ON CLASSES OF NULL SETS

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1

Results concerning classes of null sets have been obtained by various authors. See, for example, [3], [4], [6], [7]. This paper contains results concerning classes of null sets and the notion of a 'small system'. The motivation for considering 'small systems' comes from a paper by Riečan (c.f. [2]).

The main result of this paper is a natural method of constructing a class of null sets on a σ-ring. We begin with a nonempty class $\mathscr{E}$ and a sequence $\{\mathscr{N}_n\}_{n=1}^{\infty}$ of nonempty subclasses of $\mathscr{E}$. Using a method analogous to Caratheodory's method of extending measures, we construct a class of null sets on the generated σ-ring $\mathscr{S}(\mathscr{E})$.

Other results are also obtained which are generalisations of those for outer measures. Finally, the relationship between the results obtained and measure theory is indicated.

Throughout this paper, the notation E^c is used for the complement of a set and $E \Delta F$ for the symmetric difference of the sets E and F. The symbol N is used for the set of positive integers, and ϕ for the empty set. Any concept, which is not defined, is to be understood in the sense of Halmos [1].

DEFINITION. *Let X be an abstract set, $\mathscr{S}$ a σ-ring of subsets of X, and $\{\mathscr{M}_n\}_{n=1}^{\infty}$, a sequence of subclasses of $\mathscr{S}$, such that*

(A) *for each $n \in N$, $\mathscr{M}_n$ is non-empty*

(B) *for each $n \in N$, there exists a sequence $\{k_i\}_{i=1}^{\infty}$ of positive integers such that $E_i \in \mathscr{M}_{k_i}$ $(i = 1,2,\cdots)$ implies $\bigcup_{i=1}^{\infty} E_i \in \mathscr{M}_n$*

(C) *for each $n \in N$, if $E \in \mathscr{M}_n$ and $F \in \mathscr{S}$, then $E \cap F \in \mathscr{M}_n$.*

A sequence $\{\mathscr{M}_n\}_{n=1}^{\infty}$ satisfying all the above properties will be called a small system on $\mathscr{S}$.

EXAMPLE. Let X be a set, $\mathscr{S}$ a σ-ring of subsets of X and μ a measure on $\mathscr{S}$. For each $n \in N$, define

$$\mathcal{M}_n = \left\{E \in \mathcal{S} \mid \mu(E) < \frac{1}{n}\right\}.$$

Then the sequence $\{\mathcal{M}_n\}_{n=1}^{\infty}$ satisfies (A), (B) and (C). Property (B) is the replacement for the σ-subadditivity of μ, while property (C) replaces the monotonicity of μ.

If we put

$$\mathcal{M} = \{E \in \mathcal{S} \mid \mu(E) = 0\},$$

then it is easy to see that $\mathcal{M} = \bigcap_{n=1}^{\infty} \mathcal{M}_n$, and also that

(a) for each sequence $\{E_i\}_{i=1}^{\infty}$ in $\mathcal{S}$ such that $\mu(E_i) = 0$, for each i, then $\mu(\bigcup_{i=1}^{\infty} E_i) = 0$, and

(b) if $\mu(E) = 0$, $F \in \mathcal{S}$, then $\mu(E \cap F) = 0$.

Hence we are led to the following definition.

DEFINITION. *Let $\mathcal{S}$ be a σ-ring, and $\mathcal{N}$ a non-empty class such that $\mathcal{N} \subset \mathcal{S}$. Then $\mathcal{N}$ will be called a class of null sets in $\mathcal{S}$, if*

(i) $\bigcup_{i=1}^{\infty} E_i \in \mathcal{N}$, *where* $E_i \in \mathcal{N}$ $(i = 1, 2, \cdots)$

(ii) $E \cap F \in \mathcal{N}$, *where* $E \in \mathcal{N}$ *and* $F \in \mathcal{S}$.

Now let $\{\mathcal{M}_n\}_{n=1}^{\infty}$ be a small system on $\mathcal{S}$. If we put $\mathcal{M} = \bigcap_{n=1}^{\infty} \mathcal{M}_n$, then the following result holds.

THEOREM 1. *$\mathcal{M}$ is a class of null sets in $\mathcal{S}$.*

PROOF. (i) Suppose $E_i \in \mathcal{M}$ $(i = 1, 2, \cdots)$. Hence, for each $n \in N$, we have $E_i \in \mathcal{M}_n$ $(i = 1, 2, \cdots)$. Now fix n. Then, by (B), there exists a sequence $\{k_i\}_{i=1}^{\infty}$ of positive integers such that for any $F_i \in \mathcal{M}_{k_i}$, then $\bigcup_{i=1}^{\infty} F_i \in \mathcal{M}_n$.

Choose $F_i = E_i \in \mathcal{M}_{k_i}$ $(i = 1, 2, \cdots)$. Hence $\bigcup_{i=1}^{\infty} E_i \in \mathcal{M}_n$, and this is true for all $n \in N$. So $\bigcup_{i=1}^{\infty} E_i \in \mathcal{M}$.

(ii) Suppose $E \in \mathcal{M}$, $F \in \mathcal{S}$. So, for each $n \in N$, $E \in \mathcal{M}_n$, and thus by (C), $E \cap F \in \mathcal{M}_n$. That is, $E \cap F \in \mathcal{M}$. Thus the theorem is proved.

2

Let X be an abstract set, and $\mathcal{E}$ any non-empty class of subsets of X. Let $\mathcal{H}(\mathcal{E})$ be the hereditary σ-ring generated by $\mathcal{E}$, and $\{\mathcal{N}_n\}_{n=1}^{\infty}$ be any sequence of non-empty subclasses of $\mathcal{E}$.

REMARK 1. It will help the reader if he keeps the following example in mind. Let $\mathcal{E} = \mathcal{R}$, a ring, and let μ be a measure on $\mathcal{R}$. Then for each $n \in N$, define

$$\mathcal{N}_n = \left\{E \in \mathcal{R} \mid \mu(E) < \frac{1}{n}\right\}.$$

Using this example, one should see the connection between the construction to follow and Caratheodory's method of extension of measures.

