Ergod. Th. & Dynam. Sys., page 1 of 20 © The Author(s), 2025. Published by Cambridge University Press.
doi:10.1017/etds.2025.10254

Some ergodic theorems involving Omega function and their applications

RONGZHONG XIAO

University of Science and Technology of China School of Mathematical Sciences, China (e-mail: xiaorz@mail.ustc.edu.cn)

(*Received* 28 *December* 2024 *and accepted in revised form* 15 *October* 2025)

Abstract. In this paper, we build some ergodic theorems involving the function Ω, where $\Omega(n)$ denotes the number of prime factors of a natural number n counted with multiplicities. As a combinatorial application, it is shown that for any $k \in \mathbb{N}$ and every $A \subset \mathbb{N}$ with positive upper Banach density, there are $a, d \in \mathbb{N}$ such that $a, a + d, \ldots, a + kd, a + \Omega(d) \in A$.

Key words: Ergodic theorems, multiple recurrence, Omega function, polynomial Szemerédi theorem
2020 Mathematics Subject Classification: 37A30 (Primary); 37A44 (Secondary)

1. *Introduction*

Let $\Omega(n)$ denote the number of prime factors of a natural number n counted with multiplicities. In multiplicative number theory, a central topic is to study the asymptotic distribution of the values of $\Omega(n)$.

In 2022, Bergelson and Richter [2] gave an asymptotic characterization of $\Omega(n)$ from a dynamical point of view. By *topological dynamical system*, we mean a pair (X, T), where X is a compact metric space and $T : X \to X$ is a homeomorphism. We say that (X, T) is *uniquely ergodic* if there is only one T-invariant Borel probability measure on X.

THEOREM 1.1. [2, Theorem A] *Let (X, T) be a uniquely ergodic topological dynamical system with the unique T-invariant Borel probability measure μ. Then, for any $f \in C(X), x \in X$,*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} f(T^{\Omega(n)}x) = \int_X f \, d\mu.$$

Later, Loyd [19] built an analogue of Theorem 1.1 in the sense of norm convergence. By *measure-preserving system*, we mean a tuple $(X, \mathcal{X}, \mu, T)$, where $(X, \mathcal{X}, \mu)$ is a

Lebesgue space (for its definition, see [14, Definition 2.12]) and $T : X \to X$ is an invertible measure-preserving transformation. We say that $(X, \mathcal{X}, \mu, T)$ is *ergodic* if for any $A \in \mathcal{X}$ with $\mu(A \Delta T^{-1} A) = 0$, then $\mu(A) = 0$ or 1.

THEOREM 1.2. [19, Theorem 2.5] *Let $(X, \mathcal{X}, \mu, T)$ be an ergodic measure-preserving system. Then, for any $f \in L^2(\mu)$,*

$$\lim_{N\to\infty} \left\| \frac{1}{N} \sum_{n=1}^{N} f(T^{\Omega(n)}x) - \int_X f\, d\mu \right\|_{L^2(\mu)} = 0.$$

In 2024, Charamaras [5] extended Loyd's result to the double ergodic averages case.

THEOREM 1.3. [5, Corollary 1.33] *Let T and S be two invertible measure-preserving transformations acting on the Lebesgue space $(X, \mathcal{X}, \mu)$ such that $(X, \mathcal{X}, \mu, T)$ and $(X, \mathcal{X}, \mu, S)$ are ergodic. Then, for any $f, g \in L^2(\mu)$,*

$$\lim_{N\to\infty} \left\| \frac{1}{N} \sum_{n=1}^{N} f(T^n x) g(S^{\Omega(n)}x) - \int_X f\, d\mu \int_X g\, d\mu \right\|_{L^2(\mu)} = 0.$$

For any $A \subset \mathbb{Z}^k$, we define $d^*(A)$ by letting

$$d^*(A) = \sup_{\Phi} \limsup_{N\to\infty} \frac{|A \cap \Phi_N|}{|\Phi_N|},$$

where the supremum is taken over all Følner sequences $\Phi = \{\Phi_N\}_{N\in\mathbb{N}}$ in $\mathbb{Z}^k$. (A Følner sequence of $\mathbb{Z}^k$ is a sequence $\{\Phi_N\}_{N\in\mathbb{N}}$ of non-empty finite subsets of $\mathbb{Z}^k$ such that for each $h \in \mathbb{Z}^k$, $\lim_{N\to\infty}(|(\Phi_N + h)\Delta\Phi_N|/|\Phi_N|) = 0$.) If $d^*(A) > 0$, we say that A has *positive upper Banach density*. As a combinatorial application of Theorem 1.3, Charamaras [5, Corollary 1.37] showed that for any $E \subset \mathbb{N}$ with positive upper Banach density, there exist $m, n \in \mathbb{N}$ such that $m, m + n, m + \Omega(n) \in E$.

