Adv. Appl. Prob. **20**, 241–243 (1988)
Printed in N. Ireland

ASYMPTOTIC PERFORMANCE OF A MULTISTATE COHERENT SYSTEM

SRINIVAS IYER, **Indian Statistical Institute, New Delhi*

Abstract

An expression for the asymptotic or steady-state performance function is derived for a multistate coherent system when each component changes states in time according to a semi-Markov process, the stochastic processes being mutually independent. This generalizes the expression for system availability of a binary coherent system when the components are governed by mutually independent alternating renewal processes.

SEMI-MARKOV PROCESS; ASYMPTOTIC PERFORMANCE

1. Introduction

El-Neweihi et al. [3] consider a multistate coherent system which is a natural generalization of a binary coherent system. Here a dynamic version of the system is considered and an expression for the asymptotic performance function is derived when each component changes states in time according to a semi-Markov process, the stochastic processes being mutually independent. The result generalizes that for system availability in [1] where the states of the components are governed by mutually independent alternating renewal processes.

In Section 2 the notation and description of a multistate coherent system is given, along with a definition of the performance function of the system. In Section 3 the dynamic semi-Markov model is defined and an expression for the asymptotic performance function derived, while in Section 4 the steady-state expression is derived for a special case of a multistate coherent system due to Barlow and Wu [2].

2. Notation and description of a multistate coherent system

The notation and description of the system is as in [3]. For each component and for the system itself we can distinguish among $M+1$ states representing successive levels of performance ranging from perfect functioning (level M) down to complete failure (level 0). For component i, x_i denotes the corresponding state or performance level, $i=1,\dots,n$; the vector $\boldsymbol{x}=(x_1,\dots,x_n)$ denotes the vector of states of components $1,\dots,n$. We assume that the state Φ of the system is a deterministic function of the states $x_1,\dots,x_n$ of the components. Thus $\Phi=\phi(\boldsymbol{x})$, where $\boldsymbol{x}$ takes values in S^n, $S=\{0,1,\dots,M\}$ and Φ takes value in S.

The multistate coherent system (MCS) considered in [3] is a natural generalization of a binary coherent system and is defined there as follows. Let

$$(j_i,\boldsymbol{x})=(x_1,\dots,x_{i-1},j,x_{i+1},\dots,x_n)\quad\text{where}\quad j=0,1,\dots,M$$
$$(.,\boldsymbol{x})=(x_1,\dots,x_{i-1},\dots,x_{i+1},\dots,x_n)\quad\text{and}\quad \boldsymbol{j}=(j,j,\dots,j).$$

Received 21 May 1987; revision received 28 September 1987.

* Postal address: Indian Statistical Institute, Statistical Quality Control and Operations Research Unit, 7 S.J.S. Sansanwal Marg, New Delhi 110016, India.

A system of components is said to be an MCS if its structure function Φ satisfies:

(i) Φ is increasing (in each argument),

(ii) For level j of component i, there exists a vector $(\cdot_i, \boldsymbol{x})$ such that $\Phi(j_i, \boldsymbol{x}) = j$ while $\Phi(l_i, \boldsymbol{x}) \neq j$ for $l \neq j$, $i = 1, \dots, n$ and $j = 0, 1, \dots, M$.

(iii) $\Phi(\boldsymbol{j}) = j$ for $j = 0, 1, \dots, M$.

Condition (ii) may be replaced by either of the two weaker coherency conditions mentioned in Griffith [4] without affecting any of the results to follow.

In [3] the performance function of the system is defined which is a generalization of the concept of reliability for a binary system.

Let X_i denote the random state of component $i = 1, \dots, n$, with

$$P[X_i = j] = P_{ij}, \quad P[X_i \leqq j] = P_{i(j)}, \quad P[X_i \geqq j] = Q_{i(j)},$$

where $j = 0, 1, \dots, M$. P_i represents the performance distribution of component i. Let $\boldsymbol{X} = (X_1, \dots, X_n)$ be the random vector representing the states of components $1, \dots, n$ where $X_1, \dots, X_n$ are assumed to be statistically mutually independent. Then $\Phi(\boldsymbol{X})$ is the random variable representing the system state of the MCS having structure function Φ, with

$$P[\Phi(\boldsymbol{X}) = j] = P_j, \qquad P[\Phi(\boldsymbol{X}) \leqq j] = P(j), \qquad j = 0, 1, \dots, M.$$

P represents the performance distribution of the system.

In [3] the performance function h of the system is defined as

$$h = h_{\boldsymbol{p}}(\boldsymbol{p}_1, \dots, \boldsymbol{p}_n) = E[\Phi(\boldsymbol{x})]$$

where $\boldsymbol{p}_i = (p_{i0}, \dots, p_{iM})$ and $\boldsymbol{p} = (\boldsymbol{p}_1, \dots, \boldsymbol{p}_n)$, $i = 1, \dots, n$.

3. A dynamic semi-Markov model

We consider a dynamic version of the system and study the asymptotic performance function h.

Each component changes states in time according to a semi-Markov process (SMP), the stochastic processes being mutually independent. The SMP for component i has parameters $\{\Pi_j^i, \mu_j^i, \mu_{jj}^i, j = 0, 1, \dots, M\}$ (see [6]), where Π_j^i is the steady-state probability of state j for the embedded Markov chain of SMP^i, μ_j^i is the mean time in state j of SMP^i, and μ_{jj}^i is the mean-cycle time for state j of SMP^i.