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DEFINITION. *Given $E \in \mathscr{H}(\mathscr{E})$, we say the class of sets $\{E_i\}_{i \in I}$, $I \neq \phi$, $I \subset N$ is an n-cover for E, provided that $E_i \in \mathscr{N}_{k_i}$, for some $k_i \in N$ $(i \in I)$, $\bigcup_{i \in I} E_i \supset E$ and $\sum_{i \in I} 1/k_i \leqq 1/n$, where $n \in N$.*

Now we define a sequence $\{\mathscr{N}_n^*\}_{n=1}^{\infty}$ of subclasses of the class $\mathscr{H}(\mathscr{E})$ as follows:

DEFINITION. *For each $n \in N$, we define $\mathscr{N}_n^* = \{E \in \mathscr{H}(\mathscr{E}) \mid E$ has an n-cover$\}$.*

REMARK 2. Suppose $E_i \in \mathscr{N}_{k_i}^*$, where $i \in I \subset N$ and $\sum_{i \in I} 1/k_i \leqq 1/n$, then $\bigcup_{i \in I} E_i \in \mathscr{N}_n^*$.

LEMMA 1. *If the sequence $\{\mathscr{N}_n^*\}_{n=1}^{\infty}$ is defined as above, then $\mathscr{N}_n^* \supset \mathscr{N}_n$, for each $n \in N$. Further $\phi \in \mathscr{N}_n^*$, for each $n \in N$.*

PROOF. Given $n \in N$, let $E \in \mathscr{N}_n$. Then $\{E\}$ forms an *n-cover for E. Hence* $E \in \mathscr{N}_n^*$. It is clear that $\phi \in \mathscr{N}_n^*$, for each $n \in N$.

DEFINITION. *The small system $\{\mathscr{M}_n\}_{n=1}^{\infty}$ on the σ-ring $\mathscr{S}$ is said to be decreasing, if $\mathscr{M}_{n+1} \subset \mathscr{M}_n$, for each $n \in N$.*

THEOREM 2. *$\{\mathscr{N}_n^*\}_{n=1}^{\infty}$ is a decreasing small system on $\mathscr{H}(\mathscr{E})$.*

PROOF. It is clear that $\{\mathscr{N}_n^*\}_{n=1}^{\infty}$ is decreasing, since any $n+1$-cover of a set E in $\mathscr{H}(\mathscr{E})$ is also an n-cover of E.

(A) For each $n \in N$, $\mathscr{N}_n^* \neq \phi$, since $\mathscr{N}_n^* \supset \mathscr{N}_n$.

(B) We have to show that, given $n \in N$, there exists a sequence $\{k_i\}_{i=1}^{\infty}$ of positive integers such that for any $E_i \in \mathscr{N}_{k_i}^* (i = 1, 2, \cdots)$, then $\bigcup_{i=1}^{\infty} E_i \in \mathscr{N}_n^*$.

So, given $n \in N$, choose $\{k_i\}_{i=1}^{\infty}$ such that $\sum_{i=1}^{\infty} 1/k_i \leqq 1/n$. (It is sufficient to put $k_i = n.2^i (i = 1, 2, \cdots)$). Hence, for any $E_i \in \mathscr{N}_{k_i}^*$, by remark 2, we have $\bigcup_{i=1}^{\infty} E_i \in \mathscr{N}_n^*$.

(C) Given $n \in N$, let $E \in \mathscr{N}_n^*$ and $F \in \mathscr{H}(\mathscr{E})$. Then E has an n-cover and this will also be an n-cover for $E \cap F$. Thus $E \cap F \in \mathscr{N}_n^*$.

So $\{\mathscr{N}_n^*\}_{n=1}^{\infty}$ is a decreasing small system on $\mathscr{H}(\mathscr{E})$, and the theorem is proved.

DEFINITION. *We will call $\{\mathscr{N}_n^*\}_{n=1}^{\infty}$ the small system induced by $\{\mathscr{N}_n\}_{n=1}^{\infty}$.*

NOTATION. *We put $\mathscr{N}^* = \bigcap_{n=1}^{\infty} \mathscr{N}_n^*$.*

THEOREM 3. *$\mathscr{N}^*$ is a class of null sets in $\mathscr{H}(\mathscr{E})$.*

PROOF. The result follows from theorem 1, since $\{\mathscr{N}_n^*\}_{n=1}^{\infty}$ is a small system on $\mathscr{H}(\mathscr{E})$.

THEOREM 4. *$\{\mathscr{N}_n^* \cap \mathscr{S}(\mathscr{E})\}_{n=1}^{\infty}$ is a decreasing small system on $\mathscr{S}(\mathscr{E})$, the σ-ring generated by $\mathscr{E}$.*

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PROOF. (A) For each $n \in N$, $\phi \in \mathcal{N}_n^* \cap \mathcal{S}(\mathcal{E})$.

(B) We know that, given $n \in N$, there exists a sequence $\{k_i\}_{i=1}^\infty$ of positive integers such that for any $E_i \in \mathcal{N}_{k_i}^*$, then $\bigcup_{i=1}^\infty E_i \in \mathcal{N}_n^*$. This same sequence $\{k_i\}_{i=1}^\infty$ can be used for $\{\mathcal{N}_n^* \cap \mathcal{S}(\mathcal{E})\}_{n=1}^\infty$ since, given $n \in N$, for any $F_i \in \mathcal{N}_{k_i}^* \cap \mathcal{S}(\mathcal{E})$ $(i = 1,2,\cdots)$, we have $\bigcup_{i=1}^\infty F_i \in \mathcal{N}_n^*$ and $\bigcup_{i=1}^\infty F_i \in \mathcal{S}(\mathcal{E})$. So $\bigcup_{i=1}^\infty F_i \in \mathcal{N}_n^* \cap \mathcal{S}(\mathcal{E})$.

