Big[\sup_{t\le s\le T}[(L_s^1)^+]^2 + \sup_{t\le s\le T}[(L_s^2)^+]^2 + \sup_{t\le s\le T}[(U_s^1)^+]^2 + \sup_{t\le s\le T}[(U_s^2)^+]^2 \mid \mathcal{F}_t\Big] + \sum_{j=1}^{2}\sup_{\pi}\mathbb{E}\Big[\sum_{i=0}^{n}([\mathbb{E}[L_{t_{i+1}}^j \mid \mathcal{F}_{t_i}] - U_{t_i}^j]^+ + [L_{t_i}^j - \mathbb{E}[U_{t_{i+1}}^j \mid \mathcal{F}_{t_i}]]^+)^2 \mid \mathcal{F}_t\Big]\bigg)^{1/2},$$

the supremum is taken over all the partitions $\pi : t = t_0 < \cdots < t_n = T$.

We provide the comparison theorem for solutions of BSDEs with two reflecting barriers. For its proof, we refer the reader to [10] for more details.

Lemma 3. *We suppose that* (ξ^1, f^1, L^1, U^1) *and* (ξ^2, f^2, L^2, U^2) *satisfy the assumptions in Lemma 1. Let* $(Y^1, Z^1, K^{+,1}, K^{-,1})$ *and* $(Y^2, Z^2, K^{+,2}, K^{-,2})$ *be the solutions of the reflected BSDEs (1) with data* (ξ^1, f^1, L^1, U^1) *and* (ξ^2, f^2, L^2, U^2)*, respectively. If the following holds:*

(i) $\xi^1 \le \xi^2$, $\mathbb{P}$-*a.s.*

(ii) $f^1(t, y_t^2, z_t^2) \le f^2(t, y_t^2, z_t^2)$, $\mathrm{d}t\,\mathrm{d}\mathbb{P}$-*almost everywhere (a.e.)*

(iii) $L^1 \le L^2$, $U^1 \le U^2 \mathbb{P}$-*a.s.*

Then we have $Y_t^1 \le Y_t^2$, *a.s. for all* $t \in [0, T]$. *Moreover, if*

(iv) $f^1(t, y, z) \le f^2(t, y, z)$, $(t, y, z) \in [0, T] \times \mathbb{R} \times \mathbb{R}^d$, $\mathrm{d}t\,\mathrm{d}\mathbb{P}$-*a.e.*

(v) $L^1 = L^2$, $U^1 = U^2$, $\mathbb{P}$-*a.s.*

Then $K_t^{-,1} \le K_t^{-,2}$, $K_t^{+,1} \ge K_t^{+,2}$ $\mathbb{P}$-*a.s. for all* $t \in [0, T]$.

3. Nonzero-sum stochastic differential games with two reflecting barriers

In the following, let us suppose that U and V are two compact metric spaces. The space U (respectively, V) is considered as the control state-space of the first (respectively, second) player. We denote the associated sets of admissible controls by $\mathcal{U}$ and $\mathcal{V}$, respectively. The set $\mathcal{U}$ (respectively, $\mathcal{V}$) is the set of all U-valued (respectively, V-valued) $\mathbb{F}$-progressively measurable processes.

For given admissible controls $u(\cdot) \in \mathcal{U}$ and $v(\cdot) \in \mathcal{V}$, we consider the following control system for $t \in [0, T]$:

$$\mathrm{d}X_s^{t,x;u,v} = b(s, X_s^{t,x;u,v}, u_s, v_s)\,\mathrm{d}s + \sigma(s, X_s^{t,x;u,v}, u_s, v_s)\,\mathrm{d}B_s, \qquad s \in [t, T], \tag{2a}$$

$$X_t^{t,x;u,v} = x \in \mathbb{R}^n, \tag{2b}$$

where

$$b\colon [0, T] \times \mathbb{R}^n \times U \times V \to \mathbb{R}^n, \qquad \sigma\colon [0, T] \times \mathbb{R}^n \times U \times V \to \mathbb{R}^{n\times d}.$$

Let us make the following assumptions.

(H3.1) For all $x \in \mathbb{R}^n$, $b(\cdot, x, \cdot, \cdot)$ and $\sigma(\cdot, x, \cdot, \cdot)$ are continuous in (t, u, v).

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(H3.2) There exists a positive constant L such that, for all $t \in [0, T]$, $x, x' \in \mathbb{R}^n$, $u \in U$, $v \in V$,

$$|b(t, x, u, v) - b(t, x', u, v)| + |\sigma(t, x, u, v) - \sigma(t, x', u, v)| \le L|x - x'|.$$

Under the above assumptions, for any $u(\cdot) \in \mathcal{U}$ and $v(\cdot) \in \mathcal{V}$, (2) has a unique strong solution $\{X_s^{t,x;u,v}\}_{t\le s\le T}$, and the following standard estimates for solutions hold.

Lemma 4. *For all $p \ge 2$, there exists a positive constant C_p such that for all $t \in [0, T]$, $x, x' \in \mathbb{R}^n$, $u(\cdot) \in \mathcal{U}$, and $v(\cdot) \in \mathcal{V}$,*

$$\mathbb{E}\Big[\sup_{t\le s\le T} |X_s^{t,x;u,v}|^p \mid \mathcal{F}_t\Big] \le C_p(1 + |x|^p), \qquad \mathbb{P}\text{-a.s.},$$
$$\mathbb{E}\Big[\sup_{t\le s\le T} |X_s^{t,x;u,v} - X_s^{t,x';u,v}|^p \mid \mathcal{F}_t\Big] \le C_p|x - x'|^p, \qquad \mathbb{P}\text{-a.s.},$$

where the constant C_p depends only on p, the Lipschitz constant, and the linear growth of b and σ.

For given admissible controls $u(\cdot) \in \mathcal{U}$ and $v(\cdot) \in \mathcal{V}$, let us consider the following doubly controlled BSDE with two reflecting barriers for fixed $j = 1, 2$:

$$\begin{aligned} {}^jY_s^{t,x;u,v} &= \Phi_j(X_T^{t,x;u,v}) + \int_s^T f_j(r, X_r^{t,x;u,v}, {}^jY_r^{t,x;u,v}, {}^jZ_r^{t,x;u,v}, u_r, v_r)\,\mathrm{d}r + {}^jK_T^{+,t,x;u,v} \\ &\quad - {}^jK_s^{+,t,x;u,v} - {}^jK_T^{-,t,x;u,v} + {}^jK_s^{-,t,x;u,v} - \int_s^T {}^jZ_r^{t,x;u,v}\,\mathrm{d}B_r, \end{aligned} \tag{3a}$$

$$h_j(s, X_s^{t,x;u,v}) \le {}^jY_s^{t,x;u,v} \le h'_j(s, X_s^{t,x;u,v}), \qquad s \in [t, T], \tag{3b}$$

$$\int_t^T ({}^jY_r^{t,x;u,v} - h_j(r, X_r^{t,x;u,v}))\,\mathrm{d}\,{}^jK_r^{+,t,x;u,v} = 0, \tag{3c}$$

$$\int_t^T ({}^jY_r^{t,x;u,v} - h'_j(r, X_r^{t,x;u,v}))\,\mathrm{d}\,{}^jK_r^{-,t,x;u,v} = 0, \tag{3d}$$

where $X^{t,x;u,v}$ is introduced in (2) and

$$\Phi_j = \Phi_j(x)\colon \mathbb{R}^n \to \mathbb{R}, \qquad h'_j = h'_j(t, x), \qquad h_j = h_j(t, x)\colon [0, T] \times \mathbb{R}^n \to \mathbb{R},$$
$$f_j = f_j(t, x, y, z, u, v)\colon [0, T] \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^d \times U \times V \to \mathbb{R}.$$

We make the following assumptions.

(H3.3) There exists a positive constant L such that, for all $t \in [0, T]$, $x, x' \in \mathbb{R}^n$, $y, y' \in \mathbb{R}$, $z, z' \in \mathbb{R}^d$, $u \in U$ and $v \in V$,

$$\begin{aligned} &|f_j(t, x, y, z, u, v) - f_j(t, x', y', z', u, v)| + |\Phi_j(x) - \Phi_j(x')| \\ &\qquad \le L(|x - x'| + |y - y'| + |z - z'|). \end{aligned}$$

In addition, we suppose that $h_j(t, x) < h'_j(t, x)$, $h_j(T, x) \le \Phi_j(x) \le h'_j(T, x)$ for all $(t, x) \in [0, T] \times \mathbb{R}^n$.

(H3.4) For all $(x, y, z) \in \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^d$, $f_j(\cdot, x, y, z, \cdot, \cdot)$ is continuous in (t, u, v), and there exists a positive constant L such that for all $t, s \in [0, T]$, $x, y \in \mathbb{R}^n$,

$$|h_j(t, x) - h_j(s, y)| + |h'_j(t, x) - h'_j(s, y)| \le L(|x - y| + |t - s|^{1/2}).$$

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Under the above assumptions, from [10] it follows that equation (3) admits a unique solution. For given control processes $u(\cdot) \in U$ and $v(\cdot) \in V$, we introduce the associated cost functional for player j, $j = 1, 2$,

$$J_j(t, x; u, v) := {}^jY_s^{t,x;u,v}\big|_{s=t}, \qquad (t, x) \in [0, T] \times \mathbb{R}^n.$$

By virtue of [5], the following estimates for solutions hold.

Proposition 1. *Under the assumptions (H3.1)–(H3.4), there exists a positive constant C such that, for all $t \in [0, T]$, $u(\cdot) \in \mathcal{U}$ and $v(\cdot) \in \mathcal{V}$, $x, x' \in \mathbb{R}^n$,*

$$|{}^jY_t^{t,x;u,v}| \leq C(1 + |x|), \quad \mathbb{P}\text{-a.s.}, \qquad |{}^jY_t^{t,x;u,v} - {}^jY_t^{t,x';u,v}| \leq C|x - x'|, \quad \mathbb{P}\text{-a.s.}$$

Let us now give the definition of admissible controls and a nonanticipating strategy with delay (NAD strategy), which were introduced in [11].

Definition 1. The space $\mathcal{U}_{t,T}$ (respectively, $\mathcal{V}_{t,T}$) of admissible controls for Player I (respectively, Player II) on the interval $[t, T]$ is defined as the space of all processes $\{u_r\}_{r\in[t,T]}$ (respectively, $\{v_r\}_{r\in[t,T]}$), which are $\mathbb{F}$-progressively measurable and take values in U (respectively, V).

Definition 2. An NAD strategy for Player I is a measurable mapping $\alpha\colon \mathcal{V}_{t,T} \to \mathcal{U}_{t,T}$, which satisfies the following properties:

1. α is a nonanticipative strategy, i.e. for every $\mathbb{F}$-stopping time $\tau\colon \Omega \to [t, T]$ and for $v_1, v_2 \in \mathcal{V}_{t,T}$ with $v_1 = v_2$ on $[[t, \tau]]$, it holds that $\alpha(v_1) = \alpha(v_2)$ on $[[t, \tau]]$. (Recall that $[[t, \tau]] = \{(s, \omega) \in [t, T] \times \Omega, t \leq s \leq \tau(\omega)\}$).

2. α is a strategy with delay, i.e. for all $v \in \mathcal{V}_{t,T}$, there exists an increasing sequence of stopping times $\{S_n(v)\}_{n\geq 1}$ with

   (i) $t = S_0(v) \leq S_1(v) \leq \cdots \leq S_n(v) \leq \cdots \leq T$,

   (ii) $\bigcup_{n\geq 1}\{S_n(v) = T\} = \Omega$, $\mathbb{P}$-a.s. such that, for all $n \geq 1$ and $v, v' \in \mathcal{V}_{t,T}$, $\Gamma \in \mathcal{F}_t$, it holds, if $v = v'$ on $[[t, S_{n-1}(v)]]\bigcap([t, T] \times \Gamma)$, then

   (iii) $S_l(v) = S_l(v')$, on Γ, $1 \leq l \leq n$,

   (iv) $\alpha(v) = \alpha(v')$, on $[[t, S_n(v)]]\bigcap([t, T] \times \Gamma)$.

Let us denote the set of all NAD strategies for Player I for games over the time interval $[t, T]$ by $\mathcal{A}_{t,T}$. The set of all NAD strategies $\beta\colon \mathcal{U}_{t,T} \to \mathcal{V}_{t,T}$ for Player II for games over the time interval $[t, T]$ can be defined in a symmetrical way and let us denote it by $B_{t,T}$.

We have the following useful lemma, which was established in [11]. The lemma treats both players in a fair way.