Motivated by the above results, in this paper, we consider the following ergodic averages:

$$\frac{1}{N} \sum_{n=1}^{N} w(n) \prod_{i=1}^{k} f_i(T_i^{P_i(n)}x) \cdot f_{k+1}(S^{\Omega(n)}x), \tag{1.1}$$

where $S, T_1, \ldots, T_k$ is a family of invertible measure-preserving transformations acting on the Lebesgue space $(X, \mathcal{X}, \mu)$, $P_1, \ldots, P_k \in \mathbb{Z}[n]$, $f_1, \ldots, f_{k+1} \in L^\infty(\mu)$, and $w : \mathbb{N} \to \mathbb{C}$ is a sequence.

When $T_1, \ldots, T_k$ generate a nilpotent group, and f_{k+1} and w are constant, the norm convergence of (1.1) was proved by Walsh in [20].

First, we extend Theorem 1.1 to a weighted form.

THEOREM 1.4. *Let $(X, \mathcal{X}, \mu, T)$ be a measure-preserving system. Then, for any $f \in L^1(\mu)$, there is a full measure subset X_f of X such that for any $x \in X_f$, any uniquely ergodic topological dynamical system (Y, S) with the unique S-invariant Borel probability*

https://doi.org/10.1017/etds.2025.10254 Published online by Cambridge University Press

measure ν*, any* $g \in C(Y)$ *and any* $y \in Y$*,*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} f(T^n x) g(S^{\Omega(n)} y) = f^*(x) \int_Y g \, d\nu,$$

where f^* *is the conditional expectation of* f *with respect to the sub-*σ*-algebra* $\mathcal{I}(T)$ *of* $\mathcal{X}$ *generated by all T-invariant sets (for the definition of conditional expectation, see §2.2).*

Remark 1.5.

(a) Let $\mathbb{P}$ be the set of all prime numbers. Given $k \in \mathbb{N}$, let $P_1, \ldots, P_k \in \mathbb{Z}[n]$. Assume that $(X, \mathcal{X}, \mu, T)$ satisfies the following property: there is $\mathcal{P} \subset \mathbb{P}$ with positive relative density such that for any distinct $p, q \in \mathcal{P} \cup \{1\}$ and any $g_1, \ldots, g_{2k} \in L^\infty(\mu)$, the limit

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \prod_{i=1}^{k} T^{P_i(pn)} g_i \cdot \prod_{j=1}^{k} T^{P_j(qn)} g_{k+j}$$

exists for μ-almost every (a.e.) $x \in X$. (We say that $\mathcal{P} \subset \mathbb{P}$ has positive relative density if $\lim_{N\to\infty}(|\{1 \le n \le N : n \in \mathcal{P}\}|/|\{1 \le n \le N : n \in \mathbb{P}\}|) > 0$.) Then, by the method used in the proof of Theorem 1.4, and Theorems 2.1, 2.2, 5.1, we have that for any $f_1, \ldots, f_k \in L^\infty(\mu)$, there are a full measure subset $X_{f_1,\ldots,f_k}$ of X and $f^* \in L^\infty(\mu)$ such that for any $x \in X_{f_1,\ldots,f_k}$, any uniquely ergodic topological dynamical system (Y, S) with the unique S-invariant Borel probability measure ν, any $g \in C(Y)$ and any $y \in Y$,

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \prod_{i=1}^{k} f_i(T^{P_i(n)} x) \cdot g(S^{\Omega(n)} y) = f^*(x) \int_Y g \, d\nu.$$

(b) By [19, Theorem 1.2], in Theorem 1.4, when there is no continuous restriction for g, it may fail even for ν-a.e. $y \in Y$.

Next, we introduce some applications of Theorem 1.4. For any $x \in \mathbb{R}$, $[x]$ is the largest integer such that $0 \le x - [x] < 1$. When n is a non-positive integer, we set $\Omega(n) = 0$. After applying Theorem 1.4 to *rotations on tori*, we can get the following corollary.

COROLLARY 1.6. *Let* $\alpha > 0$, $\beta \in \mathbb{R}$. *Let* (Y, S) *be a uniquely ergodic topological dynamical system with the unique S-invariant Borel probability measure* ν*. Then, for any* $g \in C(Y)$ *and any* $y \in Y$*,*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} g(S^{\Omega([\alpha n+\beta])} y) = \int_Y g \, d\nu.$$

Remark 1.7.

