A REMARK ON p-VALENT FUNCTIONS

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In the theory of multivalent functions there are several different levels of postulates for p-valency. Perhaps the most well-known is the class of mean p-valent functions in the sense of Spencer [8] (we shall refer to them as areally mean p-valent functions), whose basic properties are found, e.g., in Hayman [4]. Recently Eke [1, 2] extended to these functions a number of results which had been known for circumferentially mean p-valent functions.

On the other hand, Garabedian-Royden [3] and Jenkins [5] have introduced a wider class, for which they discussed the extension of Koebe's 1/4-theorem. Functions in this class are referred to as weakly mean p-valent functions by the former, and logarithmically areally mean p-valent functions by the latter. There are various other properties of areally mean p-valent functions which are satisfied by those functions also.

In the present paper, we shall discuss a negative aspect of logarithmically areally mean p-valent functions. It will be shown that the above mentioned result of Eke cannot be extended to those functions.

We shall also give a glance at s-dimensionally mean p-valent functions, discussed in Spencer [8], which lie in between areally mean p-valent functions and logarithmically areally mean p-valent functions.

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Given a regular function f on the unit disc $|z| < 1$, let $n(w)$ be the number of w-points counted with multiplicity, and consider its circumferential mean

$$p(R) = \frac{1}{2\pi}\int_0^{2\pi} n(Re^{i\theta})d\theta,$$

$0 \leqq R < \infty$. It is a non-negative lower-semicontinuous function and is such that $p(R) > 0$ if and only if there exists z satisfying $R = |f(z)|$.

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If there exists a number $p > 0$ such that

$$\int_0^R p(R)d(R^2) \leqq p\pi R^2 \tag{A}$$

for $R > 0$, f is called an *areally mean p-valent function*. It has at most $[p]$ zeros and satisfies the following basic inequality (see Hayman [4, p. 23]):

$$\frac{1}{p}\left(\log\frac{R_2}{R_1} - \tfrac{1}{2}\right) \leqq \int_{R_1}^{R_2} \frac{dR}{Rp(R)} \tag{B}$$

for every R_1, R_2 with $0 < R_1 < R_2$.

If there exists $p > 0$ such that

$$\int_{R_1}^{R_2} \frac{p(R)}{R} dR \leqq p\left(\log\frac{R_2}{R_1} + \tfrac{1}{2}\right) \tag{L}$$

for every R_1 and R_2 with $0 < R_1 < R_2$, we shall call f *logarithmically areally mean p-valent*. As appears implicitly in Hayman [4, p. 33], (A) implies (L). Further a function with (L) has at most $[p]$ zeros and, as Schwarz's inequality

$$\left(\log\frac{R_2}{R_1}\right)^2 \leqq \int_{R_1}^{R_2} \frac{p(R)}{R} dR \cdot \int_{R_1}^{R_2} \frac{dR}{Rp(R)}$$

shows, satisfies (B). As a consequence, all the theorems in Chapter 2 of Hayman [4] are true for logarithmically areally mean p-valent functions.

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If p is a positive integer, a function f with expansion

$$f(z) = z^p + a_{p+1} z^{p+1} + \cdots \tag{N}$$

about the origin will be referred to as *normalized*.

Clearly for a function with (N) and (A) there exists an $R_0 > 0$ such that $p(R) = p$ for R with $0 \leqq R \leqq R_0$ and

$$\frac{1}{p}\log\frac{R}{R_0} \leqq \int_{R_0}^{R} \frac{dR}{Rp(R)} \tag{B*}$$

for every $R \geqq R_0$.

We shall call a function *normalized logarithmically areally mean p-valent* if it satisfies (N) and, for some R_0,

$$\begin{cases} p(R) = p \text{ for every } R \text{ with } 0 \leqq R \leqq R_0 \\ \displaystyle\int_{R_0}^{R} \frac{p(R)dR}{R} \leqq p\log\frac{R}{R_0} \end{cases} \tag{L*}$$

for every $R \geqq R_0$. It is to be noted that neither of the implications $(\mathrm{N}, \mathrm{L}) \leftrightarrows (\mathrm{N}, \mathrm{L}^*)$ holds (see 9°).

As before a function with (N, A) satisfies (L*). A function with (N, L*) has a zero only at the origin and satisfies (B*). Therefore the theorems in Chapter 2 of Hayman [4] continue to hold for normalized logarithmically areally mean p-valent functions also.

Under the assumption (N), the condition (L*) is readily seen to be equivalent to

$$\int_0^R \frac{p(R)-p}{R}\, dR \leqq 0 \qquad \text{for every } R,$$

which is the definition adopted by Garabedian-Royden [3].