Lemma 5. *Let $(\alpha, \beta) \in \mathcal{A}_{t,T} \times \mathcal{B}_{t,T}$. Then there exists a unique couple of admissible controls $(u, v) \in \mathcal{U}_{t,T} \times \mathcal{V}_{t,T}$ such that $\alpha(v) = u$ and $\beta(u) = v$.*

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For $(\alpha, \beta) \in \mathcal{A}_{t,T} \times \mathcal{B}_{t,T}$, from Lemma 5 we know that there exists a unique couple $(u, v) \in \mathcal{U}_{t,T} \times \mathcal{V}_{t,T}$ such that $(\alpha(v), \beta(u)) = (u, v)$. Thus, we write $J_j(t, x; \alpha, \beta) = J_j(t, x; u, v)$. Therefore, we define the lower and the upper value-functions W_j and U_j, respectively, associated with J_j: for all $(t, x) \in [0, T] \times \mathbb{R}^n$ and let us set

$$W_j(t, x) := \operatorname*{ess\,sup}_{\alpha \in \mathcal{A}_{t,T}} \operatorname*{ess\,inf}_{\beta \in \mathcal{B}_{t,T}} J_j(t, x; \alpha, \beta), \qquad U_j(t, x) := \operatorname*{ess\,inf}_{\beta \in \mathcal{B}_{t,T}} \operatorname*{ess\,sup}_{\alpha \in \mathcal{A}_{t,T}} J_j(t, x; \alpha, \beta).$$

Remark 1. We do not assume that the admissible control processes of both players are independent of the information at the beginning of the stochastic differential game, i.e. $\mathcal{F}_t$. The admissible control processes can depend on events occurring before the beginning of the stochastic differential game; thus, the cost functionals are not necessarily deterministic. Moreover, the cost functionals are defined by the solutions of RBSDEs with two reflecting barriers, and; thus, they are nonlinear.

Under assumptions (H3.1)–(H3.4), it follows that $W_j(t, x)$ and $U_j(t, x)$ are random variables. But, by virtue of the arguments in [2] and [11], the following proposition holds.

Proposition 2. *Let assumptions (H3.1)–(H3.4) hold. Then for all $(t, x) \in [0, T] \times \mathbb{R}^n$, the value-functions $W_j(t, x)$ and $U_j(t, x)$ are deterministic.*

We now recall the definition of stochastic backward semigroups, which was first introduced by Peng [14] in order to study a stochastic optimal control problem. For a given initial condition (t, x), a positive number $\delta \le T - t$, for admissible control processes $u(\cdot) \in \mathcal{U}_{t,t+\delta}$ and $v(\cdot) \in \mathcal{V}_{t,t+\delta}$, and a real-valued random variable $\eta \in L^2(\Omega, \mathcal{F}_{t+\delta}, \mathbb{P}; \mathbb{R})$ such that $h_j(t+\delta, X_{t+\delta}^{t,x;u,v}) \le \eta \le h'_j(t+\delta, X_{t+\delta}^{t,x;u,v})$, we define

$${}^jG_{t,t+\delta}^{t,x;u,v}[\eta] := {}^j\hat{Y}_t^{t,x;u,v},$$

where $({}^j\hat{Y}^{t,x;u,v}, {}^j\hat{Z}^{t,x;u,v}, {}^j\hat{K}^{+,t,x;u,v}, {}^j\hat{K}^{-,t,x;u,v})$ is the unique solution of the following BSDE with two reflecting barriers over the time interval $[t, t+\delta]$:

$$\begin{aligned}
{}^j\hat{Y}_s^{t,x;u,v} &= \eta + \int_s^{t+\delta} f_j(r, X_r^{t,x;u,v}, {}^j\hat{Y}_r^{t,x;u,v}, {}^j\hat{Z}_r^{t,x;u,v}, u_r, v_r)\,\mathrm{d}r \\
&\quad + {}^j\hat{K}_{t+\delta}^{+,t,x;u,v} - {}^j\hat{K}_s^{+,t,x;u,v} - {}^j\hat{K}_{t+\delta}^{-,t,x;u,v} + {}^j\hat{K}_s^{-,t,x;u,v} - \int_s^{t+\delta} {}^j\hat{Z}_r^{t,x;u,v}\,\mathrm{d}B_r, \\
&\quad h_j(s, X_s^{t,x;u,v}) \le {}^j\hat{Y}_s^{t,x;u,v} \le h'_j(s, X_s^{t,x;u,v}), \qquad s \in [t, t+\delta], \\
&\quad \int_t^{t+\delta} ({}^j\hat{Y}_r^{t,x;u,v} - h_j(r, X_r^{t,x;u,v}))\,\mathrm{d}{}^j\hat{K}_r^{+,t,x;u,v} = 0, \\
&\quad \int_t^{t+\delta} ({}^j\hat{Y}_r^{t,x;u,v} - h'_j(r, X_r^{t,x;u,v}))\,\mathrm{d}{}^j\hat{K}_r^{-,t,x;u,v} = 0,
\end{aligned}$$

and $X^{t,x;u,v}$ is the unique solution of (2).

For $(t, x) \in [0, T] \times \mathbb{R}^n$, $(u, v) \in \mathcal{U}_{t,T} \times \mathcal{V}_{t,T}$, $0 \le \delta \le T - \delta$, $j = 1, 2$, we have

$$\begin{aligned}
J_j(t, x; u, v) &= {}^jG_{t,T}^{t,x;u,v}[\Phi_j(X_T^{t,x;u,v})] \\
&= {}^jG_{t,t+\delta}^{t,x;u,v}[{}^jY_{t+\delta}^{t,x;u,v}] \\
&= {}^jG_{t,t+\delta}^{t,x;u,v}[J_j(t+\delta, X_{t+\delta}^{t,x;u,v}, u, v)].
\end{aligned}$$

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Proposition 3. *Let the assumptions (H3.1)–(H3.4) hold. Then we have the following dynamic programming principle for all* $0 < \delta \le T - t,\ x \in \mathbb{R}^n$*:*

$$W_j(t,x) = \operatorname*{ess\,sup}_{\alpha \in \mathcal{A}_{t,t+\delta}} \operatorname*{ess\,inf}_{\beta \in \mathcal{B}_{t,t+\delta}} {}^{j}G^{t,x;\alpha,\beta}_{t,t+\delta}[W_j(t+\delta, X^{t,x;\alpha,\beta}_{t+\delta})],$$

$$U_j(t,x) = \operatorname*{ess\,inf}_{\beta \in \mathcal{B}_{t,t+\delta}} \operatorname*{ess\,sup}_{\alpha \in \mathcal{A}_{t,t+\delta}} {}^{j}G^{t,x;\alpha,\beta}_{t,t+\delta}[U_j(t+\delta, X^{t,x;\alpha,\beta}_{t+\delta})].$$

For the proof of the above proposition, we can use the similar arguments to those in [2] and [11]. We omit it here. Using the standard arguments and Proposition 3, we can easily obtain the following proposition.

Proposition 4. *Let assumptions (H3.1)–(H3.4) hold. Then there exists a positive constant* C *such that for all* $t, t' \in [0, T]$ *and* $x, x' \in \mathbb{R}^n$*, we have*

(i) $W_j(t,x)$ *is* $\frac{1}{2}$*-Hölder continuous in* t*:*

$$|W_j(t,x) - W_j(t',x)| \le C(1+|x|)|t-t'|^{1/2};$$

(ii) $|W_j(t,x) - W_j(t,x')| \le C|x-x'|$.

The same properties holds for the function U_j*.*

4. Probabilistic interpretation of systems of Isaacs' equations with obstacles

The objective of this section is to give a probabilistic interpretation of systems of Isaacs' equations with obstacles, and to show that W_j and U_j introduced in Section 3 are the viscosity solutions of the following Isaacs' equations with obstacles for $(t,x) \in [0,T) \times \mathbb{R}^n$,

$$\min\Big\{W_j(t,x) - h_j(t,x), \max\Big[W_j(t,x) - h'_j(t,x), -\frac{\partial}{\partial t}W_j(t,x) \\ - H^-_j(t,x,W_j(t,x), DW_j(t,x), D^2W_j(t,x))\Big]\Big\} = 0, \tag{4a}$$

$$W_j(T,x) = \Phi_j(x), \tag{4b}$$

and

$$\min\Big\{U_j(t,x) - h_j(t,x), \max\Big[U_j(t,x) - h'_j(t,x), -\frac{\partial}{\partial t}U_j(t,x) \\ - H^+_j(t,x,U_j(t,x), DU_j(t,x), D^2U_j(t,x))\Big]\Big\} = 0, \tag{5a}$$

$$U_j(T,x) = \Phi_j(x), \tag{5b}$$

respectively, where

$$H_j(t,x,y,p,A,u,v) = \tfrac{1}{2}tr(\sigma\sigma^T(t,x,u,v)A) + pb(t,x,u,v) \\ + f_j(t,x,y,p\sigma(t,x,u,v),u,v),$$

$(t, x, y, p, u, v) \in [0, T] \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n \times U \times V$ and $A \in \mathbb{S}^n$ ($\mathbb{S}^n$ denotes all the $n \times n$ symmetric matrices),

$$H_j^-(t, x, y, p, A) = \sup_{u \in U} \inf_{v \in V} H_j(t, x, y, p, A, u, v),$$
$$H_j^+(t, x, y, p, A) = \inf_{v \in V} \sup_{u \in U} H_j(t, x, y, p, A, u, v).$$

By means of the arguments [5] and [12], we can obtain the following theorem. We omit the proof here.

Theorem 1. *Let assumptions (H3.1)–(H3.4) hold. Then the function W_j (respectively, U_j) is a viscosity solution of the system (4) (respectively, (5)).*

We now provide a comparison theorem for the viscosity solution of (4). Let us first introduce the following space:

$$\Theta := \Big\{\varphi \in C([0, T] \times \mathbb{R}^n)\colon \text{there exists a constant } A > 0 \text{ such that}$$
$$\lim_{|x|\to\infty} |\varphi(t, x)| \exp\{-A[\log((|x|^2 + 1)^{1/2})]^2\} = 0, \text{ uniformly in } t \in [0, T]\Big\}.$$

Theorem 2. *Under the assumptions (H3.1)–(H3.4), if an upper semicontinuous function $u_1 \in \Theta$ is a viscosity subsolution of (4) (respectively, (5)), and a lower semicontinuous function $u_2 \in \Theta$ is a viscosity supersolution of (4) (respectively, (5)), then we have*

$$u_1(t, x) \le u_2(t, x) \quad \textit{for all } (t, x) \in [0, T] \times \mathbb{R}^n.$$

Using the arguments in Barles *et al.* [1], we can obtain the above theorem. We omit the proof here.

Remark 2. From Proposition 4 it follows that W_j (respectively, U_j) is a viscosity solution of linear growth. Therefore, from the above theorem we see that W_j (respectively, U_j) is the unique viscosity solution in Θ of the system (4) (respectively, (5)).

Isaacs' condition: For all $(t, x, y, p, A, u, v) \in [0, T] \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{S}^n \times U \times V$, $j = 1, 2$, we have

$$\begin{aligned} &\sup_{u \in U} \inf_{v \in V} \Big\{\tfrac{1}{2} tr(\sigma\sigma^T(t, x, u, v)A) + pb(t, x, u, v) + f_j(t, x, y, p\sigma(t, x, u, v), u, v)\Big\} \\ &\quad = \inf_{v \in V} \sup_{u \in U} \Big\{\tfrac{1}{2} tr(\sigma\sigma^T(t, x, u, v)A) + pb(t, x, u, v) + f_j(t, x, y, p\sigma(t, x, u, v), u, v)\Big\}. \end{aligned} \tag{6}$$

Corollary 1. *Let Isaacs' condition (6) hold. Then we have for all $(t, x) \in [0, T] \times \mathbb{R}^n$,*

$$(U_1(t, x), U_2(t, x)) = (W_1(t, x), W_2(t, x)).$$

In a symmetric way for all $(t, x) \in [0, T] \times \mathbb{R}^n$, we write

$$\overline{W}_j(t, x) := \operatorname*{ess\,sup}_{\beta \in \mathcal{B}_{t,T}} \operatorname*{ess\,inf}_{\alpha \in \mathcal{A}_{t,T}} J_j(t, x; \alpha, \beta), \qquad \overline{U}_j(t, x) := \operatorname*{ess\,inf}_{\alpha \in \mathcal{A}_{t,T}} \operatorname*{ess\,sup}_{\beta \in \mathcal{B}_{t,T}} J_j(t, x; \alpha, \beta).$$

By virtue of the arguments in [2], we have the following propositions.