(a) Let $k \in \mathbb{N}$ and $0 \le r < k$. By applying Corollary 1.6 to $(\mathbb{Z}_k, S)$, $1_{\{r\}}$ and $y = 0$, where $S : x \mapsto x + 1$,

$$\lim_{N\to\infty} \frac{|\{1 \le n \le N : k | (\Omega([\alpha n + \beta]) - r)\}|}{N} = \frac{1}{k}.$$

https://doi.org/10.1017/etds.2025.10254 Published online by Cambridge University Press

(b) By applying Corollary 1.6 to $(\mathbb{Z}_2, S)$, $g : \mathbb{Z}_2 \to \{1, -1\}, 0 \mapsto 1, 1 \mapsto -1$ and $y = 0$, where $S : x \mapsto x + 1$,

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \lambda([\alpha n + \beta]) = 0,$$

where λ is the Liouville function, that is, $\lambda : \mathbb{N} \to \{1, -1\}, n \mapsto (-1)^{\Omega(n)}$.

We go on to apply Theorem 1.4 to *unipotent affine transformations* (see [13, pp. 67–69]) to get the following weighted ergodic theorem, which can be viewed as an analogue of Theorem 2.1.

COROLLARY 1.8. *Let $k \in \mathbb{N}$ and $\alpha \in \mathbb{R}\backslash\mathbb{Q}$. Let m be the Haar measure on $\mathbb{T}$. Then, for any $f \in L^1(m)$, there is $A_f \subset \mathbb{R}^k$ with zero Lebesgue measure such that for any $(x_0, \ldots, x_{k-1}) \notin A_f$, any uniquely ergodic topological dynamical system (Y, S) with the unique S-invariant Borel probability measure ν, any $g \in C(Y)$ and any $y \in Y$,*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} f(\alpha n^k + x_{k-1}n^{k-1} + \cdots + x_1 n + x_0) g(S^{\Omega(n)} y) = \int_{\mathbb{T}} f\, dm \int_Y g\, d\nu.$$

Now, let us come back to (1.1). For some of its cases, we can build related mean ergodic theorems, which can be viewed as extensions of Theorem 1.3.

THEOREM 1.9. *Given $k \in \mathbb{N}$, let $P_1, \ldots, P_k$ be pairwise independent polynomials with integer coefficients and any one of them is not of the form $cn^d + b$. ($P_1, \ldots, P_k$ are pairwise independent if for any $1 \le i < j \le k$, there are no non-zero $x, y \in \mathbb{Z}$ such that $xP_i + yP_j$ is a constant.) Let $S, T_1, \ldots, T_k$ be a family of invertible measure-preserving transformations acting on the Lebesgue space $(X, \mathcal{X}, \mu)$. Assume that $T_1, \ldots, T_k$ are commuting. Then, for any $f_1, \ldots, f_{k+1} \in L^\infty(\mu)$, the limit*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \prod_{i=1}^{k} f_i(T_i^{P_i(n)} x) \cdot f_{k+1}(S^{\Omega(n)} x)$$

exists in $L^2(\mu)$.

The reason why we make some restrictions on the polynomials $P_1, \ldots, P_k$ is to use the pronilfactors to control the L^2-norm of (1.1). If $T_1 = \cdots = T_k$, then we can leave out the restrictions for polynomials.

THEOREM 1.10. *Given $k \in \mathbb{N}$, let $P_1, \ldots, P_k \in \mathbb{Z}[n]$. Let T, S be two invertible measure-preserving transformations acting on the Lebesgue space $(X, \mathcal{X}, \mu)$. Then, for any $f_1, \ldots, f_{k+1} \in L^\infty(\mu)$, the limit*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \prod_{i=1}^{k} f_i(T^{P_i(n)} x) \cdot f_{k+1}(S^{\Omega(n)} x)$$

exists in $L^2(\mu)$.

Lastly, we apply Theorems 1.9 and 1.10 to search for some additive structures in the sets with positive upper Banach density. Finally, we can get the following results.

PROPOSITION 1.11. *Given $k \in \mathbb{N}$, let $P_1, \ldots, P_k$ be pairwise independent polynomials with integer coefficients and zero constant terms and any one of them is not of the form cn^d. Then, for every $A \subset \mathbb{N}^{k+1}$ with positive upper Banach density, there are $a \in \mathbb{N}^{k+1}$, $d \in \mathbb{N}$ such that*

$$a, a + P_1(d)\vec{e}_1, \ldots, a + P_k(d)\vec{e}_k, a + \Omega(d)\vec{e}_{k+1} \in A,$$

where $\{\vec{e}_1, \ldots, \vec{e}_{k+1}\}$ is the standard basis of $\mathbb{R}^{k+1}$.

PROPOSITION 1.12. *Given $k \in \mathbb{N}$, let $P_1, \ldots, P_k \in \mathbb{Z}[n]$ with zero constant terms, then for every $A \subset \mathbb{N}^2$ with positive upper Banach density, there are $(x, y) \in \mathbb{N}^2$, $d \in \mathbb{N}$ such that*

$$(x, y), (x + P_1(d), y), \ldots, (x + P_k(d), y), (x, y + \Omega(d)) \in A.$$

PROPOSITION 1.13. *Given $k \in \mathbb{N}$, let $P_1, \ldots, P_k \in \mathbb{Z}[n]$ with zero constant terms, then for every $A \subset \mathbb{N}$ with positive upper Banach density, there are $a, d \in \mathbb{N}$ such that*

$$a, a + P_1(d), \ldots, a + P_k(d), a + \Omega(d) \in A.$$

Conventionally, to prove the above results, we use Furstenberg's correspondence principle to transfer them into some multiple recurrence results, which can be deduced from Theorems 1.9, 1.10 and the polynomial Szemerédi theorem [**1**, Theorem A].