(C) Given $n \in N$, let $E \in \mathcal{N}_n^* \cap \mathcal{S}(\mathcal{E})$, and $F \in \mathcal{S}(\mathcal{E})$. Then $E \cap F \in \mathcal{N}_n^* \cap \mathcal{S}(\mathcal{E})$, since $\{\mathcal{N}_n^*\}_{n=1}^\infty$ is a small system on $\mathcal{H}(\mathcal{E})$.

Finally, $\{\mathcal{N}_n^* \cap \mathcal{S}(\mathcal{E})\}_{n=1}^\infty$ is decreasing, since $\{\mathcal{N}_n^*\}_{n=1}^\infty$ is decreasing. Thus the theorem is proved.

THEOREM 5. *$\mathcal{N}^* \cap \mathcal{S}(\mathcal{E})$ is a class of null sets in $\mathcal{S}(\mathcal{E})$.*

PROOF. The result follows from theorem 1, since

$$\mathcal{N}^* \cap \mathcal{S}(\mathcal{E}) = \bigcap_{n=1}^\infty (\mathcal{N}_n^* \cap \mathcal{S}(\mathcal{E}))$$

and $\{\mathcal{N}_n^* \cap \mathcal{S}(\mathcal{E}))_{n=1}^\infty$ is a small system on $\mathcal{S}(\mathcal{E})$.

REMARK 3. Theorems 4 and 5 remain true if $\mathcal{S}(\mathcal{E})$ is replaced by any σ-ring $\mathcal{E}$, such that $\mathcal{S}(\mathcal{E}) \subset \mathcal{S} \subset \mathcal{H}(\mathcal{E})$.

REMARK 4. Theorem 5 completes the construction of the class of null sets on $\mathcal{S}(\mathcal{E})$. As we shall show in section 4, for the special case when $\{\mathcal{N}_n\}_{n=1}^\infty$ is defined by

$$\mathcal{N}_n = \left\{E \in \mathcal{R} \,\middle|\, \mu(E) < \frac{1}{n}\right\}$$

for each $n \in N$, as in remark 1, then $\mathcal{N}^*$ is precisely the class of sets of induced outer measure zero in $\mathcal{H}(\mathcal{R})$. Unfortunately, it is not true for an arbitrary measure μ that

$$\mathcal{N}_n^* = \left\{E \in \mathcal{H}(\mathcal{R}) \,\middle|\, \mu^*(E) < \frac{1}{n}\right\}.$$

However, as the reader will see in section 4, if he thinks of $\mathcal{N}_n^*$ as the class of sets of induced outer measure $< 1/n$, it will provide motivation for the work in this and the next section.

We now consider two sequences $\{\mathcal{N}_n^1\}_{n=1}^\infty$, $\{\mathcal{N}_n^2\}_{n=1}^\infty$ of non-empty sub-classes of $\mathcal{E}$. Then we can form the induced small systems $\{\mathcal{N}_n^{1^*}\}_{n=1}^\infty$ and $\{\mathcal{N}_n^{2^*}\}_{n=1}^\infty$ on $\mathcal{H}(\mathcal{E})$.

THEOREM 6. *In the above notation, we have $\mathcal{N}_n^{1^*} = \mathcal{N}_n^{2^*}$, for each $n \in N$, if and only if both $\mathcal{N}_n^{1^*} \supset \mathcal{N}_n^2$ and $\mathcal{N}_n^{2^*} \supset \mathcal{N}_n^1$, for each $n \in N$.*

Proof. Suppose $\mathscr{N}_n^{1*} = \mathscr{N}_n^{2*}$, for each $n \in N$. Then $\mathscr{N}_n^{1*} = \mathscr{N}_n^{2*} \supset \mathscr{N}_n^2$ and $\mathscr{N}_n^{2*} = \mathscr{N}_n^{1*} \supset \mathscr{N}_n^1$, for each $n \in N$.

Conversely, given any $n \in N$, we show that $\mathscr{N}_n^{1*} = \mathscr{N}_n^{2*}$. From remark 2, it follows that $\mathscr{N}_n^1 \subset \mathscr{N}_n^{2*}$, for all $n \in N$ implies $\mathscr{N}_n^{1*} \subset \mathscr{N}_n^{2*}$, for each $n \in N$. Similarly $\mathscr{N}_n^{1*} \supset \mathscr{N}_n^{2*}$, and the theorem is proved.

Now suppose that $\mathscr{F}$ is a class of sets such that $\mathscr{E} \subset \mathscr{F} \subset \mathscr{H}(\mathscr{E})$. If $\{\mathscr{N}_n\}_{n=1}^\infty$ is a sequence of non-empty subclasses of $\mathscr{E}$ (and hence of $\mathscr{F}$), we can construct the small system $\{\mathscr{N}_n^*\}_{n=1}^\infty$ on $\mathscr{H}(\mathscr{E}) = \mathscr{H}(\mathscr{F})$. Thus $\{\mathscr{N}_n^* \cap \mathscr{F}\}_{n=1}^\infty$ is a sequence of non-empty subclasses of $\mathscr{F}$. Hence we can construct the small system $\{(\mathscr{N}_n^* \cap \mathscr{F})^*\}_{n=1}^\infty$ induced on $\mathscr{H}(\mathscr{E})$ by $\{\mathscr{N}_n^* \cap \mathscr{F}\}_{n=1}^\infty$ on $\mathscr{F}$. Then, with this notation, we have the following result.

Proposition 1. *For each* $n \in N$, $\mathscr{N}_n^* = (\mathscr{N}_n^* \cap \mathscr{F})^*$.

Proof. Given $n \in N$, we have $\mathscr{N}_n^* \supset \mathscr{N}_n^* \cap \mathscr{F}$, and also

$$(\mathscr{N}_n^* \cap \mathscr{F})^* \supset \mathscr{N}_n^* \cap \mathscr{F} \supset \mathscr{N}_n \cap \mathscr{F} = \mathscr{N}_n.$$

Hence the result follows from theorem 6.