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Proposition 5. *Let assumptions (H3.1)–(H3.4) hold. Then for all $(t,x) \in [0,T] \times \mathbb{R}^n$, the value-functions $\overline{W}_j(t,x)$ and $\overline{U}_j(t,x)$ are deterministic.*

Proposition 6. *Let assumptions (H3.1)–(H3.4) hold. Then we have the following dynamic programming principle: for all $0 < \delta \le T - t$, $x \in \mathbb{R}^n$,*

$$\overline{W}_j(t,x) = \operatorname*{ess\,sup}_{\beta \in \mathcal{B}_{t,t+\delta}} \operatorname*{ess\,inf}_{\alpha \in \mathcal{A}_{t,t+\delta}} {}^jG^{t,x;\alpha,\beta}_{t,t+\delta}[\overline{W}_j(t+\delta, X^{t,x;\alpha,\beta}_{t+\delta})],$$

$$\overline{U}_j(t,x) = \operatorname*{ess\,inf}_{\alpha \in \mathcal{A}_{t,t+\delta}} \operatorname*{ess\,sup}_{\beta \in \mathcal{B}_{t,t+\delta}} {}^jG^{t,x;\alpha,\beta}_{t,t+\delta}[\overline{U}_j(t+\delta, X^{t,x;\alpha,\beta}_{t+\delta})].$$

Isaacs' condition: For all $(t,x,y,p,A,u,v) \in [0,T] \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{S}^n \times U \times V$, $j = 1, 2$, we have

$$\begin{aligned}
&\inf_{u \in U} \sup_{v \in V} \left\{ \tfrac{1}{2} tr(\sigma\sigma^T(t,x,u,v)A) + pb(t,x,u,v) + f_j(t,x,y,p\sigma(t,x,u,v),u,v) \right\} \\
&\quad = \sup_{v \in V} \inf_{u \in U} \left\{ \tfrac{1}{2} tr(\sigma\sigma^T(t,x,u,v)A) + pb(t,x,u,v) + f_j(t,x,y,p\sigma(t,x,u,v),u,v) \right\}.
\end{aligned} \tag{7}$$

Using the above arguments in this section, we have the following proposition.

Proposition 7. *Under Isaacs' condition (7), we have for all $(t,x) \in [0,T] \times \mathbb{R}^n$,*

$$(\overline{U}_1(t,x), \overline{U}_2(t,x)) = (\overline{W}_1(t,x), \overline{W}_2(t,x)).$$

5. Nash equilibrium payoffs

The objective of this section is to obtain an existence of Nash equilibrium payoffs for nonzero-sum stochastic differential games.

In this section, let us redefine the following notation which are different from the above sections, for $(t,x) \in [0,T] \times \mathbb{R}^n$,

$$W_1(t,x) := \operatorname*{ess\,sup}_{\alpha \in \mathcal{A}_{t,T}} \operatorname*{ess\,inf}_{\beta \in \mathcal{B}_{t,T}} J_1(t,x;\alpha,\beta), \qquad W_2(t,x) := \operatorname*{ess\,sup}_{\beta \in \mathcal{B}_{t,T}} \operatorname*{ess\,inf}_{\alpha \in \mathcal{A}_{t,T}} J_2(t,x;\alpha,\beta).$$

Let us suppose that the following condition holds.

Isaacs' condition A: For all $(t,x,y,p,A,u,v) \in [0,T] \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{S}^n \times U \times V$, we have

$$\begin{aligned}
&\sup_{u \in U} \inf_{v \in V} \left\{ \tfrac{1}{2} tr(\sigma\sigma^T(t,x,u,v)A) + pb(t,x,u,v) + f_1(t,x,y,p\sigma(t,x,u,v),u,v) \right\} \\
&\quad = \inf_{v \in V} \sup_{u \in U} \left\{ \tfrac{1}{2} tr(\sigma\sigma^T(t,x,u,v)A) + pb(t,x,u,v) + f_1(t,x,y,p\sigma(t,x,u,v),u,v) \right\},
\end{aligned}$$

and

$$\begin{aligned}
&\inf_{u \in U} \sup_{v \in V} \left\{ \tfrac{1}{2} tr(\sigma\sigma^T(t,x,u,v)A) + pb(t,x,u,v) + f_2(t,x,y,p\sigma(t,x,u,v),u,v) \right\} \\
&\quad = \sup_{v \in V} \inf_{u \in U} \left\{ \tfrac{1}{2} tr(\sigma\sigma^T(t,x,u,v)A) + pb(t,x,u,v) + f_2(t,x,y,p\sigma(t,x,u,v),u,v) \right\}.
\end{aligned}$$

Under the above condition, from the previous section, we have for $(t, x) \in [0, T] \times \mathbb{R}^n$,

$$W_1(t, x) = \operatorname*{ess\,sup}_{\alpha \in \mathcal{A}_{t,T}} \operatorname*{ess\,inf}_{\beta \in \mathcal{B}_{t,T}} J_1(t, x; \alpha, \beta) = \operatorname*{ess\,inf}_{\beta \in \mathcal{B}_{t,T}} \operatorname*{ess\,sup}_{\alpha \in \mathcal{A}_{t,T}} J_1(t, x; \alpha, \beta)$$

and

$$W_2(t, x) = \operatorname*{ess\,inf}_{\alpha \in \mathcal{A}_{t,T}} \operatorname*{ess\,sup}_{\beta \in \mathcal{B}_{t,T}} J_2(t, x; \alpha, \beta) = \operatorname*{ess\,sup}_{\beta \in \mathcal{B}_{t,T}} \operatorname*{ess\,inf}_{\alpha \in \mathcal{A}_{t,T}} J_2(t, x; \alpha, \beta).$$

In order to simplify arguments, let us also assume that the coefficients b, σ, Φ_j, h_j, h'_j, and f_j satisfy assumptions (H3.1)–(H3.4) and are bounded.

We present the definition of the Nash equilibrium payoff of stochastic differential games (see [11] or [3] for more details).

Definition 3. A couple $(e_1, e_2) \in \mathbb{R}^2$ is called a Nash equilibrium payoff at the point (t, x) if for any $\varepsilon > 0$, there exists $(\alpha_\varepsilon, \beta_\varepsilon) \in \mathcal{A}_{t,T} \times \mathcal{B}_{t,T}$ such that for all $(\alpha, \beta) \in \mathcal{A}_{t,T} \times \mathcal{B}_{t,T}$,

$$\begin{aligned} J_1(t, x; \alpha_\varepsilon, \beta_\varepsilon) &\geq J_1(t, x; \alpha, \beta_\varepsilon) - \varepsilon, \\ J_2(t, x; \alpha_\varepsilon, \beta_\varepsilon) &\geq J_2(t, x; \alpha_\varepsilon, \beta) - \varepsilon, \qquad \mathbb{P}\text{-a.s.}, \end{aligned} \tag{8}$$

and

$$|\mathbb{E}[J_j(t, x; \alpha_\varepsilon, \beta_\varepsilon)] - e_j| \leq \varepsilon, \qquad j = 1, 2.$$

By Lemma 5, we have the following lemma.

Lemma 6. *For any $\varepsilon > 0$ and $(\alpha_\varepsilon, \beta_\varepsilon) \in \mathcal{A}_{t,T} \times \mathcal{B}_{t,T}$, (8) holds if and only if for all $(u, v) \in \mathcal{U}_{t,T} \times \mathcal{V}_{t,T}$,*

$$\begin{aligned} J_1(t, x; \alpha_\varepsilon, \beta_\varepsilon) &\geq J_1(t, x; u, \beta_\varepsilon(u)) - \varepsilon, \qquad \mathbb{P}\text{-a.s.}, \\ J_2(t, x; \alpha_\varepsilon, \beta_\varepsilon) &\geq J_2(t, x; \alpha_\varepsilon(v), v) - \varepsilon, \qquad \mathbb{P}\text{-a.s.} \end{aligned}$$

We first provide the following lemma, which we shall need in what follows.

Lemma 7. *Let $(t, x) \in [0, T] \times \mathbb{R}^n$ and $u \in \mathcal{U}_{t,T}$ be arbitrarily fixed. Then*

(i) *for all $\delta \in [0, T - t]$ and $\varepsilon > 0$, there exists a NAD strategy $\alpha \in \mathcal{A}_{t,T}$ such that for all $v \in \mathcal{V}_{t,T}$,*

$$\alpha(v) = u, \text{ on } [t, t + \delta], \qquad {}^2Y^{t,x;\alpha(v),v}_{t+\delta} \leq W_2(t + \delta, X^{t,x;\alpha(v),v}_{t+\delta}) + \varepsilon, \qquad \mathbb{P}\text{-a.s.}$$

(ii) *for all $\delta \in [0, T - t]$ and $\varepsilon > 0$, there exists a NAD strategy $\alpha \in \mathcal{A}_{t,T}$ such that for all $v \in \mathcal{V}_{t,T}$,*

$$\alpha(v) = u, \text{ on } [t, t + \delta], \qquad {}^1Y^{t,x;\alpha(v),v}_{t+\delta} \geq W_1(t + \delta, X^{t,x;\alpha(v),v}_{t+\delta}) - \varepsilon, \qquad \mathbb{P}\text{-a.s.}$$

For the proof of the above lemma, see [11]. We also need the following lemma. And we easily prove it by standard arguments for SDEs.

Lemma 8. *There exists a positive constant C such that for all $(u, v), (u', v') \in \mathcal{U}_{t,T} \times \mathcal{V}_{t,T}$, and for all $\mathcal{F}_r$-stopping times $S \colon \Omega \to [t, T]$ such that $X^{t,x;u,v}_S = X^{t,x;u',v'}_S$, $\mathbb{P}$-a.s., it holds for all real $\tau \in [0, T]$,*

$$\mathbb{E}[\sup_{0 \leq s \leq \tau} |X^{t,x;u,v}_{(S+s)\wedge T} - X^{t,x;u',v'}_{(S+s)\wedge T}|^2 \mid \mathcal{F}_t] \leq C\tau, \qquad \mathbb{P}\text{-a.s.}$$

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We now provide one of the main results of this section: the characterization theorem of Nash equilibrium payoffs for nonzero-sum stochastic differential games with two reflecting barriers as follows.

Theorem 3. *Under Isaacs' condition A, for $(t, x) \in [0, T] \times \mathbb{R}^n$, and for all $\varepsilon > 0$, if there exists $(u^\varepsilon, v^\varepsilon) \in \mathcal{U}_{t,T} \times \mathcal{V}_{t,T}$ such that for all $s \in [t, T]$ and $j = 1, 2$,*

$$\mathbb{P}({}^jY_s^{t,x;u^\varepsilon,v^\varepsilon} \geq W_j(s, X_s^{t,x;u^\varepsilon,v^\varepsilon}) - \varepsilon \mid \mathcal{F}_t) \geq 1 - \varepsilon, \qquad \mathbb{P}\text{-a.s.}, \tag{9}$$

and

$$|\mathbb{E}[J_j(t, x; u^\varepsilon, v^\varepsilon)] - e_j| \leq \varepsilon, \tag{10}$$

then $(e_1, e_2) \in \mathbb{R}^2$ is a Nash equilibrium payoff at point (t, x).