In fact, by our method and [**2**, Theorem B and Corollary 1.27], Omega function in all results built by us in the paper can be replaced with any completely additive function $a : \mathbb{N} \to \mathbb{N} \cup \{0\}$ satisfying the following property:

For any uniquely ergodic additive topological dynamical system (X, T) with the unique T-invariant Borel probability measure μ, $(X, (T^{a(n)})_{n\in\mathbb{N}})$ is an aperiodic, finitely generated and strongly uniquely ergodic multiplicative topological dynamical system with the unique Borel probability measure μ that pretends to be invariant under $(T^{a(n)})_{n\in\mathbb{N}}$. (For the definitions of aperiodic, finitely generated and strongly uniquely ergodic multiplicative topological dynamical system and Borel probability measures that pretend to be invariant, see [**2**, Definitions 1.11, 1.13, 1.22].)

1.1. *Organization of the paper.* In §2, we recall some notions and results. In §3, we prove Theorem 1.4. In §4, we show Corollaries 1.6 and 1.8. In §5, we prove Theorems 1.9 and 1.10. In §6, we show Propositions 1.11–1.13. In §7, we list some questions.

2. *Preliminaries*

2.1. *Isomorphism and factors.* We say that measure-preserving systems $(X, \mathcal{X}, \mu, T)$ and $(Y, \mathcal{Y}, \nu, S)$ are *isomorphic* if there exists an invertible measure-preserving transformation $\Phi : X_0 \to Y_0$ with $\Phi \circ T = S \circ \Phi$, where X_0 is a T-invariant full measure subset of X and Y_0 is an S-invariant full measure subset of Y.

A *factor* of a measure-preserving system $(X, \mathcal{X}, \mu, T)$ is a T-invariant sub-σ-algebra of $\mathcal{X}$. A *factor map* from $(X, \mathcal{X}, \mu, T)$ to $(Y, \mathcal{Y}, \nu, S)$ is a measurable map $\pi : X_0 \to Y_0$ with $\pi \circ T = S \circ \pi$ and such that ν is the image of μ under π, where X_0 is a T-invariant full measurable subset of X and Y_0 is an S-invariant full measurable subset of Y. In this

https://doi.org/10.1017/etds.2025.10254 Published online by Cambridge University Press

case, $\pi^{-1}(\mathcal{Y})$ is a factor of $(X, \mathcal{X}, \mu, T)$ and every factor of $(X, \mathcal{X}, \mu, T)$ can be obtained in this way.

2.2. *Conditional expectation and ergodic decomposition.* Given a Lebesgue space $(X, \mathcal{X}, \mu)$, let $\mathcal{Y}$ be a sub-σ-algebra of $\mathcal{X}$. For any $f \in L^1(X, \mathcal{X}, \mu)$, the *conditional expectation of f with respect to* $\mathcal{Y}$ is the function $\mathbb{E}_\mu(f|\mathcal{Y})$, defined in $L^1(X, \mathcal{Y}, \mu)$, such that for any $A \in \mathcal{Y}$, $\int_A f\, d\mu = \int_A \mathbb{E}_\mu(f|\mathcal{Y})\, d\mu$. Then, there exists a unique $\mathcal{Y}$-measurable map $X \to \mathcal{M}(X, \mathcal{X})$, $x \mapsto \mu_x$, called the *disintegration of* μ *with respect to* $\mathcal{Y}$, up to μ-null sets such that for any $f \in L^\infty(X, \mathcal{X}, \mu)$, $\mathbb{E}_\mu(f|\mathcal{Y})(x) = \int_X f\, d\mu_x$ for μ-a.e. $x \in X$, where $\mathcal{M}(X, \mathcal{X})$ is the collection of probability measures on $(X, \mathcal{X})$, endowed with the standard Borel structure.

Let $(X, \mathcal{X}, \mu, T)$ be a measure-preserving system and $\mathcal{I}(T)$ be the sub-σ-algebra of $\mathcal{X}$, generated by all T-invariant sets. The disintegration of μ with respect to $\mathcal{I}(T)$, denoted by $X \to \mathcal{M}(X, \mathcal{X})$, $x \mapsto \mu_x$, is called the *ergodic decomposition of* μ *with respect to T*. Then, for μ-a.e. $x \in X$, $(X, \mathcal{X}, \mu_x, T)$ is an ergodic measure-preserving system.