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We shall say that a p-valent function f *attains maximum growth in the direction* φ if

$$\overline{\lim_{r\to 1}}\,(1-r)^{2p}|f(re^{i\varphi})| > 0. \tag{1}$$

For circumferentially mean p-valent functions (i.e., $p(R) \leqq p$ for every $R > 0$) Hayman [4], and for areally p-valent functions Eke [1, 2] recently, proved that (1) implies regularity of growth, namely the existence of the finite non-zero limit

$$\lim_{r\to 1}\,(1-r)^{2p}|f(re^{i\varphi})|. \tag{2}$$

We shall show that this conclusion does not hold for a function with (L) or (N, L*):

THEOREM 1. *There exists a logarithmically areally mean p-valent function as well as a normalized logarithmically areally mean p-valent function which attain maximum growth in direction* φ *yet do not have the limit* (2).

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To prove Theorem 1 by constructing counter-examples we need some preparation, which begins with the quotation of a result of Eke [1, Theorem 3] as follows:

If a regular function on $|z| < 1$ has only a finite number of zeros and satisfies

$$\overline{\lim_{r\to 1}}\,|f(re^{i\varphi})| = \infty, \quad \int_{R_0}^{\infty} \frac{dR}{Rp(R)} = \infty \tag{3}$$

for some R_0 with $p(R_0) > 0$, then the limit

$$\alpha = \lim_{r\to 1}\left(\int_{R_0}^{|f(re^{i\varphi})|} \frac{dR}{Rp(R)} - 2\log\frac{1}{1-r}\right) \tag{4}$$

exists including the possibility of $\alpha = -\infty$.

Notice that, if (3) holds for some R_0, then it does also for every R_0 with $p(R_0) > 0$. The value of α depends on R_0, but whether or not $\alpha > -\infty$ is independent of R_0.

This result indicates that if the growth is measured by means of the integral in (4), then the growth is regular whenever $\alpha > -\infty$, and the case $\alpha > -\infty$ corresponds to the case where f attains maximum growth so measured in direction φ.

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We now compare these with (2) and (1).

LEMMA 1. *For a function f satisfying (B), a necessary and sufficient condition for* (1) *is the validity of*

$$\overline{\lim_{r\to 1}}\left(\int_{R_0}^{|f(re^{i\varphi})|}\frac{dR}{Rp(R)}-2\log\frac{1}{1-r}\right) > -\infty \tag{5}$$

and

$$\overline{\lim_{R\to\infty}}\left(\frac{1}{p}\log R-\int_{R_0}^{R}\frac{dR}{Rp(R)}\right) > -\infty \tag{6}$$

for every, of equivalently some, R_0 with $p(R_0) > 0$.

For a function f satisfying (B^*), the same is true with respect to the R_0 involved in (B^*).

PROOF. It is immediate if we compare (4) with

$$\overline{\lim_{r\to 1}}\left(\frac{1}{p}\log|f(re^{i\varphi})|-2\log\frac{1}{1-r}\right) > -\infty, \tag{1'}$$

which is equivalent to (1). Notice that either (1′) or (5), (6) implies (3), so that the limit of (5) always exists.

LEMMA 2. *For a function f satisfying* (1) *and (L), the existence of the finite non-zero limit* (2) *is equivalent to the existence of the finite limit*

$$\lim_{R\to\infty}\left(\int_{R_0}^{R}\frac{p(R)}{R}\,dR-p\log R\right) \tag{7}$$

for every, or equivalently some, R_0 with $p(R_0) > 0$.

For a function f satisfying (1) *and (L^*), the same is true with respect to the R_0 involved in (L^*).*

PROOF. On comparing (4) with

$$\lim_{r\to 1}\left(\frac{1}{p}\log|f(re^{i\varphi})|-2\log\frac{1}{1-r}\right) > -\infty, \tag{2'}$$

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which is equivalent to the existence of the finite non-zero limit (2), we conclude as before that (2′) is equivalent to the existence of the finite limit

$$\lim_{R\to\infty}\left(\frac{1}{p}\log R-\int_{R_0}^{R}\frac{dR}{Rp(R)}\right). \tag{8}$$

On the other hand, the right-hand side of

$$\begin{aligned}\int_{R_0}^{R}\frac{(p(R)-p)^2}{Rp(R)}\,dR \\ = \left(\int_{R_0}^{R}\frac{p(R)}{R}\,dR-p\log\frac{R}{R_0}\right)+p^2\left(\int_{R_0}^{R}\frac{dR}{Rp(R)}-\frac{1}{p}\log\frac{R}{R_0}\right)\end{aligned} \tag{9}$$

is bounded since (6) and either (L) or (L*) are satisfied. Since the integrand of the left-hand side is non-negative, the limit for $R\to\infty$ of the integral exists. Accordingly the existence of (8) is equivalent to that of the first term of the right-hand side of (9). Q.E.D.