Proof. For arbitrarily fixed $\varepsilon > 0$ and some $\varepsilon_0 > 0$ (ε_0 depends on ε), we suppose that $(u^{\varepsilon_0}, v^{\varepsilon_0}) \in \mathcal{U}_{t,T} \times \mathcal{V}_{t,T}$ satisfies (9) and (10), i.e. for all $s \in [t, T]$ and $j = 1, 2$,

$$\mathbb{P}({}^jY_s^{t,x;u^{\varepsilon_0},v^{\varepsilon_0}} \geq W_j(s, X_s^{t,x;u^{\varepsilon_0},v^{\varepsilon_0}}) - \varepsilon_0 \mid \mathcal{F}_t) \geq 1 - \varepsilon_0, \qquad \mathbb{P}\text{-a.s.}, \tag{11}$$

as well as

$$|\mathbb{E}[J_j(t, x; u^{\varepsilon_0}, v^{\varepsilon_0})] - e_j| \leq \varepsilon_0. \tag{12}$$

Let us fix some partition $t = t_0 \leq t_1 \leq \cdots \leq t_m = T$ of $[t, T]$ and write $\tau = \sup_i |t_i - t_{i+1}|$. Applying Lemma 7 to u^{ε_0} and $t + \delta = t_1, \ldots, t_m$, successively, for $\varepsilon_1 > 0$ (ε_1 depends on ε and is specified later), we have the existence of NAD strategies $\alpha_i \in \mathcal{A}_{t,T}$, $i = 1, \ldots, m$, such that for all $v \in \mathcal{V}_{t,T}$,

$$\alpha_i(v) = u^{\varepsilon_0}, \text{ on } [t, t_i],$$
$${}^2Y_{t_i}^{t,x;\alpha_i(v),v} \leq W_2(t_i, X_{t_i}^{t,x;\alpha_i(v),v}) + \varepsilon_1, \qquad \mathbb{P}\text{-a.s.} \tag{13}$$

For all $v \in \mathcal{V}_{t,T}$, let

$$S^v = \inf\{s \geq t \mid \lambda(\{r \in [t, s]\colon v_r \neq v_r^{\varepsilon_0}\}) > 0\},$$
$$t^v = \inf\{t_i \geq S^v \mid i = 1, \ldots, m\} \wedge T,$$

where λ denotes the Lebesgue measure on the real line $\mathbb{R}$. It can be checked that S^v and t^v are stopping times such that $S^v \leq t^v \leq S^v + \tau$. Let

$$\alpha_\varepsilon(v) = \begin{cases} u^{\varepsilon_0}, & \text{on } [[t, t^v]], \\ \alpha_i(v), & \text{on } (t_i, T] \times \{t^v = t_i\},\ 1 \leq i \leq m. \end{cases}$$

Then α_ε is a NAD strategy. From (13), we obtain

$$\begin{aligned} {}^2Y_{t^v}^{t,x;\alpha_\varepsilon(v),v} &= \sum_{i=1}^m {}^2Y_{t_i}^{t,x;\alpha_\varepsilon(v),v}\, \mathbf{1}_{\{t^v=t_i\}} \\ &\leq \sum_{i=1}^m W_2(t_i, X_{t_i}^{t,x;\alpha_\varepsilon(v),v})\, \mathbf{1}_{\{t^v=t_i\}} + \varepsilon_1 \\ &= W_2(t^v, X_{t^v}^{t,x;\alpha_\varepsilon(v),v}) + \varepsilon_1, \qquad \mathbb{P}\text{-a.s.} \end{aligned} \tag{14}$$

We claim that for all $\varepsilon > 0$ and $v \in \mathcal{V}_{t,T}$,

$$J_2(t, x; \alpha_\varepsilon(v), v) \leq J_2(t, x; u^{\varepsilon_0}, v^{\varepsilon_0}) + \varepsilon, \qquad \alpha_\varepsilon(v^{\varepsilon_0}) = u^{\varepsilon_0}. \tag{15}$$

The proof of the above inequality is presented later. By a symmetric argument, we can construct $\beta_\varepsilon \in \mathcal{B}_{t,T}$ such that for all $u \in \mathcal{U}_{t,T}$,

$$J_1(t, x; u, \beta_\varepsilon(u)) \le J_1(t, x; u^{\varepsilon_0}, v^{\varepsilon_0}) + \varepsilon, \qquad \beta_\varepsilon(u^{\varepsilon_0}) = v^{\varepsilon_0}. \tag{16}$$

Finally, from (15), (16), (12), and Lemma 6, we see that $(\alpha_\varepsilon, \beta_\varepsilon)$ satisfies Definition 3. Therefore, (e_1, e_2) is a Nash equilibrium payoff.

We now provide the proof of (15). For this, we need the following estimate. There exists a positive constant C such that

$$\begin{aligned} J_2(t, x, \alpha_\varepsilon(v), v) &= {}^2G^{t,x;\alpha_\varepsilon(v),v}_{t,t^v}[{}^2Y^{t,x,\alpha_\varepsilon(v),v}_{t^v}] \\ &\le {}^2G^{t,x;\alpha_\varepsilon(v),v}_{t,t^v}[W_2(t^v, X^{t,x;\alpha_\varepsilon(v),v}_{t^v})] + C\varepsilon_1^{1/2}. \end{aligned} \tag{17}$$

Indeed, we consider the following BSDEs:

$$\begin{aligned} {}^2Y^{t,x;\alpha_\varepsilon(v),v}_s &= {}^2Y^{t,x;\alpha_\varepsilon(v),v}_{t^v} + k^+_{t^v} - k^+_s - k^-_{t^v} + k^-_s - \int_s^{t^v} {}^2Z^{t,x;\alpha_\varepsilon(v),v}_r \,\mathrm{d}B_r \\ &\quad + \int_s^{t^v} f_2(r, X^{t,x;\alpha_\varepsilon(v),v}_r, {}^2Y^{t,x;\alpha_\varepsilon(v),v}_r, {}^2Z^{t,x;\alpha_\varepsilon(v),v}_r, \alpha_\varepsilon(v_r), v_r)\,\mathrm{d}r, \\ h_2(s, X^{t,x;\alpha_\varepsilon(v),v}_s) &\le {}^2Y^{t,x;\alpha_\varepsilon(v),v}_s \le h_2'(s, X^{t,x;\alpha_\varepsilon(v),v}_s), \qquad s \in [t, t^v], \\ &\int_t^{t^v} ({}^2Y^{t,x;\alpha_\varepsilon(v),v}_r - h_2(r, X^{t,x;\alpha_\varepsilon(v),v}_r))\,\mathrm{d}k^+_r = 0, \\ &\int_t^{t^v} ({}^2Y^{t,x;\alpha_\varepsilon(v),v}_r - h_2'(r, X^{t,x;\alpha_\varepsilon(v),v}_r))\,\mathrm{d}k^-_r = 0, \end{aligned}$$

and

$$\begin{aligned} y_s &= W_2(t^v, X^{t,x;\alpha_\varepsilon(v),v}_{t^v}) + \varepsilon_1 + \int_s^{t^v} f_2(r, X^{t,x;\alpha_\varepsilon(v),v}_r, y_r, z_r, \alpha_\varepsilon(v_r), v_r)\,\mathrm{d}r \\ &\quad + k^+_{t^v} - k^+_s - k^-_{t^v} + k^-_s - \int_s^{t^v} z_r \,\mathrm{d}B_r, \\ &h_2(s, X^{t,x;\alpha_\varepsilon(v),v}_s) \le y_s \le h_2'(s, X^{t,x;\alpha_\varepsilon(v),v}_s) + \varepsilon_1, \qquad s \in [t, t^v], \\ &\int_t^{t^v} (y_r - h_2(r, X^{t,x;\alpha_\varepsilon(v),v}_r))\,\mathrm{d}k^+_r = \int_t^{t^v} (y_r - h_2'(r, X^{t,x;\alpha_\varepsilon(v),v}_r) - \varepsilon_1)\,\mathrm{d}k^-_r = 0, \end{aligned}$$

and

$$\begin{aligned} \hat{y}_s &= W_2(t^v, X^{t,x;\alpha_\varepsilon(v),v}_{t^v}) + \int_s^{t^v} f_2(r, X^{t,x;\alpha_\varepsilon(v),v}_r, \check{y}_r, \hat{z}_r, \alpha_\varepsilon(v_r), v_r)\,\mathrm{d}r \\ &\quad + \hat{k}^+_{t^v} - \hat{k}^+_s - \hat{k}^-_{t^v} + \hat{k}^-_s - \int_s^{t^v} \hat{z}_r \,\mathrm{d}B_r, \\ &h_2(s, X^{t,x;\alpha_\varepsilon(v),v}_s) \le \hat{y}_s \le h_2'(s, X^{t,x;\alpha_\varepsilon(v),v}_s), \qquad s \in [t, t^v], \\ &\int_t^{t^v} (\hat{y}_r - h_2(r, X^{t,x;\alpha_\varepsilon(v),v}_r))\,\mathrm{d}\hat{k}^+_r = \int_t^{t^v} (\hat{y}_r - h_2'(r, X^{t,x;\alpha_\varepsilon(v),v}_r))\,\mathrm{d}\hat{k}^-_r = 0. \end{aligned}$$

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Thanks to (14), using Lemma 3 we obtain ${}^2Y_t^{t,x;\alpha_\varepsilon(v),v} \leq y_t$, $\mathbb{P}$-a.s. From Lemma 2 it follows that there exists a constant C such that $|\hat{y}_t - y_t| \leq C\varepsilon_1^{1/2}$, $\mathbb{P}$-a.s., where we use the fact that for all the partitions $\pi : t = t_0 < \cdots < t_n = \sup_{\omega\in\Omega} t^v(\omega)$,

$$\begin{aligned}
&[\mathbb{E}[h_2(t_{i+1}, X_{t_{i+1}}^{t,x;\alpha_\varepsilon(v),v}) \mid \mathcal{F}_{t_i}] - h_2'(t_i, X_{t_i}^{t,x;\alpha_\varepsilon(v),v})]^+ \\
&\quad \leq \mathbb{E}[|h_2(t_{i+1}, X_{t_{i+1}}^{t,x;\alpha_\varepsilon(v),v}) - h_2(t_i, X_{t_i}^{t,x;\alpha_\varepsilon(v),v})| \mid \mathcal{F}_{t_i}] \\
&\qquad + [h_2(t_i, X_{t_i}^{t,x;\alpha_\varepsilon(v),v} - h_2'(t_i, X_{t_i}^{t,x;\alpha_\varepsilon(v),v})]^+ \\
&\quad \leq L|t_{i+1} - t_i|^{1/2} + L\mathbb{E}[|X_{t_{i+1}}^{t,x;\alpha_\varepsilon(v),v} - X_{t_i}^{t,x;\alpha_\varepsilon(v),v})| \mid \mathcal{F}_{t_i}],
\end{aligned}$$

similarly,

$$\begin{aligned}
&[h_2(t_i, X_{t_i}^{t,x;\alpha_\varepsilon(v),v}) - \mathbb{E}[h_2'(t_{i+1}, X_{t_{i+1}}^{t,x;\alpha_\varepsilon(v),v}) \mid \mathcal{F}_{t_i}]]^+ \\
&\quad \leq L|t_{i+1} - t_i|^{1/2} + L\mathbb{E}[|X_{t_{i+1}}^{t,x;\alpha_\varepsilon(v),v} - X_{t_i}^{t,x;\alpha_\varepsilon(v),v})| \mid \mathcal{F}_{t_i}].
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\sup_\pi \mathbb{E}\bigg[\sum_{i=0}^n ([\mathbb{E}[h_2(t_{i+1}, X_{t_{i+1}}^{t,x;\alpha_\varepsilon(v),v}) \mid \mathcal{F}_{t_i}] - h_2'(t_i, X_{t_i}^{t,x;\alpha_\varepsilon(v),v})]^+ \\
&\qquad\qquad + [h_2(t_i, X_{t_i}^{t,x;\alpha_\varepsilon(v),v}) - \mathbb{E}[h_2'(t_{i+1}, X_{t_{i+1}}^{t,x;\alpha_\varepsilon(v),v}) \mid \mathcal{F}_{t_i}]]^+)^2 \mid \mathcal{F}_t\bigg] \\
&\quad \leq C \sup_\pi \mathbb{E}\bigg[\sum_{i=0}^n (|t_{i+1} - t_i| + \mathbb{E}[|X_{t_{i+1}}^{t,x;\alpha_\varepsilon(v),v} - X_{t_i}^{t,x;\alpha_\varepsilon(v),v})|^2 \mid \mathcal{F}_{t_i}]) \mid \mathcal{F}_t\bigg] \\
&\quad \leq C \sup_\pi \mathbb{E}\bigg[\sum_{i=0}^n (t_{i+1} - t_i) \mid \mathcal{F}_t\bigg] \\
&\quad \leq C,
\end{aligned}$$

where C is a constant. Then we can easily obtain (17).