2.3. *Nilsequences.* Let $k \in \mathbb{N}$. A *k-step nilmanifold* X is a quotient space G/Γ, where G is a k-step nilpotent Lie group and Γ is a cocompact discrete subgroup of G. A *basic k-step nilsequence* is the sequence $\{f(a^n \cdot x)\}_{n\in\mathbb{Z}}$, where $f \in C(X)$, $a \in G$, $x \in X$. A *k-step nilsequence* is a uniform limit of basic k-step nilsequences. Clearly, all k-step nilsequences are an invariant algebra of $l^\infty(\mathbb{Z})$ under translation (for more details, see [16, Ch. 11]). By [16, Proposition 11.13], we have that for any nilsequence $\{b_n\}_{n\in\mathbb{Z}}$, the limit $\lim_{N\to\infty}(1/N)\sum_{n=1}^N b_n$ exists.

For nilsequences, Bergelson and Richter [2] built a disjointness result on it.

THEOREM 2.1. [2, Corollary 1.27 and Lemma 6.3] *Let* (X, T) *be a uniquely ergodic topological dynamical system with the unique T-invariant Borel probability measure* μ. *Let* $\{b_n\}_{n\in\mathbb{Z}}$ *be a nilsequence. Then, for any* $f \in C(X)$, $x \in X$,

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} b_n f(T^{\Omega(n)}x) = \int_X f\, d\mu \cdot \lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N} b_n.$$

2.4. *Pronilfactors.* Fix a measure-preserving system $(X, \mathcal{X}, \mu, T)$. For any $f \in L^\infty(\mu)$, we let $|||f|||_1 = \|\mathbb{E}_\mu(f|\mathcal{I}(T))\|_{L^2(\mu)}$. Next, we define $|||\cdot|||_k$ inductively. For each $k \geq 1$ and any $f \in L^\infty(\mu)$, we let

$$|||f|||_{k+1} = \bigg(\lim_{H\to\infty}\frac{1}{H}\sum_{h=0}^{H-1} |||f \cdot T^h\bar{f}|||_k^{2^k}\bigg)^{1/2^{k+1}}.$$

By [16, Proposition 8.16], the limit in the above equation always exists. So, $|||\cdot|||_k$ is well defined for each $k \in \mathbb{N}$. By [16, Theorem 9.7], there exists a factor $\mathcal{Z}_k(T)$ of $(X, \mathcal{X}, \mu, T)$, called the *k-step factor*, such that for any $f \in L^\infty(\mu)$, $|||f|||_{k+1} = 0$ if and only if $\mathbb{E}(f|\mathcal{Z}_k(T)) = 0$. By [16, (8.15)], we know that for any $k \in \mathbb{N}$, $\mathcal{Z}_k(T) \subset \mathcal{Z}_{k+1}(T)$. So, we can define the factor $\mathcal{Z}_\infty(T)$ of $(X, \mathcal{X}, \mu, T)$, called the ∞-*step factor*, by letting it be the smallest σ-algebra containing $\bigcup_{k=1}^\infty \mathcal{Z}_k(T)$ (for more details, see [16, Chs. 8, 9, 16]).

https://doi.org/10.1017/etds.2025.10254 Published online by Cambridge University Press

Next, we introduce a structure theorem for general measure-preserving systems, which will be used in the proofs of the main results of this paper.

THEOREM 2.2. [16, Theorem 16.10] *Let $k \in \mathbb{N}$ and let $(X, \mathcal{X}, \mu, T)$ be a measure-preserving system. Then, for all $p \geq 1$ and $\epsilon > 0$, every $f \in L^\infty(\mu)$ admits a decomposition*

$$f = f_{\mathrm{unif}} + f_{\mathrm{nil}} + f_{\mathrm{sml}},$$

where:

(i) $|||f_{\mathrm{unif}}|||_{k+1} = 0$;

(ii) *for μ-a.e. $x \in X$, the sequence $\{f_{\mathrm{nil}}(T^n x)\}_{n\in\mathbb{Z}}$ is a k-step nilsequence;*

(iii) $f_{\mathrm{sml}} \in L^\infty(\mu)$ *satisfies* $\|f_{\mathrm{sml}}\|_{L^p(\mu)} < \epsilon$.

Furthermore, $\|f_{\mathrm{nil}}\|_{L^\infty(\mu)} \leq \|f\|_{L^\infty(\mu)}$ *and* $\|f_{\mathrm{nil}} + f_{\mathrm{sml}}\|_{L^\infty(\mu)} \leq \|f\|_{L^\infty(\mu)}$.

2.5. *An orthogonality criterion.* The following orthogonality criterion (see [6, Lemma 1], [17, (3.1)], [4, Theorem 2]) is an application of the Turán–Kubilius inequality.