Let X_i^t denote the state of component i at time t with $p_{ij}^t = \Pr[X_i^t = j]$, $i = 1, \dots, n$; $j = 0, 1, \dots, M$. Then [6], $p_{ij}^t \to p_{ij}^*$, as $t \to \infty$, where p_{ij}^* is the steady-state probability of being in state j for component i and is given by

$$p_{ij}^* = \frac{\mu_j^i}{\mu_{jj}^i} = \frac{\Pi_j^i \mu_j^i}{\sum_{k=0}^{M} \Pi_k^i \mu_k^i}. \tag{1}$$

For a continuous-time Markov chain these could be calculated from the rate or balance equations.

Then since [3], $h(\boldsymbol{p})$ is continuous (in fact, differentiable) with respect to p, see [3], $h(\boldsymbol{p}^t) \to h(\boldsymbol{p}^*)$, as $t \to \infty$, where $\boldsymbol{p}^* = (\boldsymbol{p}_1^*, \dots, \boldsymbol{p}_n^*)$ is the vector of steady-state probabilities

$$\boldsymbol{p}_k^* = (p_{k0}^*, \dots, p_{kM}^*), \qquad k = 1, \dots, n.$$

Thus the asymptotic system performance function, $h^*(\boldsymbol{p})$ is given by

$$h^*(\boldsymbol{p}) = h(\boldsymbol{p}^*) = h\left(\frac{\mu_0^1}{\mu_{00}^1}, \frac{\mu_1^1}{\mu_{11}^1}, \dots, \frac{\mu_M^1}{\mu_{MM}^1}, \dots, \frac{\mu_0^n}{\mu_{00}^n}, \frac{\mu_1^n}{\mu_{11}^n}, \dots, \frac{\mu_M^n}{\mu_{MM}^n}\right) \tag{2}$$

where each $\mu_{jj}^i = \sum_k \Pi_k^i \mu_k^i / \Pi_j^i$, $i = 1, \dots, n$, $j = 0, 1, \dots, M$. This is a generalization of the result mentioned in [1] for the system availability for a coherent binary system of n

https://doi.org/10.2307/1427281 Published online by Cambridge University Press

components with structure function Φ and reliability h_ϕ governed by n mutually independent ARPs, namely, system availability

$$h' = h\left(\frac{\mu_1}{\mu_{1+v_1}}, \dots, \frac{\mu_n}{\mu_{n+v_n}}\right) \tag{3}$$

where μ_i is the mean time in state 0 ('on state') for component i and v_i is the mean time in state 1 ('off state') for component i, $i = 1, \dots, n$.

4. The Barlow–Wu model

As a special case we consider the MCS studied in [2]. Here we have p min-path sets $\{P_1, \dots, P_p\}$ defined as for a coherent binary system. The system state function $\Phi(\boldsymbol{x})$ for the MCS is defined by

$$\Phi(\boldsymbol{x}) = \max_{1 \leqq r \leqq p} \min_{i \in P_r} x_i. \tag{4}$$

Let Ψ represent the coherent structure function (as in the binary case) corresponding to the min-path sets $\{P_1, \dots, P_p\}$, and let h_Ψ represent the reliability polynomial (as in the binary case) corresponding to Ψ. Then, as shown in [2],

$$P[\Phi(\boldsymbol{x}) \geqq j] = h_\Psi(\boldsymbol{Q}_j), \qquad \boldsymbol{Q}_j = (Q_{1(j)}, \dots, Q_{n(j)}). \tag{5}$$

Hence the performance function h_Φ, or simply h for the MCS as defined in [2] is in this case given by

$$h(\boldsymbol{p}) = \sum_{j=1}^{M} P[\Phi(\boldsymbol{X}) \geqq j] = \sum_{j=1}^{M} h_\Psi(\boldsymbol{Q}_j). \tag{6}$$

For the dynamic version of the model in [2], since, as $t \to \infty$,

$$\boldsymbol{Q}_j^t = \left(\sum_{k=j}^{M} p_{1k}^t, \dots, \sum_{k=j}^{M} p_{nk}^t\right) \to \left(\sum_{k=j}^{M} p_{1k}^*, \dots, \sum_{k=j}^{M} p_{nk}^*\right)$$

the asymptotic system performance function, $h(\boldsymbol{p})$ is in this case given by

$$h^*(\boldsymbol{p}) = \sum_{j=1}^{M} h_\Psi\left(\sum_{k=j}^{M} p_{1k}^*, \dots, \sum_{k=j}^{M} p_{nk}^*\right)$$

or

$$\sum_{j=1}^{M} h_{\Psi_j}\left(\sum_{k=j}^{M} p_{1k}^*, \dots, \sum_{k=j}^{M} p_{nk}^*\right)$$

if ψ varies with j in the more general model of Natvig [5].

References

[1] Barlow, R. E. and Proschan, F. (1981) *Statistical Theory of Reliability and Life Testing.* To Begin With, Silver Spring, Md.

[2] Barlow, R. E. and Wu, A. S. (1978) Coherent systems with multistate components. *Math. Operat. Res.* **3,** 275–281.

[3] El-Neweihi, E., Proschan, F. and Sethuraman, J. (1978) Multistate Coherent Systems. *J. Appl. Prob.* **15,** 675–688.

[4] Griffith, W. S. (1980) Multistate reliability models. *J. Appl. Prob.* **17,** 735–744.

[5] Natvig, B. (1982) Two suggestions of how to define a multistate coherent system. *Adv. Appl. Prob.* **14,** 434–455.

[6] Ross, S. M. (1982) *Stochastic Processes.* Wiley, New York.