Using Lemma 2 again, we have

$$\begin{aligned}
&|{}^2G_{t,t^v}^{t,x;\alpha_\varepsilon(v),v}[W_2(t^v, X_{t^v}^{t,x;u^{\varepsilon_0},v^{\varepsilon_0}})] - {}^2G_{t,t^v}^{t,x;\alpha_\varepsilon(v),v}[W_2(t^v, X_{t^v}^{t,x;\alpha_\varepsilon(v),v})]| \\
&\quad \leq C\mathbb{E}[|W_2(t^v, X_{t^v}^{t,x;u^{\varepsilon_0},v^{\varepsilon_0}}) - W_2(t^v, X_{t^v}^{t,x;\alpha_\varepsilon(v),v})|^2 \mid \mathcal{F}_t]^{1/2} \\
&\quad \leq C\mathbb{E}[|X_{t^v}^{t,x;u^{\varepsilon_0},v^{\varepsilon_0}} - X_{t^v}^{t,x;\alpha_\varepsilon(v),v}|^2 \mid \mathcal{F}_t]^{1/2} \\
&\quad \leq C\tau^{1/2}, \qquad \mathbb{P}\text{-a.s.}
\end{aligned}$$

For the last two inequalities we have used Proposition 4 and Lemma 8. Then by (17), we have

$$\begin{aligned}
&J_2(t, x, \alpha_\varepsilon(v), v) \\
&\quad \leq {}^2G_{t,t^v}^{t,x;\alpha_\varepsilon(v),v}[W_2(t^v, X_{t^v}^{t,x;u^{\varepsilon_0},v^{\varepsilon_0}})] + C\varepsilon_1 \\
&\qquad + |{}^2G_{t,t^v}^{t,x;\alpha_\varepsilon(v),v}[W_2(t^v, X_{t^v}^{t,x;u^{\varepsilon_0},v^{\varepsilon_0}})] - {}^2G_{t,t^v}^{t,x;\alpha_\varepsilon(v),v}[W_2(t^v, X_{t^v}^{t,x;\alpha_\varepsilon(v),v})]| \\
&\quad \leq {}^2G_{t,t^v}^{t,x;\alpha_\varepsilon(v),v}[W_2(t^v, X_{t^v}^{t,x;u^{\varepsilon_0},v^{\varepsilon_0}})] + C\varepsilon_1^{1/2} + C\tau^{1/2}.
\end{aligned}$$

Let us write

$$\Omega_s = \{{}^2Y_s^{t,x;u^{\varepsilon 0},v^{\varepsilon 0}} \geq W_2(s, X_s^{t,x;u^{\varepsilon 0},v^{\varepsilon 0}}) - \varepsilon_0\}, \qquad s \in [t, T]. \tag{18}$$

Then we have

$$\begin{aligned} J_2(t, x; \alpha_\varepsilon(v), v) & \\ &\leq {}^2G_{t,t^v}^{t,x;\alpha_\varepsilon(v),v}\Bigg[\sum_{i=1}^m W_2(t_i, X_{t_i}^{t,x;u^{\varepsilon 0},v^{\varepsilon 0}})\,\mathbf{1}_{\{t^v=t_i\}}\Bigg] + C\varepsilon_1^{1/2} + C\tau^{1/2} \\ &\leq {}^2G_{t,t^v}^{t,x;\alpha_\varepsilon(v),v}\Bigg[\sum_{i=1}^m W_2(t_i, X_{t_i}^{t,x;u^{\varepsilon 0},v^{\varepsilon 0}})\,\mathbf{1}_{\{t^v=t_i\}}\,\mathbf{1}_{\Omega_{t_i}}\Bigg] + C\varepsilon_1^{1/2} + C\tau^{1/2} + I, \end{aligned} \tag{19}$$

where

$$\begin{aligned} I = \Bigg| & {}^2G_{t,t^v}^{t,x;\alpha_\varepsilon(v),v}\Bigg[\sum_{i=1}^m W_2(t_i, X_{t_i}^{t,x;u^{\varepsilon 0},v^{\varepsilon 0}})\,\mathbf{1}_{\{t^v=t_i\}}\Bigg] \\ & - {}^2G_{t,t^v}^{t,x;\alpha_\varepsilon(v),v}\Bigg[\sum_{i=1}^m W_2(t_i, X_{t_i}^{t,x;u^{\varepsilon 0},v^{\varepsilon 0}})\,\mathbf{1}_{\{t^v=t_i\}}\,\mathbf{1}_{\Omega_{t_i}}\Bigg]\Bigg|. \end{aligned}$$

Since h_2 and h_2' are bounded, it follows that W_2 is bounded. Thus, by Lemma 2, we have

$$\begin{aligned} I &\leq \mathbb{E}\Bigg[\sum_{i=1}^m |W_2(t_i, X_{t_i}^{t,x;u^{\varepsilon 0},v^{\varepsilon 0}})|^2\,\mathbf{1}_{\{t^v=t_i\}}\,\mathbf{1}_{\Omega_{t_i}^c} \mid \mathcal{F}_t\Bigg]^{1/2} \\ &\leq C\sum_{i=1}^m \mathbb{P}(\Omega_{t_i}^c \mid \mathcal{F}_t)^{1/2} \\ &\leq Cm\varepsilon_0^{1/2}. \end{aligned} \tag{20}$$

Here, we have used (11) for the latter estimate. Using (18) and a similar argument in the proof of (17), we have that there exists a constant $C > 0$ such that

$$\begin{aligned} & {}^2G_{t,t^v}^{t,x;\alpha_\varepsilon(v),v}\Bigg[\sum_{i=1}^m W_2(t_i, X_{t_i}^{t,x;u^{\varepsilon 0},v^{\varepsilon 0}})\,\mathbf{1}_{\{t^v=t_i\}}\,\mathbf{1}_{\Omega_{t_i}}\Bigg] \\ &\qquad \leq {}^2G_{t,t^v}^{t,x;\alpha_\varepsilon(v),v}\Bigg[\sum_{i=1}^m {}^2Y_{t_i}^{t,x;u^{\varepsilon 0},v^{\varepsilon 0}}\,\mathbf{1}_{\{t^v=t_i\}}\,\mathbf{1}_{\Omega_{t_i}}\Bigg] + C\varepsilon_0, \end{aligned}$$

and, using the above arguments, we also have

$$\begin{aligned} & \Bigg|{}^2G_{t,t^v}^{t,x;\alpha_\varepsilon(v),v}\Bigg[\sum_{i=1}^m {}^2Y_{t_i}^{t,x;u^{\varepsilon 0},v^{\varepsilon 0}}\,\mathbf{1}_{\{t^v=t_i\}}\,\mathbf{1}_{\Omega_{t_i}}\Bigg] - {}^2G_{t,t^v}^{t,x;\alpha_\varepsilon(v),v}\Bigg[\sum_{i=1}^m {}^2Y_{t_i}^{t,x;\alpha_\varepsilon(v),v}\,\mathbf{1}_{\{t^v=t_i\}}\Bigg]\Bigg| \\ &\quad \leq Cm\varepsilon_0^{1/2}. \end{aligned}$$

Therefore,

$$\begin{aligned} & {}^2G_{t,t^v}^{t,x;\alpha_\varepsilon(v),v}\Bigg[\sum_{i=1}^m W_2(t_i, X_{t_i}^{t,x;u^{\varepsilon 0},v^{\varepsilon 0}})\,\mathbf{1}_{\{t^v=t_i\}}\,\mathbf{1}_{\Omega_{t_i}}\Bigg] \\ &\qquad \leq {}^2G_{t,t^v}^{t,x;\alpha_\varepsilon(v),v}[{}^2Y_{t^v}^{t,x;u^{\varepsilon 0},v^{\varepsilon 0}}] + C\varepsilon_0 + Cm\varepsilon_0^{1/2} \end{aligned}$$

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$$
\begin{aligned}
&\le |{}^2G^{t,x;\alpha_\varepsilon(v),v}_{t,t^v}[{}^2Y^{t,x;u^{\varepsilon_0},v^{\varepsilon_0}}_{t^v}] - {}^2G^{t,x;u^{\varepsilon_0},v^{\varepsilon_0}}_{t,t^v}[{}^2Y^{t,x;u^{\varepsilon_0},v^{\varepsilon_0}}_{t^v}]| \\
&\quad + {}^2G^{t,x;u^{\varepsilon_0},v^{\varepsilon_0}}_{t,t^v}[{}^2Y^{t,x;u^{\varepsilon_0},v^{\varepsilon_0}}_{t^v}] + C\varepsilon_0 + Cm\varepsilon_0^{1/2} \\
&= |{}^2G^{t,x;\alpha_\varepsilon(v),v}_{t,t^v}[{}^2Y^{t,x;u^{\varepsilon_0},v^{\varepsilon_0}}_{t^v}] - {}^2G^{t,x;u^{\varepsilon_0},v^{\varepsilon_0}}_{t,t^v}[{}^2Y^{t,x;u^{\varepsilon_0},v^{\varepsilon_0}}_{t^v}]| \\
&\quad + J_2(t,x;u^{\varepsilon_0},v^{\varepsilon_0}) + C\varepsilon_0 + Cm\varepsilon_0^{1/2} \\
&\le J_2(t,x;u^{\varepsilon_0},v^{\varepsilon_0}) + C\varepsilon_0 + Cm\varepsilon_0^{1/2} + C\tau^{1/2}.
\end{aligned}
$$

Here we have used the fact that

$$
|{}^2G^{t,x;\alpha_\varepsilon(v),v}_{t,t^v}[{}^2Y^{t,x;u^{\varepsilon_0},v^{\varepsilon_0}}_{t^v}] - {}^2G^{t,x;u^{\varepsilon_0},v^{\varepsilon_0}}_{t,t^v}[{}^2Y^{t,x;u^{\varepsilon_0},v^{\varepsilon_0}}_{t^v}]| \le C\tau^{1/2}.
$$

In fact, let us consider the following BSDEs:

$$
\begin{aligned}
y_s &= {}^2Y^{t,x;u^{\varepsilon_0},v^{\varepsilon_0}}_{t^v} + \int_s^{t^v} f_2(r, X^{t,x;\alpha_\varepsilon(v),v}_r, y_r, z_r, \alpha_\varepsilon(v_r), v_r)\,\mathrm{d}r \\
&\qquad + k^+_{t^v} - k^+_s - k^-_{t^v} + k^-_s - \int_s^{t^v} z_r\,\mathrm{d}B_r, \\
&h_2(s, X^{t,x;\alpha_\varepsilon(v),v}_s) \le y_s \le h_2'(s, X^{t,x;\alpha_\varepsilon(v),v}_s), \qquad s\in[t,t^v], \\
&\int_t^{t^v}(y_r - h_2(r, X^{t,x;\alpha_\varepsilon(v),v}_r))\,\mathrm{d}k^+_r = \int_t^{t^v}(y_r - h_2'(r, X^{t,x;\alpha_\varepsilon(v),v}_r))\,\mathrm{d}k^-_r = 0,
\end{aligned}
$$

and

$$
\begin{aligned}
{}^2Y^{t,x;u^{\varepsilon_0},v^{\varepsilon_0}}_s &= {}^2Y^{t,x;u^{\varepsilon_0},v^{\varepsilon_0}}_{t^v} - \int_s^{t^v} {}^2Z^{t,x;u^{\varepsilon_0},v^{\varepsilon_0}}_r\,\mathrm{d}B_r \\
&\quad + \int_s^{t^v} f_2(r, X^{t,x;u^{\varepsilon_0},v^{\varepsilon_0}}_r, {}^2Y^{t,x;u^{\varepsilon_0},v^{\varepsilon_0}}_r, {}^2Z^{t,x;u^{\varepsilon_0},v^{\varepsilon_0}}_r, u^{\varepsilon_0}_r, v^{\varepsilon_0}_r)\,\mathrm{d}r \\
&\quad + {}^2K^{+,t,x;u^{\varepsilon_0},v^{\varepsilon_0}}_{t^v} - {}^2K^{+,t,x;u^{\varepsilon_0},v^{\varepsilon_0}}_s - {}^2K^{-,t,x;u^{\varepsilon_0},v^{\varepsilon_0}}_{t^v} + {}^2K^{-,t,x;u^{\varepsilon_0},v^{\varepsilon_0}}_s, \\
h_2(s, X^{t,x;u^{\varepsilon_0},v^{\varepsilon_0}}_s) &\le {}^2Y^{t,x;u^{\varepsilon_0},v^{\varepsilon_0}}_s \le h_2'(s, X^{t,x;u^{\varepsilon_0},v^{\varepsilon_0}}_s), \qquad s\in[t,t^v], \\
&\int_t^{t^v}({}^2Y^{t,x;u^{\varepsilon_0},v^{\varepsilon_0}}_r - h_2(r, X^{t,x;u^{\varepsilon_0},v^{\varepsilon_0}}_r))\,\mathrm{d}\,{}^2K^{+,t,x;u^{\varepsilon_0},v^{\varepsilon_0}}_r = 0, \\
&\int_t^{t^v}({}^2Y^{t,x;u^{\varepsilon_0},v^{\varepsilon_0}}_r - h_2'(r, X^{t,x;u^{\varepsilon_0},v^{\varepsilon_0}}_r))\,\mathrm{d}\,{}^2K^{-,t,x;u^{\varepsilon_0},v^{\varepsilon_0}}_r = 0.
\end{aligned}
$$

We note that $\alpha_\varepsilon(v) = u^{\varepsilon_0}$, on $[[t,t^v]]$, and $v = v^{\varepsilon_0}$, on $[[t,S^v]]$. By Lemma 2 and the boundedness of f_2, b, and σ, we have

$$
\begin{aligned}
&|{}^2G^{t,x;\alpha_\varepsilon(v),v}_{t,t^v}[{}^2Y^{t,x;u^{\varepsilon_0},v^{\varepsilon_0}}_{t^v}] - {}^2G^{t,x;u^{\varepsilon_0},v^{\varepsilon_0}}_{t,t^v}[{}^2Y^{t,x;u^{\varepsilon_0},v^{\varepsilon_0}}_{t^v}]|^2 \\
&\le C\mathbb{E}\bigg[\int_t^{t^v} |f_2(r, X^{t,x;\alpha_\varepsilon(v),v}_r, y_r, z_r, \alpha_\varepsilon(v)_r, v_r)\,\mathrm{d}r \\
&\qquad\qquad - f_2(r, X^{t,x;u^{\varepsilon_0},v^{\varepsilon_0}}_r, y_r, z_r, u^{\varepsilon_0}_r, v^{\varepsilon_0}_r)|^2 \mid \mathcal{F}_t\bigg]
\end{aligned}
$$