LEMMA 2.3. [5, Lemma 2.14] *Let $\{A(n)\}_{n\geq 1}$ be a bounded sequence in Hilbert space $\mathcal{H}$. If $\mathcal{P} \subset \mathbb{P}$ with positive relative density such that for any distinct $p, q \in \mathcal{P}$,*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \langle A(pn), A(qn)\rangle = 0,$$

then

$$\lim_{N\to\infty} \left\| \frac{1}{N} \sum_{n=1}^{N} A(n) \right\| = 0.$$

3. *Proof of Theorem 1.4*

First, let us recall Bourgain's double recurrence theorem.

THEOREM 3.1. [3, Main Theorem and (2.3)] *Let $(X, \mathcal{X}, \mu, T)$ be an ergodic measure-preserving system. Fix two distinct non-zero integers a and b. Fix $f, g \in L^\infty(\mu)$. If f or g is orthogonal to the closed subspace of $L^2(\mu)$ spanned by all eigenfunctions with respect to T, then*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} f(T^{an}x)g(T^{bn}x) = 0$$

for μ-a.e. $x \in X$.

Now, we prove Theorem 1.4.

Proof of Theorem 1.4. The whole proof is divided into three steps.

Step I. Reduction to the ergodic case. In this step, we show that to verify that Theorem 1.4 holds, it suffices to prove that Theorem 1.4 holds for all ergodic measure-preserving systems.

Suppose that Theorem 1.4 holds for all ergodic measure-preserving systems. Next, we use contradiction argument to verify that Theorem 1.4 holds.

If Theorem 1.4 does not hold, then there is a measure-preserving system $(X, \mathcal{X}, \mu, T)$ and $f \in L^1(\mu)$ such that there is a measurable set $X' \subset X$ of positive μ-measure such that for each $x \in X'$, there are a uniquely ergodic topological dynamical system (Y_x, S_x) with the unique S_x-invariant Borel probability measure ν_x, $g_x \in C(Y)$ and $y_x \in Y$ such that

$$\limsup_{N\to\infty} \left| \frac{1}{N} \sum_{n=1}^{N} f(T^n x) g_x(S_x^{\Omega(n)} y_x) - f^*(x) \int_{Y_x} g_x \, d\nu_x \right| > 0, \tag{3.1}$$

where

$$f^*(x) = \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} f(T^n x).$$

Let $X \to \mathcal{M}(X)$, $z \to \mu_z$ be the ergodic decomposition of μ with respect to T. Then, there is $z \in X$ such that the following hold:

(1) $(X, \mathcal{X}, \mu_z, T)$ is ergodic;
(2) $f \in L^1(\mu_z)$;
(3) $\mu_z(X') > 0$.

Then, by the hypothesis and Birkhoff's ergodic theorem, we know that there is a measurable set $X'' \subset X$ of full μ_z-measure such that for each $x \in X'' \cap X'$,

$$\lim_{N\to\infty} \left| \frac{1}{N} \sum_{n=1}^{N} f(T^n x) g_x(S_x^{\Omega(n)} y_x) - \int_X f \, d\mu_z \int_{Y_x} g_x \, d\nu_x \right| = 0 \tag{3.2}$$

and $\int_X f \, d\mu_z = f^*(x)$. Then, (3.2) contradicts with (3.1). So, Theorem 1.4 can be reduced to the ergodic case.

Step II. Reduction to the L^∞-functions. Based on Step I, we fix an ergodic measure-preserving system $(X, \mathcal{X}, \mu, T)$. In this step, we show that to verify that Theorem 1.4 holds for $(X, \mathcal{X}, \mu, T)$, it suffices to prove that Theorem 1.4 holds for all elements of $L^\infty(\mu)$.

Suppose that Theorem 1.4 holds for all elements of $L^\infty(\mu)$. Fix $f \in L^1(\mu)$. Then, there is a sequence of functions $\{f_j\}_{j\geq 1} \subset L^\infty(\mu)$ such that the following hold:

(1) for each $j \geq 1$, $\|f - f_j\|_{L^1(\mu)} < 1/2^j$;
(2) for each $j \geq 1$, there is a full measure subset X_j of X such that Theorem 1.4 holds for f_j.

Let

$$X_0 = \bigcap_{j\geq 1} X_j \cap \left\{ x \in X : \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} f(T^n x) = \int_X f \, d\mu \right\}$$
$$\cap \bigcap_{j\geq 1} \left\{ x \in X : \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} |f - f_j|(T^n x) = \|f - f_j\|_{L^1(\mu)} \right\}.$$

By Birkhoff's ergodic theorem, we have that $\mu(X_0) = 1$.