Corollary 1. *For each* $n \in N$, $\mathscr{N}_n^* = (\mathscr{N}_n^*)^*$.

Proof. Put $\mathscr{F} = \mathscr{H}(\mathscr{E})$ in proposition 1.

3

Let X be an abstract set. Throughout this section let $\mathscr{E}$ be a non-empty class of subsets of X and $\{\mathscr{N}_n\}_{n=0}^\infty$ a sequence of non-empty subclasses of $\mathscr{E}$ such that

(i) $\mathscr{N}_0 \supset \mathscr{N}_n$, for each $n \in N$
(ii) $E \in \mathscr{N}_0$, $F \in \mathscr{N}_0$ implies $E \Delta F \in \mathscr{N}_0$
(iii) $E \in \mathscr{N}_0$, $F \in \mathscr{E}$ implies $E \cap F \in \mathscr{N}_0$.

Remark 5. From (iii), we see that $E \in \mathscr{N}_0$, $F \in \mathscr{N}_0$ implies $E \cap F \in \mathscr{N}_0$. Hence (ii) and (iii) imply that $\mathscr{N}_0$ is a ring.

Remark 6. With $\{\mathscr{N}_n\}_{n=1}^\infty$ defined as in remark 1, put $\mathscr{N}_0 = \{E \in \mathscr{R} \mid \mu(E) < \infty\}$. Then $\{\mathscr{N}_n\}_{n=0}^\infty$ satisfies the conditions (i), (ii) and (iii) above.

As in section 2, we can define the induced small system $\{\mathscr{N}_n^*\}_{n=1}^\infty$ on $\mathscr{H}(\mathscr{E})$, from the sequence $\{\mathscr{N}_n\}_{n=1}^\infty$ on $\mathscr{E}$. We can also define the class $\mathscr{N}_0^* \subset \mathscr{H}(\mathscr{E})$ as follows:

Definition. $\mathscr{N}_0^* = \{E \in \mathscr{H}(\mathscr{E}) \mid$ *for each* $n \in N$, *there exists* $F \in \mathscr{N}_0$ *such that* $E - F \in \mathscr{N}_n^*\}$.

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REMARK 7. It is clear that $\mathcal{N}_0 \subset \mathcal{N}_0^*$ and also $\mathcal{N}_0^*$ is hereditary, in the sense that $E \in \mathcal{N}_0^*$ and $F \subset E$ imply $F \in \mathcal{N}_0^*$.

REMARK 8. We will see later that for the case when $\{\mathcal{N}_n\}_{n=0}^{\infty}$ is defined as in remark 6, $\mathcal{N}_0^*$ is precisely the class of sets of finite outer measure in $\mathcal{H}(\mathcal{R})$.

PROPOSITION 2. *For each* $m \in N$, $\mathcal{N}_0^* \supset \mathcal{N}_m^*$.

PROOF. Given $m \in N$, let $E \in \mathcal{N}_m^*$. Hence E has an m-cover $\{E_i\}_{i \in I}$. Suppose $n \in N$ is given. If I is finite, choose $F = \bigcup_{i \in I} E_i$. Otherwise choose i_0 such that $\sum_{i > i_0} 1/k_i \leqq 1/n$, where $E_i \in \mathcal{N}_{k_i}$, and put $F = \bigcup_{i=1}^{i_0} E_i$. In either case, $F \in \mathcal{N}_0$ and $E - F \in \mathcal{N}_n^*$. Thus the proposition is proved.

PROPOSITION 3. $\mathcal{E} \subset \mathcal{H}(\mathcal{N}_0)$ *implies* $\mathcal{H}(\mathcal{E}) \subset \mathcal{H}(\mathcal{N}_0^*)$, *where* $\mathcal{H}(\mathcal{N}_0)$ and $\mathcal{H}(\mathcal{N}_0^*)$ are *the hereditary σ-rings generated by* $\mathcal{N}_0$ *and* $\mathcal{N}_0^*$ *respectively.*

PROOF. $\mathcal{E} \subset \mathcal{H}(\mathcal{N}_0)$ implies $\mathcal{H}(\mathcal{E}) \subset \mathcal{H}(\mathcal{N}_0)$. Then, since $\mathcal{N}_0^* \supset \mathcal{N}_0$, we have $\mathcal{H}(\mathcal{N}_0^*) \supset \mathcal{H}(\mathcal{N}_0) \supset \mathcal{H}(\mathcal{E})$.

To motivate the next two definitions, we remind the reader of the following measure-theoretic results.

PROPOSITION 4. *Let* $E \in \mathcal{H}(\mathcal{R})$. *Then* E *is* μ^**-measurable and* $\mu^*(E) < \infty$, *if and only if, given* $\varepsilon > 0$, *there exists* $F \in \mathcal{R}$ *siuch that* $\mu(F) < \infty$ *and* $\mu^*(E \Delta F) < \varepsilon$.

PROPOSITION 5. *If* $E \in \mathcal{H}(\mathcal{R})$, *then* E *is* μ^**-measurable, if and only if, given* $\varepsilon > 0$ *and* $A \in \mathcal{R}$ *such that* $\mu(A) < \infty$, *there exists* $F \in \mathcal{R}$ *such that* $\mu(F) < \infty$ *and* $\mu^*[(E \cap A) \Delta F] < \varepsilon$.

With these results in mind, we make the following definitions.

DEFINITION. $\mathcal{S}_0^* = \{E \in \mathcal{H}(\mathcal{E}) \mid$ *given* $n \in N$, *there exists* $F \in \mathcal{N}_0$ *such that* $E \Delta F \in \mathcal{N}_n^*\}$.

DEFINITION. $\mathcal{S}^* = \{E \in \mathcal{H}(\mathcal{E}) \mid$ *given* $n \in N$ *and* $A \in \mathcal{N}_0$, *there exists* $F \in \mathcal{N}_0$ *such that* $(E \cap A) \Delta F \in \mathcal{N}_n^*\}$.