Actually Eke [2, Theorem 1] showed the existence of (7) for areally mean p-valent functions with (1), and proved that (1) implies (2).

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In order to prove Theorem 1, it suffices to construct a function f which satisfies (5), (6), and either (L) or (L*), yet does not have the limit (7).

Except for (5), these conditions are geometric properties of the Riemann surface covering the w-plane. There is a case, even though very limited, where a sufficient condition for (5) (i.e., $\alpha > -\infty$ in (4)) also is obtained in geometric terms.

Suppose a regular function f on $|z| < 1$ has only a finite number of zeros. Take $r_0 > 0$ such that f does not vanish on the annulus $r_0 < |z| < 1$. Let D be the domain obtained from this annulus cut along the ray $\arg z = \varphi+\pi$. Clearly

$$\zeta = \log f(z)$$

is single valued and regular on D. We assume that this function is univalent. Furthermore we require the image domain to have the following shape: there exist positive lower-semicontinuous functions θ_1 and θ_2 on $-\infty < \xi < \infty$ having the property that the image domain is contained in $\Delta = \{\zeta = \xi+i\eta \mid -\infty < \xi < \infty,\ -\theta_1(\xi) < \eta < \theta_2(\xi)\}$, contains $\{\zeta\in\Delta \mid \xi_0 \leqq \xi\}$ for some ξ_0, and such that the point $z = e^{i\varphi}$ corresponds to $\zeta = +\infty$.

LEMMA 3. *If a function f with the above properties satisfies in addition the following conditions, then* (5) *holds*: *There exist* $0 < m$ *and* $M < \infty$ *such that*

$$m < \theta_k(\xi) < M, \qquad k = 1, 2 \tag{10}$$

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for $\xi \geqq \xi_0$, *and there exists* $V < \infty$ *such that the total variation* $V_k(\xi_1, \xi_2)$ *of* θ_k *over the interval* $[\xi_1, \xi_2]$ *satisfies*

$$(11) \qquad V_k(\xi_1, \xi_2) \leqq V, \qquad k = 1, 2$$

for every ξ_1, ξ_2 *with* $\xi_0 \leqq \xi_1 < \xi_2$.

Proof. Map the unit disk cut along the radius $\arg z = \varphi + \pi$ by

$$Z = \tfrac{1}{2} \log \frac{ze^{-i\varphi}}{(1-ze^{-i\varphi})^2}$$

onto the strip $S = \{Z \mid |\mathrm{Im}\, Z| < \frac{1}{2}\pi\}$. Apply the Second Fundamental Inequality of Ahlfors (see Jenkins-Oikawa [6]) to the conformal mapping $Z \to \zeta$. On setting $X'(\xi) = \inf\{\mathrm{Re}\, Z \mid \mathrm{Re}\, \zeta(Z) = \xi\}$, $X''(\xi) = \sup\{\mathrm{Re}\, Z \mid \mathrm{Re}\, \zeta(Z) = \xi\}$, and $\Theta = \theta_1 + \theta_2$, we obtain

$$\frac{X''(\xi) - X'(\xi^*)}{\pi} \leqq \int_{\xi^*}^{\xi} \frac{d\xi}{\Theta(\xi)} + \frac{VM}{m^2} + \frac{4M}{m}$$

for every ξ and ξ^* such that $\xi_0 + 2M < \xi^* < \xi$. Observe that $\Theta(\xi) = 2\pi p(R)$ if $\xi = \log R$ and $\frac{1}{2}\log(r(1-r)^{-2}) \leqq X''(\xi)$ if $R = |f(re^{i\varphi})|$. On taking R_0 with $\xi_0 < \log R_0$ and then ξ^* with $\log R_0 < \xi^*$, we obtain

$$2 \log \frac{r^{\frac{1}{2}}}{1-r} \leqq \int_{R_0}^{|f(re^{i\varphi})|} \frac{dR}{Rp(R)} + \text{const},$$

which implies (5).

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Now we are in a position to prove Theorem 1. Consider a positive lower-semi-continuous function $\Theta(\xi)$ on $-\infty < \xi < \infty$ such that $\Theta(\xi) = 2\pi p = \text{const.}$ for $\xi < 0$. Let g be a conformal mapping of $|z| < 1$ onto the domain $\Delta = \{\zeta = \xi + i\eta \mid -\infty < \xi < \infty, |\eta| < \Theta(\xi)/2\}$ such that $z = e^{i\varphi}$ corresponds to $\zeta = +\infty$. Set

$$f = \exp g.$$

Next, let g^* be a conformal mapping of $|z| < 1$ cut along the radius $\arg z = \varphi + \pi$ onto Δ such that $z = 0$, $e^{i\varphi}$ correspond to $\zeta = -\infty, +\infty$, respectively, and that the radius $\arg z = \varphi$ corresponds to the real-axis in the ζ-plane. If p is a positive integer, the symmetry of g^* guarantees the regularity of $\exp g^*$ on $|z| < 1$, which then has a zero of multiplicity p at the origin. On taking a constant c_0 suitably we can make

$$f^* = \exp(g^* + c_0)$$

satisfy the condition (N).