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$$
\begin{aligned}
&+ C\mathbb{E}\Big[\sup_{r\in[t,t^v]}(|h_2(r, X_r^{t,x;\alpha_\varepsilon(v),v}) - h_2(r, X_r^{t,x;u^{\varepsilon_0},v^{\varepsilon_0}})|^2 \\
&\qquad + |h_2'(r, X_r^{t,x;\alpha_\varepsilon(v),v}) - h_2'(r, X_r^{t,x;u^{\varepsilon_0},v^{\varepsilon_0}})|^2) \mid \mathcal{F}_t\Big]^{1/2} \\
= {} & C\mathbb{E}\bigg[\int_{S^v}^{t^v} |f_2(r, X_r^{t,x;\alpha_\varepsilon(v),v}, y_r, z_r, \alpha_\varepsilon(v)_r, v_r)\,\mathrm{d}r \\
&\qquad - f_2(r, X_r^{t,x;u^{\varepsilon_0},v^{\varepsilon_0}}, y_r, z_r, u_r^{\varepsilon_0}, v_r^{\varepsilon_0})|^2 \mid \mathcal{F}_t\bigg] \\
&+ C\mathbb{E}\Big[\sup_{r\in[S^v,t^v]}|X_r^{t,x;\alpha_\varepsilon(v),v} - X_r^{t,x;u^{\varepsilon_0},v^{\varepsilon_0}}|^2 \mid \mathcal{F}_t\Big]^{1/2} \\
\le {} & C\mathbb{E}\bigg[\int_{S^v}^{t^v}\mathbf{1}_{\{v_r\neq v_r^{\varepsilon_0}\}} \mid \mathcal{F}_t\bigg] + C\tau^{1/2} \le C\mathbb{E}[t^v - S^v \mid \mathcal{F}_t] + C\tau^{1/2} \le C\tau^{1/2}.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&{}^2G_{t,t^v}^{t,x;\alpha_\varepsilon(v),v}\bigg[\sum_{i=1}^m W_2(t_i, X_{t_i}^{t,x;u^{\varepsilon_0},v^{\varepsilon_0}})\,\mathbf{1}_{\{t^v=t_i\}}\,\mathbf{1}_{\Omega_{t_i}}\bigg] \\
&\qquad \le C\tau^{1/2} + J_2(t,x;u^{\varepsilon_0},v^{\varepsilon_0}) + C\varepsilon_0 + Cm\varepsilon_0^{1/2}.
\end{aligned}
$$

Thus, (19) and (20) yield

$$
J_2(t,x;\alpha_\varepsilon(v),v) \le J_2(t,x;u^{\varepsilon_0},v^{\varepsilon_0}) + C\varepsilon_0 + Cm\varepsilon_0^{1/2} + C\varepsilon_1^{1/2} + C\tau^{1/4}.
$$

Let us choose $\tau > 0$, $\varepsilon_0 > 0$, and $\varepsilon_1 > 0$ such that $C\varepsilon_0 + Cm\varepsilon_0^{1/2} + C\varepsilon_1^{1/2} + C\tau^{1/4} \le \varepsilon$ and $\varepsilon_0 < \varepsilon$. Consequently,

$$
J_2(t,x;\alpha_\varepsilon(v),v) \le J_2(t,x;u^{\varepsilon_0},v^{\varepsilon_0}) + \varepsilon, \qquad v \in \mathcal{V}_{t,T}.
$$

The proof is complete.

Let us provide some preliminaries for the existence of a Nash equilibrium payoff.

Proposition 8. *Under the assumptions of Theorem 3 for all $\varepsilon > 0$, there exists $(u^\varepsilon, v^\varepsilon) \in \mathcal{U}_{t,T} \times \mathcal{V}_{t,T}$ independent of $\mathcal{F}_t$ such that for all $t \le s_1 \le s_2 \le T$, $j = 1, 2$,*

$$
\mathbb{P}(W_j(s_1, X_{s_1}^{t,x;u^\varepsilon,v^\varepsilon}) - \varepsilon \le {}^jG_{s_1,s_2}^{t,x;u^\varepsilon,v^\varepsilon}[W_j(s_2, X_{s_2}^{t,x;u^\varepsilon,v^\varepsilon})] \mid \mathcal{F}_t) > 1 - \varepsilon.
$$

The proof of the above proposition needs the following two lemmas. Since the proof of the following lemma is similar to that in [11], we omit it here.

Lemma 9. *For all $\varepsilon > 0$, all $\delta \in [0, T-t]$, and $x \in \mathbb{R}^n$, there exists $(u^\varepsilon, v^\varepsilon) \in \mathcal{U}_{t,T} \times \mathcal{V}_{t,T}$ independent of $\mathcal{F}_t$ such that $j = 1, 2$,*

$$
W_j(t,x) - \varepsilon \le {}^jG_{t,t+\delta}^{t,x;u^\varepsilon,v^\varepsilon}[W_j(t+\delta, X_{t+\delta}^{t,x;u^\varepsilon,v^\varepsilon})], \qquad \mathbb{P}\text{-a.s.}
$$

We also need the following lemma.

Lemma 10. *Let $n \ge 1$ and let us fix some partition $t = t_0 < t_1 < \cdots < t_n = T$ of the interval $[t, T]$. Then for all $\varepsilon > 0$, there exists $(u^\varepsilon, v^\varepsilon) \in \mathcal{U}_{t,T} \times \mathcal{V}_{t,T}$ independent of $\mathcal{F}_t$ such that for all $i = 0, \dots, n-1$,*

$$
W_j(t_i, X_{t_i}^{t,x;u^\varepsilon,v^\varepsilon}) - \varepsilon \le {}^jG_{t_i,t_{i+1}}^{t,x;u^\varepsilon,v^\varepsilon}[W_j(t_{i+1}, X_{t_{i+1}}^{t,x;u^\varepsilon,v^\varepsilon})], \qquad \mathbb{P}\text{-a.s.}
$$

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Proof. We provide the proof by induction. By the above lemma, it is obvious for $i = 0$. We now suppose that $(u^\varepsilon, v^\varepsilon)$ independent of $\mathcal{F}_t$, is constructed on the interval $[t, t_i)$ and we shall define it on $[t_i, t_{i+1})$. From the above lemma, we have for all $y \in \mathbb{R}^n$, there exists $(u^y, v^y) \in \mathcal{U}_{t_i,T} \times \mathcal{V}_{t_i,T}$ independent of $\mathcal{F}_t$ such that,

$$W_j(t_i, y) - \frac{\varepsilon}{2} \leq {}^jG^{t_i,y;u^y,v^y}_{t_i,t_{i+1}}[W_j(t_{i+1}, X^{t,y;u^y,v^y}_{t_{i+1}})], \qquad \mathbb{P}\text{-a.s.} \quad j = 1, 2. \tag{21}$$

For arbitrarily $j = 1, 2$, for all $y, z \in \mathbb{R}^n$ and $s \in [t_i, t_{i+1}]$, write

$$y^1_s = {}^jG^{t_i,y;u^y,v^y}_{s,t_{i+1}}[W_j(t_{i+1}, X^{t_i,y;u^y,v^y}_{t_{i+1}})] \quad \text{and} \quad y^2_s = {}^jG^{t_i,z;u^y,v^y}_{s,t_{i+1}}[W_j(t_{i+1}, X^{t_i,z;u^y,v^y}_{t_{i+1}})].$$

Then let us consider the following BSDEs:

$$\begin{aligned}
y^1_s &= W_j(t_{i+1}, X^{t_i,y;u^y,v^y}_{t_{i+1}}) + \int_s^{t_{i+1}} f_j(r, X^{t_i,y;u^y,v^y}_r, y^1_r, z^1_r, u^y_r, v^y_r)\,\mathrm{d}r \\
&\quad + {}^1K^+_{t_{i+1}} - {}^1K^+_s - {}^1K^-_{t_{i+1}} + {}^1K^-_s - \int_s^{t_{i+1}} z^1_r\,\mathrm{d}B_r, \\
&h_j(s, X^{t_i,y;u^y,v^y}_s) \leq y^1_s \leq h'_j(s, X^{t_i,y;u^y,v^y}_s), \qquad s \in [t_i, t_{i+1}], \\
&\int_{t_i}^{t_{i+1}} (y^1_s - h_j(s, X^{t_i,y;u^y,v^y}_s))\,\mathrm{d}\,{}^1K^+_r = \int_{t_i}^{t_{i+1}} (y^1_s - h'_j(s, X^{t_i,y;u^y,v^y}_s))\,\mathrm{d}\,{}^1K^-_r = 0,
\end{aligned}$$

and

$$\begin{aligned}
y^2_s &= W_j(t_{i+1}, X^{t_i,z;u^y,v^y}_{t_{i+1}}) + \int_s^{t_{i+1}} f_j(r, X^{t_i,z;u^y,v^y}_r, y^2_r, z^2_r, u^y_r, v^y_r)\,\mathrm{d}r \\
&\quad + {}^2K^+_{t_{i+1}} - {}^2K^+_s - {}^2K^-_{t_{i+1}} + {}^2K^-_s - \int_s^{t_{i+1}} z^2_r\,\mathrm{d}B_r, \\
&h_j(s, X^{t_i,z;u^y,v^y}_s) \leq y^2_s \leq h'_j(s, X^{t_i,z;u^y,v^y}_s), \qquad s \in [t_i, t_{i+1}], \\
&\int_{t_i}^{t_{i+1}} (y^2_s - h_j(s, X^{t_i,z;u^y,v^y}_s))\,\mathrm{d}^2K^+_s = \int_{t_i}^{t_{i+1}} (y^2_s - h'_j(s, X^{t_i,z;u^y,v^y}_s))\,\mathrm{d}^2K^-_s = 0.
\end{aligned}$$

By Lemmas 2 and 4, we deduce that

$$\begin{aligned}
&|{}^jG^{t_i,y;u^y,v^y}_{t_i,t_{i+1}}[W_j(t_{i+1}, X^{t,y;u^y,v^y}_{t_{i+1}})] - {}^jG^{t_i,z;u^y,v^y}_{t_i,t_{i+1}}[W_j(t_{i+1}, X^{t_i,z;u^y,v^y}_{t_{i+1}})]|^2 \\
&\quad \leq C\mathbb{E}[|W_j(t_{i+1}, X^{t_i,y;u^y,v^y}_{t_{i+1}}) - W_j(t_{i+1}, X^{t_i,z;u^y,v^y}_{t_{i+1}})|^2 \mid \mathcal{F}_{t_i}] \\
&\qquad + C\mathbb{E}\bigg[\bigg(\int_{t_i}^{t_{i+1}} |f_j(r, X^{t_i,y;u^y,v^y}_r, y^1_r, z^1_r, u^y_r, v^y_r) \\
&\qquad\qquad - f_j(r, X^{t_i,z;u^y,v^y}_r, y^1_r, z^1_r, u^y_r, v^y_r)|\,\mathrm{d}r\bigg)^2 \mid \mathcal{F}_{t_i}\bigg]
\end{aligned}$$

$$
\begin{aligned}
&+C\mathbb{E}\Big[\sup_{t_i\le s\le t_{i+1}}(|h_j(s,X_s^{t_i,y;u^y,v^y})-h_j(s,X_s^{t_i,z;u^y,v^y})|^2\\
&\qquad\qquad+|h'_j(s,X_s^{t_i,y;u^y,v^y})-h'_j(s,X_s^{t_i,z;u^y,v^y})|^2)\mid\mathcal{F}_{t_i}\Big]^{1/2}\\
&\le C\mathbb{E}[|X_{t_{i+1}}^{t_i,y;u^y,v^y}-X_{t_{i+1}}^{t_i,z;u^y,v^y}|^2\mid\mathcal{F}_{t_i}]+C\mathbb{E}\bigg[\int_{t_i}^{t_{i+1}}|X_r^{t_i,y;u^y,v^y}-X_r^{t_i,z;u^y,v^y}|^2\,\mathrm{d}r\mid\mathcal{F}_{t_i}\bigg]\\
&+C\mathbb{E}\Big[\sup_{t_i\le s\le t_{i+1}}|X_s^{t_i,y;u^y,v^y}-X_s^{t_i,z;u^y,v^y}|^2\mid\mathcal{F}_{t_i}\Big]^{1/2}\\
&\le C|y-z|.
\end{aligned}
$$

Thus, from Proposition 4 and (21), we have

$$
\begin{aligned}
W_j(t_i,z)-\varepsilon&\le W_j(t_i,y)-\varepsilon+C|y-z|^{1/2}\\
&\le {}^jG_{t_i,t_{i+1}}^{t_i,y;u^y,v^y}[W_j(t_{i+1},X_{t_{i+1}}^{t,y;u^y,v^y})]-\frac{\varepsilon}{2}+C|y-z|^{1/2}\\
&\le {}^jG_{t_i,t_{i+1}}^{t,z;u^y,v^y}[W_j(t_{i+1},X_{t_{i+1}}^{t,z;u^y,v^y})]-\frac{\varepsilon}{2}+C|y-z|^{1/2}\\
&\le {}^jG_{t_i,t_{i+1}}^{t,z;u^y,v^y}[W_j(t_{i+1},X_{t_{i+1}}^{t,z;u^y,v^y})],\qquad \mathbb{P}\text{-a.s.}
\end{aligned}
$$

for $C|y-z|^{1/2}\le\varepsilon/2$.