REMARK 9. It is clear that $E \in \mathcal{S}^*$ if and only if $E \cap A \in \mathcal{S}_0^*$, for all $A \in \mathcal{N}_0$. Also we will see later that with $\{\mathcal{N}_n\}_{n=0}^{\infty}$ defined as in remark 6, $\mathcal{S}_0^*$ is the class of measurable sets of finite outer measure in $\mathcal{H}(\mathcal{R})$, and $\mathcal{S}^*$ is the class of measurable sets in $\mathcal{H}(\mathcal{R})$.

THEOREM 7. $\mathcal{S}_0^* = \mathcal{S}^* \cap \mathcal{N}_0^*$.

PROOF. Let $E \in \mathcal{S}_0^*$, $A \in \mathcal{N}_0$ and $n \in N$. Hence there exists $F_1 \in \mathcal{N}_0$ such that $E \Delta F_1 \in \mathcal{N}_n^*$. Now

$$(E \cap A) \Delta (F_1 \cap A) = (E \Delta F_1) \cap A \subset E \Delta F_1.$$

So if we put $F = F_1 \cap A \in \mathcal{N}_0$, then $(E \cap A) \Delta F \in \mathcal{N}_n^*$. That is $E \in \mathcal{S}^*$. Hence $\mathcal{S}_0^* \subset \mathcal{S}^*$.

Further, since $E - F \subset E \Delta F$, we have $E \in \mathcal{N}_0^*$. Thus $\mathcal{S}_0^* \subset \mathcal{N}_0^*$, and so $\mathcal{S}_0^* \subset \mathcal{S}^* \cap \mathcal{N}_0^*$.

Now suppose $E \in \mathcal{S}^* \cap \mathcal{N}_0^*$. Hence, given $n \in N$, there exists $F_1 \in \mathcal{N}_0$ such that $E - F_1 \in \mathcal{N}_{2n}^*$. Then since E also belongs to $\mathcal{S}^*$, there exists $F \in \mathcal{N}_0$ such that

$$(E \cap F_1) \Delta F \in \mathcal{N}_{2n}^*.$$

Then

$$E \Delta F = [(E \cap F_1) \cup (E - F_1)] \Delta F \subset [(E \cap F_1) \Delta F] \cup (E - F_1) \in \mathcal{N}_n^*.$$

Thus $E \in \mathcal{S}_0^*$. That is $\mathcal{S}^* \cap \mathcal{N}_0^* \subset \mathcal{S}_0^*$, and the theorem is proved.

DEFINITION. *We say the sequence $\{\mathcal{N}_n\}_{n=0}^\infty$ satisfies the finiteness condition, if given $n \in N$, $A \in \mathcal{N}_0$ and $\{E_i\}_{i=1}$ such that $E_i \in \mathcal{S}_0^*$ $(i = 1, 2, \cdots)$, where the E_i are pairwise disjoint and $\bigcup_{i=1}^\infty E_i \subset A$, then there exists $i_0 \in N$ such that $\bigcup_{i \geqq i_0} E_i \in \mathcal{N}_n^*$.*

THEOREM 8. *If $\{\mathcal{N}_n\}_{n=0}^\infty$ satisfies the finiteness condition, then $\mathcal{S}^*$ is a σ-ring.*

PROOF. Suppose $E_1 \in \mathcal{S}^*$ and $E_2 \in \mathcal{S}^*$. Hence given $n \in N$ and $A \in \mathcal{N}_0$, there exists $F_1 \in \mathcal{N}_0$ and $F_2 \in \mathcal{N}_0$ such that

$$(E_1 \cap A) \Delta F_1 \in \mathcal{N}_{2n}^* \text{ and } (E_2 \cap A) \Delta F_2 \in \mathcal{N}_{2n}^*.$$

Then $F_1 \cup F_2 \in \mathcal{N}_0$ and

$$[(E_1 \cup E_2) \cap A] \Delta (F_1 \cup F_2) \subset [(E_1 \cap A) \Delta F_1] \cup [(E_2 \cap A) \Delta F_2] \in \mathcal{N}_n^*.$$

Hence $E_1 \cup E_2 \in \mathcal{S}^*$.

Further, $F_1 - F_2 \in \mathcal{N}_0$ and

$$[(E_1 - E_2) \cap A] \Delta (F_1 - F_2) \subset [(E_1 \cap A) \Delta F_1] \cup [(E_2 \cap A) \Delta F_2] \in \mathcal{N}_n^*.$$

Thus $E_1 - E_2 \in \mathcal{S}^*$, and so $\mathcal{S}^*$ is a ring.

Now let $\{E_i\}_{i=1}^\infty$ be a sequence of pairwise disjoint sets from $\mathcal{S}^*$. If we can show that $\bigcup_{i=1}^\infty E_i \in \mathcal{S}^*$, then $\mathcal{S}^*$ will be a σ-ring.

Suppose we are given $A \in \mathcal{N}_0$. Then

$$\left(\bigcup_{i=1}^\infty E_i\right) \cap A = \bigcup_{i=1}^\infty (E_i \cap A) \subset A.$$

Since the $E_i \in \mathcal{S}^*$, we have that $E_i \cap A \in \mathcal{S}_0^* (i = 1, 2, \cdots)$. Hence, by the finiteness condition, given $n \in N$, there exists $i_0 \in N$ such that $\bigcup_{i \geqq i_0} (E_i \cap A) \in \mathcal{N}_{2n}^*$.