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For both these functions f and f^*, we have

$$2\pi p(R) = \Theta(\xi) \qquad \text{if } \xi = \log R.$$

Accordingly the conditions (L), (L*), (6), (10), and (11) with respect to $R_0 = 1$ for f and $R_0 = \exp c_0$ for f^* are respectively expressed as follows:

(12) $$\int_a^b (\Theta(\xi) - 2\pi p) d\xi \leqq \pi p \quad \text{for } 0 < a < b < \infty$$

(12*) $$\int_0^b (\Theta(\xi) - 2\pi p) d\xi \leqq 0 \quad \text{for } 0 < b < \infty$$

(13) $$\varliminf_{b \to \infty} \int_0^b \left(\frac{1}{\Theta(\xi)} - \frac{1}{2\pi p} \right) d\xi < \infty$$

(14) $$0 < \inf \Theta(\xi), \quad \sup \Theta(\xi) < \infty$$

(15) The total variation of $\Theta(\xi)$ on any closed subinterval is bounded by a constant V.

The non-existence of the limit (7) is equivalent to

(16) $$\text{Non-existence of } \lim_{b \to \infty} \int_0^b (\Theta(\xi) - 2\pi p) d\xi.$$

An example of a function $\Theta(\xi)$ with these properties is obtained by considering a step function as follows: Prepare sequences $\{\xi_\nu\}$ and $\{\varepsilon_\nu\}$ with $0 = \xi_0 < \xi_1 < \cdots \to \infty$ and $0 < \varepsilon_\nu < 1$, and set $\Theta(\xi) = 2\pi p$ if $\xi < 0$, $\Theta(\xi) = 2\pi p(1 + (-1)^\nu \varepsilon_\nu)$ if $\xi_{\nu-1} < \xi < \xi_\nu$, $\nu = 1, 2, \cdots$, and $\Theta(\xi_\nu)$ suitably so that the resulting function $\Theta(\xi)$ on $-\infty < \xi < \infty$ is positive and lower-semicontinuous. If the sequences satisfy, e.g.,

$$\sum_{\nu=1}^{\infty} \varepsilon_\nu < \infty, \ \varepsilon_\nu(\xi_\nu - \xi_{\nu-1}) = \tfrac{1}{6}, \quad \nu = 1, 2, \cdots,$$

then it is not difficult to see that $\Theta(\xi)$ satisfies (12), (12*), (13)–(16). The proof of Theorem 1 is herewith complete.

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Incidentally, for a positive integer p, we can construct a $\Theta(\xi)$ with (12) but not (12*), and also one with (12*) but not (12). Thus neither of (N, L) and (N, L*) implies the other.

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Spencer's paper [9] contains a suggestion for another possible extension of areal mean p-valency. Let us call a regular function f on the unit disc $|z| < 1$

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s-dimensionally mean p-valent $(s > 0)$ if

$$\int_0^R p(R)d(R^s) \leqq p\pi R^s \tag{A_s}$$

for $R > 0$.

Spencer [9] showed $(A_s) \Rightarrow (A_{s'})$ if $0 < s' \leqq s$. On the other hand, by an argument similar to Hayman [4, p. 33] we see that (A_s) implies

$$\int_{R_1}^{R_2} \frac{p(R)}{R} dR \leqq p\left(\log \frac{R_2}{R_1} + \frac{1}{s}\right) \tag{L_s}$$

for every R_1, R_2 with $0 < R_1 < R_2$. On disregarding the last constant of (L_s), we may say that s-dimensionally mean p-valent functions with $0 < s < 2$ are more general than areally mean p-valent functions, and essentially less general then logarithmically areally mean p-valent functions.

Observe that (L_s) implies as before

$$\frac{1}{p}\left(\log \frac{R_2}{R_1} - \frac{1}{s}\right) \leqq \int_{R_1}^{R_2} \frac{dR}{Rp(R)}, \tag{B_s}$$

so that the theorems in Chapter 2 of Hayman [4] are true for these functions.

For a function with (A_s) the reasoning of Eke [2, Theorem 1] is applicable mutatis mutandis to prove

THEOREM 2. *For an s-dimensionally mean p-valent function* $(0 < s \leqq 2)$ *which attains maximum growth in direction* φ, *the finite non-zero limit* (2) *exists.*

REMARK. W. K. Hayman has informed the authors that closely related results have been obtained independently by V. R. Eke.

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