Let $\{O_i\}_{i\ge1}\subset\mathcal{B}(\mathbb{R}^n)$ be a partition of $\mathbb{R}^n$ with $\operatorname{diam}(O_i)<\varepsilon/2C$ and let $y_l\in O_l$. Then for $z\in O_l$,

$$
W_j(t_i,z)-\varepsilon\le {}^jG_{t_i,t_{i+1}}^{t,z;u^{y_l},v^{y_l}}[W_j(t_{i+1},X_{t_{i+1}}^{t,z;u^{y_l},v^{y_l}})],\qquad \mathbb{P}\text{-a.s.}\tag{22}
$$

Write

$$
u^\varepsilon=\sum_{l\ge1}\mathbf{1}_{O_l}(X^{t,x;u^\varepsilon,v^\varepsilon})u^{y_l},\qquad v^\varepsilon=\sum_{l\ge1}\mathbf{1}_{O_l}(X^{t,x;u^\varepsilon,v^\varepsilon})v^{y_l}.
$$

Then

$$
\begin{aligned}
&{}^jG_{t_i,t_{i+1}}^{t,x;u^\varepsilon,v^\varepsilon}[W_j(t_{i+1},X_{t_{i+1}}^{t,x;u^\varepsilon,v^\varepsilon})]\\
&\quad={}^jG_{t_i,t_{i+1}}^{t_i,X_{t_i}^{t,x;u^\varepsilon,v^\varepsilon};u^\varepsilon,v^\varepsilon}\bigg[\sum_{l\ge1}W_j(t_{i+1},X_{t_{i+1}}^{t_i,X_{t_i}^{t,x;u^\varepsilon,v^\varepsilon};u^\varepsilon,v^\varepsilon})\mathbf{1}_{O_l}(X_{t_i}^{t,x;u^\varepsilon,v^\varepsilon})\bigg]\\
&\quad={}^jG_{t_i,t_{i+1}}^{t_i,X_{t_i}^{t,x;u^\varepsilon,v^\varepsilon};u^\varepsilon,v^\varepsilon}\bigg[\sum_{l\ge1}W_j(t_{i+1},X_{t_{i+1}}^{t_i,X_{t_i}^{t,x;u^\varepsilon,v^\varepsilon};u^{y_l},v^{y_l}})\mathbf{1}_{O_l}(X_{t_i}^{t,x;u^\varepsilon,v^\varepsilon})\bigg]\\
&\quad=\sum_{l\ge1}{}^jG_{t_i,t_{i+1}}^{t_i,X_{t_i}^{t,x;u^\varepsilon,v^\varepsilon};u^{y_l},v^{y_l}}[W_j(t_{i+1},X_{t_{i+1}}^{t_i,X_{t_i}^{t,x;u^\varepsilon,v^\varepsilon};u^{y_l},v^{y_l}})]\mathbf{1}_{O_l}(X_{t_i}^{t,x;u^\varepsilon,v^\varepsilon}).
\end{aligned}
$$

Consequently, from (22), we have

$$
\begin{aligned}
{}^jG_{t_i,t_{i+1}}^{t,x;u^\varepsilon,v^\varepsilon}[W_j(t_{i+1},X_{t_{i+1}}^{t,x;u^\varepsilon,v^\varepsilon})]&\ge\sum_{l\ge1}[W_j(t_i,X_{t_i}^{t,x;u^{y_l},v^{y_l}})-\varepsilon]\mathbf{1}_{O_l}(X_{t_i}^{t,x;u^\varepsilon,v^\varepsilon})\\
&=\sum_{l\ge1}W_j(t_i,X_{t_i}^{t,x;u^{y_l},v^{y_l}})\mathbf{1}_{O_l}(X_{t_i}^{t,x;u^\varepsilon,v^\varepsilon})-\varepsilon\\
&=W_j(t_i,X_{t_i}^{t,x;u^\varepsilon,v^\varepsilon})-\varepsilon,
\end{aligned}
$$

from which we obtain the desired result.

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We now provide the proof of Proposition 8.

Proof. Let $t = t_0 < t_1 < \cdots < t_n = T$ be a partition of $[t, T]$ and $\tau = \sup_i(t_{i+1} - t_i)$. From Proposition 4 and Lemma 8, it follows that for all $j = 1, 2$, $0 \le k \le n$, $s \in [t_k, t_{k+1})$ and $(u, v) \in \mathcal{U}_{t,T} \times \mathcal{V}_{t,T}$,

$$\begin{aligned}
&\mathbb{E}[|W_j(t_k, X_{t_k}^{t,x;u,v}) - W_j(s, X_s^{t,x;u,v})|^2] \\
&\quad \le 2\mathbb{E}[|W_j(t_k, X_{t_k}^{t,x;u,v}) - W_j(s, X_{t_k}^{t,x;u,v})|^2] \\
&\qquad + 2\mathbb{E}[|W_j(s, X_{t_k}^{t,x;u,v}) - W_j(s, X_s^{t,x;u,v})|^2] \\
&\quad \le C|s - t_k|(1 + \mathbb{E}[|X_{t_k}^{t,x;u,v}|^2]) + C\mathbb{E}[|X_{t_k}^{t,x;u,v} - X_s^{t,x;u,v}|^2] \\
&\quad \le C\tau.
\end{aligned} \tag{23}$$

In what follows, C represents a generic constant which may be different at different places.

Let $(u^\varepsilon, v^\varepsilon) \in \mathcal{U}_{t,T} \times \mathcal{V}_{t,T}$ be defined as in Lemma 10 for $\varepsilon = \varepsilon_0$, where $\varepsilon_0 > 0$ will be specified later. Then for all i, $0 \le i \le n$,

$$W_j(t_i, X_{t_i}^{t,x;u^\varepsilon,v^\varepsilon}) - \varepsilon_0 \le {}^jG_{t_i,t_{i+1}}^{t,x;u^\varepsilon,v^\varepsilon}[W_j(t_{i+1}, X_{t_{i+1}}^{t,x;u^\varepsilon,v^\varepsilon})], \qquad \mathbb{P}\text{-a.s.}$$

For $t \le s_1 \le s_2 \le T$, without loss of generality, we assume that $t_{i-1} \le s_1 < t_i$ and $t_k < s_2 \le t_{k+1}$ for some $1 \le i < k \le n-1$. Thus, from Lemmas 2 and 3, it follows that

$$\begin{aligned}
{}^jG_{t_i,t_{k+1}}^{t,x;u^\varepsilon,v^\varepsilon}[W_j(t_{k+1}, X_{t_{k+1}}^{t,x;u^\varepsilon,v^\varepsilon})] &= {}^jG_{t_i,t_k}^{t,x;u^\varepsilon,v^\varepsilon}[{}^jG_{t_k,t_{k+1}}^{t,x;u^\varepsilon,v^\varepsilon}[W_j(t_{k+1}, X_{t_{k+1}}^{t,x;u^\varepsilon,v^\varepsilon})]] \\
&\ge {}^jG_{t_i,t_k}^{t,x;u^\varepsilon,v^\varepsilon}[W_j(t_k, X_{t_k}^{t,x;u^\varepsilon,v^\varepsilon}) - \varepsilon_0] \\
&\ge {}^jG_{t_i,t_k}^{t,x;u^\varepsilon,v^\varepsilon}[W_j(t_k, X_{t_k}^{t,x;u^\varepsilon,v^\varepsilon})] - C\varepsilon_0 \\
&\ge \cdots \ge {}^jG_{t_i,t_{i+1}}^{t,x;u^\varepsilon,v^\varepsilon}[W_j(t_{i+1}, X_{t_{i+1}}^{t,x;u^\varepsilon,v^\varepsilon})] - C(k-i)\varepsilon_0 \\
&\ge W_j(t_i, X_{t_i}^{t,x;u^\varepsilon,v^\varepsilon}) - C(k-i+1)\varepsilon_0.
\end{aligned}$$

Consequently, from the above inequality and a similar argument in the proof of (17), it follows that there exists a constant $C > 0$ such that

$$\begin{aligned}
{}^jG_{s_1,t_{k+1}}^{t,x;u^\varepsilon,v^\varepsilon}[W_j(t_{k+1}, X_{t_{k+1}}^{t,x;u^\varepsilon,v^\varepsilon})] &= {}^jG_{s_1,t_i}^{t,x;u^\varepsilon,v^\varepsilon}[{}^jG_{t_i,t_{k+1}}^{t,x;u^\varepsilon,v^\varepsilon}[W_j(t_{k+1}, X_{t_{k+1}}^{t,x;u^\varepsilon,v^\varepsilon})]] \\
&\ge {}^jG_{s_1,t_i}^{t,x;u^\varepsilon,v^\varepsilon}[W_j(t_i, X_{t_i}^{t,x;u^\varepsilon,v^\varepsilon})] - C(k-i+2)\varepsilon_0 \\
&\ge {}^jG_{s_1,t_i}^{t,x;u^\varepsilon,v^\varepsilon}[W_j(t_i, X_{t_i}^{t,x;u^\varepsilon,v^\varepsilon})] - \frac{\varepsilon}{2},
\end{aligned}$$

where we write $\varepsilon_0 = \varepsilon/2Cn$. We set

$$\begin{aligned}
I_1 &= {}^jG_{s_1,t_{k+1}}^{t,x;u^\varepsilon,v^\varepsilon}[W_j(t_{k+1}, X_{t_{k+1}}^{t,x;u^\varepsilon,v^\varepsilon})] - {}^jG_{s_1,t_i}^{t,x;u^\varepsilon,v^\varepsilon}[W_j(t_i, X_{t_i}^{t,x;u^\varepsilon,v^\varepsilon})] + \frac{\varepsilon}{2} \ge 0, \\
I_2 &= {}^jG_{s_1,s_2}^{t,x;u^\varepsilon,v^\varepsilon}[W_j(s_2, X_{s_2}^{t,x;u^\varepsilon,v^\varepsilon})] - W_j(s_1, X_{s_1}^{t,x;u^\varepsilon,v^\varepsilon}) + \frac{\varepsilon}{2}.
\end{aligned} \tag{24}$$

We claim that

$$\mathbb{E}[|I_1 - I_2|^2] \le C\tau.$$

Indeed, we write

$$y_s = {}^jG_{s,t_i}^{t,x;u^\varepsilon,v^\varepsilon}[W_j(t_i, X_{t_i}^{t,x;u^\varepsilon,v^\varepsilon})], \qquad s \in [s_1, t_i].$$

We consider the following BSDEs:

$$y_s = W_j(t_i, X_{t_i}^{t,x;u^\varepsilon,v^\varepsilon}) + \int_s^{t_i} f_j(r, X_r^{t,x;u^\varepsilon,v^\varepsilon}, y_r, z_r, u_r^\varepsilon, v_r^\varepsilon)\,\mathrm{d}r \\ + k_{t_i}^+ - k_s^+ - k_{t_i}^- + k_s^- - \int_s^{t_i} z_r\,\mathrm{d}B_r,$$

$$h_j(s, X_s^{t,x;u^\varepsilon,v^\varepsilon}) \le y_s \le h_j'(s, X_s^{t,x;u^\varepsilon,v^\varepsilon}), \qquad s \in [s_1, t_i],$$

$$\int_{t_i}^{t_{i+1}} (y_r - h_j(r, X_r^{t,x;u^\varepsilon,v^\varepsilon}))\,\mathrm{d}k_r^+ = \int_{t_i}^{t_{i+1}} (y_r - h_j'(r, X_r^{t,x;u^\varepsilon,v^\varepsilon}))\,\mathrm{d}k_r^- = 0,$$

and

$$y_s' = W_j(s_1, X_{s_1}^{t,x;u^\varepsilon,v^\varepsilon}), \qquad s \in [s_1, t_i].$$