Further, since $\mathcal{S}^*$ is a ring, $\bigcup_{i=1}^{i_0 - 1} E_i \in \mathcal{S}^*$. Hence, there is $F \in \mathcal{N}_0$ such that

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$$\left[\left(\bigcup_{i=1}^{i_0-1} E_i\right) \cap A\right] \Delta F \in \mathscr{N}_{2n}^*.$$

Now

$$\left[\left(\bigcup_{i=1}^{\infty} E_i\right) \cap A\right] \Delta F \subset \left\{\left[\left(\bigcup_{i=1}^{i_0-1} E_i\right) \cap A\right] \Delta F\right\} \cup \left\{\left(\bigcup_{i=i_0}^{\infty} E_i\right) \cap A\right\} \in \mathscr{N}_n^*.$$

Hence, $\bigcup_{i=1}^{\infty} E_i \in \mathscr{S}^*$, and the theorem is proved.

PROPOSITION 6. *If* $\{\mathscr{N}_n\}_{n=0}^{\infty}$ *satisfies the finiteness condition, then* $\mathscr{S}(\mathscr{E}) \subset \mathscr{S}^*$.

PROOF. Suppose $E \in \mathscr{E}$. Then given $A \in \mathscr{N}_0$, we have $E \cap A \in \mathscr{N}_0$, by the properties of $\mathscr{N}_0$. Then

$$(E \cap A) \Delta (E \cap A) = \phi \in \mathscr{N}_n^*,$$

for all $n \in N$. So $E \in \mathscr{S}^*$. Hence $\mathscr{E} \subset \mathscr{S}^*$, and since $\mathscr{S}^*$ is a σ-ring, we have $\mathscr{S}(\mathscr{E}) \subset \mathscr{S}^*$.

PROPOSITION 7. $\mathscr{N}^* \subset \mathscr{S}_0^*$.

PROOF. Suppose $E \in \mathscr{N}^*$. Then, given $n \in N$, $E \in \mathscr{N}_n^*$. Now $\phi \in \mathscr{N}_0$ and $E \Delta \phi = E \in \mathscr{N}_n^*$. So $E \in \mathscr{S}_0^*$.

PROPOSITION 8. *If* $E \in \mathscr{S}^*$, *and* $E^c \in \mathscr{H}(\mathscr{E})$, *then* $E^c \in \mathscr{S}^*$.

PROOF. $E \in \mathscr{S}^*$ implies, given $n \in N$ and $A \in \mathscr{N}_0$, there exists $F \in \mathscr{N}_0$ such that $(E \cap A) \Delta F \in \mathscr{N}_n^*$. Then $A - F \in \mathscr{N}_0$, $E^c \in \mathscr{H}(\mathscr{E})$ and

$$(E^c \cap A) \Delta (A - F) \subset (E \cap A) \Delta F \in \mathscr{N}_n^*.$$

Hence $E^c \in \mathscr{S}^*$.

PROPOSITION. 9. $E \in \mathscr{H}(\mathscr{E})$ *and* $E \cap A \in \mathscr{N}^*$, *for all* $A \in \mathscr{N}_0$ *implies*

(i) $E \in \mathscr{S}^*$

and (ii) *either* $E \in \mathscr{N}^*$ *or* $E \in \mathscr{H}(\mathscr{E}) - \mathscr{N}_0^*$.

PROOF. (i) We have that $E \cap A \in \mathscr{N}^* \subset \mathscr{S}_0^*$, for all $A \in \mathscr{N}_0$. Hence $E \in \mathscr{S}^*$.

(ii) Assume $E \notin \mathscr{H}(\mathscr{E}) - \mathscr{N}_0^*$. Hence $E \in \mathscr{N}_0^*$ and, given $n \in N$, there exists $F \in \mathscr{N}_0$ such that $E - F \in \mathscr{N}_{2n}^*$. Also, since $F \in \mathscr{N}_0$, $E \cap F \in \mathscr{N}_{2n}^*$. Hence

$$E = (E - F) \cup (E \cap F) \in \mathscr{N}_n^*,$$

and since this is true for all $n \in N$, $E \in \mathscr{N}^*$.

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4

Throughout this section we suppose that we have a set X, a ring $\mathscr{R}$ of subsets of X, and a measure μ on $\mathscr{R}$. Let μ^* be the induced outer measure of μ on $\mathscr{H}(\mathscr{R})$, the hereditary σ-ring generated by $\mathscr{R}$. We define the sequence $\{\mathscr{N}_n\}_{n=0}^{\infty}$ on $\mathscr{R}$ by

$$\mathscr{N}_n = \left\{E \in \mathscr{R} \,\middle|\, \mu(E) < \frac{1}{n}\right\},$$

for each $n \in N$, and $\mathscr{N}_0 = \{E \in \mathscr{R} \mid \mu(E) < \infty\}$. Note that, for each $n \in N$, $\mathscr{N}_n$ is non-empty. Hence the small system $\{\mathscr{N}_n^*\}_{n=1}^{\infty}$ on $\mathscr{H}(\mathscr{R})$ can be induced from the sequence $\{\mathscr{N}_n\}_{n=1}^{\infty}$ on $\mathscr{R}$, by the method of section 2. We can also construct $\mathscr{N}_0^*, \mathscr{S}_0^*$ and $\mathscr{S}^*$, as in section 3. We will be concerned with the relationships between $\{\mathscr{N}_n^*\}_{n=0}^{\infty}$, $\mathscr{S}_0^*$, $\mathscr{S}^*$ and certain classes of sets, which are defined by means of μ^*.

First of all, note that, in general, it is not true that

$$\mathscr{N}_n^* = \left\{E \in \mathscr{H}(\mathscr{R}) \,\middle|\, \mu^*(E) < \frac{1}{n}\right\},$$

for each $n \in N$. For let X be any countably infinite set $\{a_i\}_{i=1}^{\infty}$. Let $\mathscr{R}$ be the ring of all finite subsets of X. We specify a measure μ on $\mathscr{R}$, by assigning $\mu(a_1) = 0.21$, $\mu(a_2) = 0.26$, $\mu(a_i) = 0$, $(i \neq 1,2)$. Then $\mu^*(X) < \frac{1}{2}$, but $X \notin \mathscr{N}_2^*$. However, the following results do hold.

PROPOSITION 10. *For each* $n \in N$, $\mathscr{N}_n^* \subset \{E \in \mathscr{H}(\mathscr{R}) \mid \mu^*(E) < 1/n\}$.