Fix $x \in X_0$. Fix a uniquely ergodic topological dynamical system (Y, S) with the unique S-invariant Borel probability measure ν, $g \in C(Y)$ and $y \in Y$. Then,

https://doi.org/10.1017/etds.2025.10254 Published online by Cambridge University Press

$$\begin{aligned}
&\limsup_{N\to\infty}\left|\frac{1}{N}\sum_{n=1}^{N} f(T^n x)g(S^{\Omega(n)}y) - \int_X f\,d\mu \int_Y g\,d\nu\right| \\
&\le \limsup_{j\to\infty}\limsup_{N\to\infty}\left|\frac{1}{N}\sum_{n=1}^{N}(f-f_j)(T^n x)g(S^{\Omega(n)}y)\right| + \limsup_{j\to\infty}\left|\int_X (f-f_j)\,d\mu \int_Y g\,d\nu\right| \\
&\le 2\|g\|_{L^\infty(\nu)} \limsup_{j\to\infty}\|f-f_j\|_{L^1(\mu)} \\
&= 0.
\end{aligned}$$

So, Theorem 1.4 can be reduced to the L^∞-functions.

Step III. Proving Theorem 1.4 for the ergodic case and the L^∞-functions. Fix an ergodic measure-preserving system $(X, \mathcal{X}, \mu, T)$ and 1-bounded $f \in L^\infty(\mu)$. Let Z be the sub-σ-algebra of $\mathcal{X}$ generated by all eigenfunctions with respect to T. Then, we can write f as $\mathbb{E}_\mu(f|Z) + (f - \mathbb{E}_\mu(f|Z))$.

By repeating the argument in Step II and the linear property of ergodic averages, we know that to prove that for $\mathbb{E}_\mu(f|Z)$, Theorem 1.4 holds, it suffices to show that Theorem 1.4 holds for all eigenfunctions with respect to T. Fix a non-constant eigenfunction h with eigenvalue λ. Clearly, $\lambda \in \mathbb{T}$, $\lambda \neq 1$ and $\int_X h\,d\mu = 0$. Fix

$$x \in \{y \in X : h(T^n y) = \lambda^n h(y) \text{ for each } n \ge 1\} \cap \{y \in X : |h(y)| < \infty\}.$$

Fix a uniquely ergodic topological dynamical system (Y, S) with the unique S-invariant Borel probability measure ν, $g \in C(Y)$ and $y \in Y$. Then,

$$\frac{1}{N}\sum_{n=1}^{N} h(T^n x)g(S^{\Omega(n)}y) = h(x)\frac{1}{N}\sum_{n=1}^{N}\lambda^n g(S^{\Omega(n)}y).$$

By [2, Corollary 1.25] and the fact that $\int_X h\,d\mu = 0$,

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N} h(T^n x)g(S^{\Omega(n)}y) = \int_X h\,d\mu \int_Y g\,d\nu.$$

So, for such h, Theorem 1.4 holds.

When h is a constant, by Theorem 1.1, we know that Theorem 1.4 holds. To sum up, Theorem 1.4 holds for $\mathbb{E}_\mu(f|Z)$.

Next, we prove that Theorem 1.4 holds for $\tilde{f} := f - \mathbb{E}_\mu(f|Z)$. Clearly, $\tilde{f}$ is orthogonal to the closed subspace of $L^2(\mu)$ spanned by all eigenfunctions with respect to T and $\int_X \tilde{f}\,d\mu = 0$.

Let

$$\begin{aligned}
\bar{X} = &\left\{x \in X : \lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\tilde{f}(T^n x) = 0\right\} \\
&\cap \bigcap_{\substack{p,q\in\mathbb{P},\\ p\neq q}}\left\{x \in X : \lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\tilde{f}(T^{pn}x)\bar{\tilde{f}}(T^{qn}x) = 0\right\}.
\end{aligned}$$

By Birkhoff's ergodic theorem and Theorem 3.1, $\mu(\bar{X}) = 1$.

https://doi.org/10.1017/etds.2025.10254 Published online by Cambridge University Press

Fix $x \in \bar{X}$. Fix a uniquely ergodic topological dynamical system (Y, S) with the unique S-invariant Borel probability measure ν, $g \in C(Y)$ and $y \in Y$. Note that when $0 \leq g \leq 1$, g can be written as

$$\tfrac{1}{2}((g + i\sqrt{1-g^2}) + (g - i\sqrt{1-g^2})).$$

So, by the linear property of ergodic averages, we can assume that $|g| \equiv 1$.

Fix two distinct prime numbers p and q. Then,

$$\begin{aligned}
&\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} \tilde{f}(T^{pn}x)g(S^{\Omega(pn)}y) \cdot \bar{\tilde{f}}(T^{qn}x)\bar{g}(S^{\Omega(qn)}y) \\
&= \lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} \tilde{f}(T^{pn}x)\bar{\tilde{f}}(T^{qn}x)(g\cdot\bar{g})(S^{\Omega(n)+1}y) \\
&= \lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} \tilde{f}(T^{pn}x)\bar{\tilde{f}}(T^{qn}x) \\
&= 0. \qquad (3.3)
\end{aligned}$$

By combining (3.3), Lemma 2.3 and the fact that $\int_X \tilde{f}\, d\mu = 0$, we know that

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} \tilde{f}(T^{n}x)g(S^{\Omega(n)}y) = \int_X \tilde{f}\, d\mu \int_Y g\, d\nu.$$

So, Theorem 1.4 holds for $\tilde{f}$. This finishes the whole proof. □

4. *Applications of Theorem 1.4*

First, we prove Corollary 1.6.