By Lemma 2, we obtain

$$\begin{aligned}
&|{}^jG_{s_1,t_i}^{t,x;u^\varepsilon,v^\varepsilon}[W_j(t_i, X_{t_i}^{t,x;u^\varepsilon,v^\varepsilon})] - W_j(s_1, X_{s_1}^{t,x;u^\varepsilon,v^\varepsilon})|^2 \\
&\quad \le C\mathbb{E}[|W_j(t_i, X_{t_i}^{t,x;u^\varepsilon,v^\varepsilon}) - W_j(s_1, X_{s_1}^{t,x;u^\varepsilon,v^\varepsilon})|^2 \mid \mathcal{F}_{s_1}] \\
&\qquad + C\mathbb{E}\bigg[\int_{s_1}^{t_i} |f_j(r, X_r^{t,x;u^\varepsilon,v^\varepsilon}, y_r, z_r, u_r^\varepsilon, v_r^\varepsilon)|^2 \mid \mathcal{F}_{s_1}\bigg] \\
&\qquad + C\mathbb{E}\bigg[\sup_{s_1\le s\le t_i} |h_j(s, X_s^{t,x;u^\varepsilon,v^\varepsilon}) - h_j(s_1, X_{s_1}^{t,x;u^\varepsilon,v^\varepsilon})|^2 dr \mid \mathcal{F}_{s_1}\bigg]^{1/2} \\
&\quad \le C\mathbb{E}[|W_j(t_i, X_{t_i}^{t,x;u^\varepsilon,v^\varepsilon}) - W_j(s_1, X_{s_1}^{t,x;u^\varepsilon,v^\varepsilon})|^2 \mid \mathcal{F}_{s_1}] \\
&\qquad + C(t_i - s_1) + C\mathbb{E}\bigg[\sup_{s_1\le s\le t_i} |X_s^{t,x;u^\varepsilon,v^\varepsilon} - X_{s_1}^{t,x;u^\varepsilon,v^\varepsilon}|^2 \mid \mathcal{F}_{s_1}\bigg]^{1/2}.
\end{aligned}$$

Here, we have used assumptions (H3.3) and (H3.4) and the boundedness of f_j. Since $(u^\varepsilon, v^\varepsilon) \in \mathcal{U}_{t,T} \times \mathcal{V}_{t,T}$ is independent of $\mathcal{F}_t$, we have

$$\begin{aligned}
&\mathbb{E}[|{}^jG_{s_1,t_i}^{t,x;u^\varepsilon,v^\varepsilon}[W_j(t_i, X_{t_i}^{t,x;u^\varepsilon,v^\varepsilon})] - W_j(s_1, X_{s_1}^{t,x;u^\varepsilon,v^\varepsilon})|^2 \mid \mathcal{F}_t] \\
&\quad \le C\mathbb{E}[|W_j(t_i, X_{t_i}^{t,x;u^\varepsilon,v^\varepsilon}) - W_j(s_1, X_{s_1}^{t,x;u^\varepsilon,v^\varepsilon})|^2] + C(t_i - s_1) \\
&\qquad + C\mathbb{E}\bigg[\sup_{s_1\le s\le t_i} |X_s^{t,x;u^\varepsilon,v^\varepsilon} - X_{s_1}^{t,x;u^\varepsilon,v^\varepsilon}|^2 \mid \mathcal{F}_t\bigg]^{1/2}.
\end{aligned}$$

From (23), we have

$$\mathbb{E}[|{}^jG_{s_1,t_i}^{t,x;u^\varepsilon,v^\varepsilon}[W_j(t_i, X_{t_i}^{t,x;u^\varepsilon,v^\varepsilon})] - W_j(s_1, X_{s_1}^{t,x;u^\varepsilon,v^\varepsilon})|^2] \le C\tau^{1/2}. \tag{25}$$

Using a similar argument, we have

$$\mathbb{E}[|{}^jG_{s_2,t_{k+1}}^{t,x;u^\varepsilon,v^\varepsilon}[W_j(t_{k+1}, X_{t_{k+1}}^{t,x;u^\varepsilon,v^\varepsilon})] - W_j(s_2, X_{s_2}^{t,x;u^\varepsilon,v^\varepsilon})|^2] \le C\tau^{1/2}. \tag{26}$$

For $s \in [s_1, s_2]$, we write

$$y_s^1 = {}^jG_{s,t_{k+1}}^{t,x;u^\varepsilon,v^\varepsilon}[W_j(t_{k+1}, X_{t_{k+1}}^{t,x;u^\varepsilon,v^\varepsilon})] = {}^jG_{s,s_2}^{t,x;u^\varepsilon,v^\varepsilon}[{}^jG_{s_2,t_{k+1}}^{t,x;u^\varepsilon,v^\varepsilon}[W_j(t_{k+1}, X_{t_{k+1}}^{t,x;u^\varepsilon,v^\varepsilon})]],$$

and

$$y_s^2 = {}^jG_{s,s_2}^{t,x;u^\varepsilon,v^\varepsilon}[W_j(s_2, X_{s_2}^{t,x;u^\varepsilon,v^\varepsilon})].$$

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Let us consider the associated BSDEs:

$$y_s^1 = {}^jG^{t,x;u^\varepsilon,v^\varepsilon}_{s_2,t_{k+1}}[W_j(t_{k+1}, X^{t,x;u^\varepsilon,v^\varepsilon}_{t_{k+1}})] + \int_s^{s_2} f_j(r, X^{t,x;u^\varepsilon,v^\varepsilon}_r, y^1_r, z^1_r, u^\varepsilon_r, v^\varepsilon_r)\,\mathrm{d}r$$
$$+ k^{1,+}_{s_2} - k^{1,+}_s - k^{1,-}_{s_2} + k^{1,-}_s - \int_s^{s_2} z^1_r\,\mathrm{d}B_r,$$
$$h_j(s, X^{t,x;u^\varepsilon,v^\varepsilon}_s) \le y^1_s \le h'_j(s, X^{t,x;u^\varepsilon,v^\varepsilon}_s), \qquad s\in[s_1,s_2],$$
$$\int_{s_1}^{s_2} (y_r - h_j(r, X^{t,x;u^\varepsilon,v^\varepsilon}_r))\,\mathrm{d}k^{1,+}_r = \int_{s_1}^{s_2} (y_r - h'_j(r, X^{t,x;u^\varepsilon,v^\varepsilon}_r))\,\mathrm{d}k^{1,-}_r = 0,$$

and

$$y_s^2 = W_j(s_2, X^{t,x;u^\varepsilon,v^\varepsilon}_{s_2}) + \int_s^{s_2} f_j(r, X^{t,x;u^\varepsilon,v^\varepsilon}_r, y^2_r, z^2_r, u^\varepsilon_r, v^\varepsilon_r)\,\mathrm{d}r$$
$$+ k^{2,+}_{s_2} - k^{2,+}_s - k^{2,-}_{s_2} + k^{2,-}_s - \int_s^{s_2} z^2_r\,\mathrm{d}B_r,$$
$$h_j(s, X^{t,x;u^\varepsilon,v^\varepsilon}_s) \le y^2_s \le h'_j(s, X^{t,x;u^\varepsilon,v^\varepsilon}_s), \qquad s\in[s_1,s_2],$$
$$\int_{s_1}^{s_2} (y_r - h_j(r, X^{t,x;u^\varepsilon,v^\varepsilon}_r))\,\mathrm{d}k^{2,+}_r = \int_{s_1}^{s_2} (y_r - h'_j(r, X^{t,x;u^\varepsilon,v^\varepsilon}_r))\,\mathrm{d}k^{2,+}_r = 0,$$

By Lemmas 2 and 4, it follows that

$$|{}^jG^{t,x;u^\varepsilon,v^\varepsilon}_{s_1,t_{k+1}}[W_j(t_{k+1}, X^{t,x;u^\varepsilon,v^\varepsilon}_{t_{k+1}})] - {}^jG^{t,x;u^\varepsilon,v^\varepsilon}_{s_1,s_2}[W_j(s_2, X^{t,x;u^\varepsilon,v^\varepsilon}_{s_2})]|^2$$
$$\le C\mathbb{E}[|{}^jG^{t,x;u^\varepsilon,v^\varepsilon}_{s_2,t_{k+1}}[W_j(t_{k+1}, X^{t,x;u^\varepsilon,v^\varepsilon}_{t_{k+1}})] - W_j(s_2, X^{t,x;u^\varepsilon,v^\varepsilon}_{s_2})|^2 \mid \mathcal{F}_{s_1}].$$

Hence, (26) yields

$$\mathbb{E}[|{}^jG^{t,x;u^\varepsilon,v^\varepsilon}_{s_1,t_{k+1}}[W_j(t_{k+1}, X^{t,x;u^\varepsilon,v^\varepsilon}_{t_{k+1}})] - {}^jG^{t,x;u^\varepsilon,v^\varepsilon}_{s_1,s_2}[W_j(s_2, X^{t,x;u^\varepsilon,v^\varepsilon}_{s_2})]|^2] \le C\tau^{1/2}.$$

The above inequality and (25) yield

$$\mathbb{E}[|I_1 - I_2|^2] \le C\tau^{1/2}.$$

Therefore,

$$\mathbb{P}\bigg(I_2 \le -\frac{\varepsilon}{2}\bigg) \le \mathbb{P}\bigg(|I_1 - I_2| \ge \frac{\varepsilon}{2}\bigg) \le \frac{4\mathbb{E}[|I_1-I_2|^2]}{\varepsilon^2} \le \frac{4C\tau^{1/2}}{\varepsilon^2} \le \varepsilon,$$

where we choose $\tau \le (\varepsilon^3/4C)^2$, and, by (24), we have

$$\mathbb{P}(W_j(s_1, X^{t,x;u^\varepsilon,v^\varepsilon}_{s_1}) - \varepsilon \le {}^jG^{t,x;u^\varepsilon,v^\varepsilon}_{s_1,s_2}[W_j(s_2, X^{t,x;u^\varepsilon,v^\varepsilon}_{s_2})]) \ge 1-\varepsilon.$$

We also refer to the fact that since $(u^\varepsilon, v^\varepsilon)$ is independent of $\mathcal{F}_t$, the conditional probability $\mathbb{P}(\cdot \mid \mathcal{F}_t)$ of the event $\{W_j(s_1, X^{t,x;u^\varepsilon,v^\varepsilon}_{s_1}) - \varepsilon \le {}^jG^{t,x;u^\varepsilon,v^\varepsilon}_{s_1,s_2}[W_j(s_2, X^{t,x;u^\varepsilon,v^\varepsilon}_{s_2})]\}$ coincides with its probability. Indeed, also

$$\{W_j(s_1, X^{t,x;u^\varepsilon,v^\varepsilon}_{s_1}) - \varepsilon \le {}^jG^{t,x;u^\varepsilon,v^\varepsilon}_{s_1,s_2}[W_j(s_2, X^{t,x;u^\varepsilon,v^\varepsilon}_{s_2})]\}$$

is independent of $\mathcal{F}_t$. The proof is complete.

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We now provide another main result in this section: the existence theorem of a Nash equilibrium payoff.

Theorem 4. *Let Isaacs' condition A hold. Then for all $(t, x) \in [0, T] \times \mathbb{R}^n$, there exists a Nash equilibrium payoff at (t, x).*

Proof. By Theorem 3 we have to prove only that for all $\varepsilon > 0$, there exists $(u^{\varepsilon}, v^{\varepsilon}) \in \mathcal{U}_{t,T} \times \mathcal{V}_{t,T}$ which satisfies (9) and (10) for $s \in [t, T]$, $j = 1, 2$. For $\varepsilon > 0$, let us consider $(u^{\varepsilon}, v^{\varepsilon}) \in \mathcal{U}_{t,T} \times \mathcal{V}_{t,T}$ given by Proposition 8, i.e. in particular, $(u^{\varepsilon}, v^{\varepsilon})$ is independent of $\mathcal{F}_t$. Setting $s_1 = t$ and $s_2 = T$ in Proposition 8, we obtain (9). Since $(u^{\varepsilon}, v^{\varepsilon})$ is independent of $\mathcal{F}_t$, $J_j(t, x; u^{\varepsilon}, v^{\varepsilon})$, $j = 1, 2$, are deterministic and $\{(J_1(t, x; u^{\varepsilon}, v^{\varepsilon}), J_2(t, x; u^{\varepsilon}, v^{\varepsilon})), \varepsilon > 0\}$ is a bounded sequence. Consequently, let us choose an accumulation point of this sequence, as $\varepsilon \to 0$. We denote this point by (e_1, e_2). From Theorem 3, it follows that (e_1, e_2) is a Nash equilibrium payoff at (t, x). The proof is complete.

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