PROOF. Given $n \in N$, let $E \in \mathscr{N}_n^*$. Hence there exists a class $\{E_i\}_{i \in I}$, $I \neq \phi$, $I \subset N$, such that $E_i \in \mathscr{N}_{k_i}$, some $k_i \in N$ $(i \in I)$, $\bigcup_{i \in I} E_i \supset E$ and $\sum_{i \in I} 1/k_i \leqq 1/n$.
Hence $\mu(E_i) < 1/k_i$, for each $i \in I$. If $N - I \neq \phi$, put $E_i = \phi$, for $i \in N-I$. So $\sum_{i=1}^{\infty} \mu(E_i) < \sum_{i \in I} 1/k_i \leqq 1/n$. That is $\mu^*(E) < 1/n$, and the proposition is proved.

PROPOSITION 11. *For each* $n \in N$, $\mathscr{N}_n^* \supset \{E \in \mathscr{H}(\mathscr{R}) \mid \mu^*(E) < 1/2n\}$.

PROOF. Given $n \in N$, let $E \in \mathscr{H}(\mathscr{R})$ with $\mu^*(E) < 1/2n$. Hence there exists a sequence $\{E_i\}_{i=1}^{\infty}$ such that $E_i \in \mathscr{R}$ $(i = 1,2,\cdots)$, $\bigcup_{i=1}^{\infty} E_i \supset E$ and $\sum_{i=1}^{\infty} \mu(E_i) < 1/2n$.

Now, for each $\mu(E_i) > 0$, define $1/k_i$ as the smallest number of the form $1/p$ (where p is a positive integer) strictly greater than $\mu(E_i)$. Then, by the definition of $1/k_i$, we have $1/2k_i \leqq \mu(E_i) < 1/k_i$, for $i \in M = \{i \in N \mid \mu(E_i) > 0\}$. Hence

$$\frac{1}{2} \sum_{i \in M} \frac{1}{k_i} \leqq \sum_{i \in M} \mu(E_i) = \sum_{i=1}^{\infty} \mu(E_i) < \frac{1}{2n}.$$

That is, $\sum_{i \in M} 1/k_i < 1/n$.

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Now, if $N - M \neq \phi$, we choose a set $\{k_i\}_{i \in N-M}$ of positive integers such that

$$\sum_{i \in N-M} \frac{1}{k_i} \leqq \frac{1}{n} - \sum_{i \in M} \frac{1}{k_i}.$$

Hence we have a sequence $\{E_i\}_{i=1}^{\infty}$ such that

$$E_i \in \mathcal{N}_{k_i} \ (i = 1, 2, \cdots), \ \bigcup_{i=1}^{\infty} E_i \supset E \ \text{ and } \ \sum_{i=1}^{\infty} \frac{1}{k_i} \leqq \frac{1}{n}.$$

So $E \in \mathcal{N}_n^*$, and the proposition is proved.

THEOREM 9. *If* $\mathcal{N}^* = \bigcap_{n=1}^{\infty} \mathcal{N}_n^*$, *then* $\mathcal{N}^* = \{E \in \mathscr{H}(\mathscr{R}) \mid \mu^*(E) = 0\}$.

PROOF. From propositions 10 and 11, we have, for each $n \in N$,

$$\left\{E \in \mathscr{H}(\mathscr{R}) \,\middle|\, \mu^*(E) < \frac{1}{n}\right\} \supset \mathcal{N}_n^* \supset \left\{E \in \mathscr{H}(\mathscr{R}) \,\middle|\, \mu^*(E) < \frac{1}{2n}\right\}.$$

Hence

$$\bigcap_{n=1}^{\infty} \left\{E \in \mathscr{H}(\mathscr{R}) \,\middle|\, \mu^*(E) < \frac{1}{n}\right\} \supset \mathcal{N}^* \supset \bigcap_{n=1}^{\infty} \left\{E \in \mathscr{H}(\mathscr{R}) \,\middle|\, \mu^*(E) < \frac{1}{2n}\right\}.$$

That is, $\mathcal{N}^* = \{E \in \mathscr{H}(\mathscr{R}) \mid \mu^*(E) = 0\}$.

For a certain class of measures, including Lebesgue measure, it is true that

$$\mathcal{N}_n^* = \left\{E \in \mathscr{H}(\mathscr{R}) \,\middle|\, \mu^*(E) < \frac{1}{n}\right\},$$

for each $n \in N$.

DEFINITION. *If* μ *is a measure on a ring* $\mathscr{R}$, *a set* $E \in \mathscr{R}$ *of positive measure is called an atom if, given* $F \in \mathscr{R}$ *such that* $F \subset E$, *then either* $\mu(F) = 0$ *or* $\mu(E - F) = 0$.

LEMMA 2. (c.f. [5], p. 272). *Let* $\mathscr{R}$ *be a ring, and* μ *a measure on* $\mathscr{R}$. *If* $E \in \mathscr{R}$ *is of finite positive measure and* E *does not contain any atoms, then for any real number* β, *such that* $0 < \beta < \mu(E)$, *there exists a subset* F *of* E *such that* $F \in \mathscr{R}$ *and* $\mu(F) = \beta$.

REMARK 9. We see from lemma 2, that for $\alpha_i \geqq 0$ $(i = 1, 2, \cdots, n)$ and $\sum_{i=1}^{n} \alpha_i = \mu(E)$, there exist disjoint sets $E_i \in \mathscr{R}$, such that $\bigcup_{i=1}^{n} E_i = E$ and $\mu(E_i) = \alpha_i$.

THEOREM 10. *If* μ *has no atoms, then* $\mathcal{N}_n^* = \{E \in \mathscr{H}(\mathscr{R}) \mid \mu^*(E) < 1/n\}$, *for each* $n \in N$.

PROOF. In view of proposition 10, we need only show that

$$\mathcal{N}_n^* \supset \left\{E \in \mathscr{H}(\mathscr{R}) \,\middle|\, \mu^*(E) < \frac{1}{n}\right\},$$

for each $n \in N$.

Given $n \in N$, let $E \in \mathscr{H}(\mathscr{R})$ with $\mu^*(E) < 1/n$. Then there exists a sequence $\{E_i\}_{i=1}^{\infty}$ such that $E_i \in \mathscr{R}$ $(i = 1,2,\cdots)$, $\bigcup_{i=1}^{\infty} E_i \supset E$ and $\sum_{i=1}^{\infty} \mu(E_i) < 1/n$. Now, for each i, choose p_i/q_i (p_i and q_i are positive integers) such that $\mu(E_i) < p_i/q_i$ and $\sum_{i=1}^{\infty} \mu(E_i) < \sum_{i=1}^{\infty} p_i/q_i \leqq 1/n$.

This can always be done for the following reasons. Choose $\varepsilon > 0$, such that $\varepsilon \leqq 1/n - \sum_{i=1}^{\infty} \mu(E_i)$, and then choose p_i/q_i such that

$$\mu(E_i) < \frac{p_i}{q_i} \leqq \mu(E_i) + \frac{\varepsilon}{2^i} \quad (i = 1,2,\cdots).$$

Then

$$\sum_{i=1}^{\infty} \mu(E_i) < \sum_{i=1}^{\infty} \frac{p_i}{q_i} \leqq \sum_{i=1}^{\infty} \mu(E_i) + \varepsilon \leqq \frac{1}{n}.$$

For each i, from remark 9, we can choose p_i disjoint subsets $\{E_i^j\}_{j=1}^{p_i}$ of E_i such that $\bigcup_{j=1}^{p_i} E_i^j = E_i$ and $\mu(E_i^j) = \mu(E_i)/p_i$. Hence

$$\mu(E_i^j) = \frac{\mu(E_i)}{p_i} < \frac{p_i}{q_i} \cdot \frac{1}{p_i} = \frac{1}{q_i}, \text{ for } j = 1,2,\cdots,p_i.$$

That is, for each i, $E_i^j \in \mathscr{N}_{q_i}$, for $j = 1,2,\cdots,p_i$. Further,

$$\sum_{i=1}^{\infty} \sum_{j=1}^{p_i} \frac{1}{q_i} = \sum_{i=1}^{\infty} \frac{p_i}{q_i} \leqq \frac{1}{n}.$$

Hence the class $\{E_i^j\}_{i=1}^{\infty}{}_{j=1}^{p_i}$ forms an n-cover for E. Hence $E \in \mathscr{N}_n^*$, and the theorem is proved.

We remind the reader of the following measure-theoretic result. If $E \in \mathscr{H}(\mathscr{R})$, then $\mu^*(E) < \infty$, if and only if there exists $F \in \mathscr{R}$ such that $\mu(F) < \infty$ and $\mu^*(E - F) < \varepsilon$. Then with this and propositions 10 and 11 in mind, it is easy to see that

$$\mathscr{N}_0^* = \{E \in \mathscr{H}(\mathscr{R}) \,|\, \mu^*(E) < \infty\}.$$

Proposition 3 is then the generalisation of the result: 'μ is σ-finite, implies μ^* is σ-finite'.

Comparing propositions 10 and 11 and propositions 4 and 5, it is easy to see that

$$\mathscr{S}_0^* = \{E \in \mathscr{H}(\mathscr{R}) \,|\, \mu^*(E) < \infty \text{ and } E \text{ is } \mu^*\text{-measurable}\}$$

and

$$\mathscr{S}^* = \{E \in \mathscr{H}(\mathscr{R}) \,|\, E \text{ is } \mu^*\text{-measurable}\}.$$

Finally we have the following result.

PROPOSITION 12. *$\{\mathscr{N}_n\}_{n=0}^{\infty}$ satisfies the finiteness condition.*

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PROOF. Let $A \in \mathscr{R}$, where $\mu(A) < \infty$ and $\{E_i\}_{i=1}^{\infty}$ be a pairwise disjoint sequence of sets in $\mathscr{S}_0^*$, such that $\bigcup_{i=1}^{\infty} E_i \subset A$. Then, for each i, E_i is μ^*-measurable and

$$\sum_{i=1}^{\infty} \mu^*(E_i) = \mu^*\left(\bigcup_{i=1}^{\infty} E_i\right) \leqq \mu(A) < \infty .$$

Hence, we can choose $i_0 \in N$ such that

$$\sum_{i=i_0+1}^{\infty} \mu^*(E_i) < \frac{1}{2n}.$$

Then for $i \geqq i_0$, we have $E_i \in \mathscr{N}_{k_i}^*$, where $\sum_{i \geqq i_0} 1/k_i \leqq 1/n$. Hence $\bigcup_{i \geqq i_0} E_i \in \mathscr{N}_n^*$, and the proposition is proved.

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References

[1] P. R. Halmos, *Measure Theory* (Van Nostrand, New York, 1950).
[2] B. Riečan, 'Abstract Formulation of Some Theorems of Measure Theory', *Mat. -Fyz. Časopis Slovan Akad. Vied* 16,3, (1966), 268–273.
[3] V. Ficker,'Dominated Classes and Related Questions', *Acta Fac. Rerum Natur. Univ. Comenian* 10, 7, (1966), 3–18.
[4] V. Ficker, 'An Abstract Formulation of the Lebesgue Decomposition Theorem', *J. Aust. Math. Soc.* 12 (1971), 101–105.
[5] A. Zaanen, *Integration* (North-Holland, Amsterdam, 1967).
[6] L. Sucheston, 'A Note on Conservative Transformations and the Recurrence Theorem', *Amer. J. Math.* 79 (1957, 444–447.
[7] T. Neubrunn, 'Remark on Absolute Continuity of Measures', (Russian) *Mat. -Fyz. Časopis Slovan. Akad. Vied* 16 (1966), 21–30.

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