Proof of Corollary 1.6. Note that for any bounded sequence $\{a_n\}_{n\geq 1} \subset \mathbb{C}$ and each $k \in \mathbb{N}$,

$$\lim_{N\to\infty}\bigg|\frac{1}{N}\sum_{n=1}^{N} a_n - \frac{1}{N}\sum_{n=1}^{N} a_{n+k}\bigg| = \lim_{N\to\infty}\bigg|\frac{1}{N}\sum_{n=1}^{N} a_n - \frac{1}{k}\sum_{i=0}^{k-1}\frac{1}{[N/k]}\sum_{n=1}^{[N/k]} a_{kn+i}\bigg| = 0. \qquad (4.1)$$

Then, we can assume that $\beta > 0$ and $\alpha + \beta \geq 1$.

If $\alpha \in \mathbb{Q}$, then there are $q, p \in \mathbb{N}$ such that for each $0 \leq i < q$, there are $d_i \geq 0$, $0 \leq c_i < p$ such that for any $n \in \mathbb{N}$, $[\alpha(qn+i)+\beta] = p(n+d_i)+c_i$. By combining (4.1) and [2, Corollary 1.16], we have that Corollary 1.6 holds.

Now, fix $\alpha \in \mathbb{R}\backslash\mathbb{Q}$. By (4.1), we can assume that $\alpha > 100$ and for each $n \in \mathbb{N}$, $\alpha n + \beta \notin \mathbb{N}$. For any $t \in \mathbb{R}$, let $\{t\} = t - [t]$. Note that $m \in \{[\alpha n+\beta] : n \in \mathbb{N}\}$ if and only if

$$\left\{\frac{m-\beta}{\alpha}\right\} \in \left(1 - \frac{1}{\alpha}, 1\right).$$

https://doi.org/10.1017/etds.2025.10254 Published online by Cambridge University Press

Then, there exists an open interval $A_{\alpha,\beta}$ of $\mathbb{T}$ with length α^{-1} such that $m \in \{[\alpha n + \beta] : n \in \mathbb{N}\}$ if and only if $m/\alpha \in A_{\alpha,\beta}$.

Fix a uniquely ergodic topological dynamical system (Y, S) with the unique S-invariant Borel probability measure ν, $g \in C(Y)$ and $y \in Y$. Let $T : \mathbb{T} \to \mathbb{T}$, $x \mapsto x + \alpha^{-1}$. Then, $(\mathbb{T}, T)$ is uniquely ergodic and the unique T-invariant Borel probability measure is the Haar measure μ on $\mathbb{T}$. Let $\mathcal{B}(\mathbb{T})$ be the Borel σ-algebra on $\mathbb{T}$. After applying Theorem 1.4 to $(\mathbb{T}, \mathcal{B}(\mathbb{T}), \mu, T)$ and $1_{A_{\alpha,\beta}}$, we have that for any $\epsilon \in (0, (100\alpha)^{-1})$, there is $x_\epsilon \in (0, \epsilon)$ such that

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} 1_{A_{\alpha,\beta}}(x_\epsilon + n/\alpha) g(S^{\Omega(n)} y) = \frac{1}{\alpha} \int_Y g \, d\nu. \tag{4.2}$$

By Weyl's uniform distribution theorem,

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} |1_{A_{\alpha,\beta}}(x_\epsilon + n/\alpha) - 1_{A_{\alpha,\beta}}(n/\alpha)| \le 2\epsilon. \tag{4.3}$$

Note that

$$\frac{1}{N} \sum_{n=1}^{N} g(S^{\Omega([\alpha n+\beta])} y) = \frac{[\alpha N + \beta]}{N} \cdot \frac{1}{[\alpha N + \beta]} \sum_{n=1}^{[\alpha N+\beta]} 1_{A_{\alpha,\beta}}(n/\alpha) g(S^{\Omega(n)} y). \tag{4.4}$$

By combining (4.2)–(4.4), we conclude that

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} g(S^{\Omega([\alpha n+\beta])} y) = \int_Y g \, d\nu.$$

This finishes the proof. □

Next, we show Corollary 1.8.

Proof of Corollary 1.8. For any $\beta \in \mathbb{R}$, we define $T_\beta : \mathbb{T}^k \to \mathbb{T}^k$ by putting

$$T(x_1, \ldots, x_k) = (x_1 + \beta, x_2 + x_1, \ldots, x_k + x_{k-1})$$

for any $(x_1, \ldots, x_k) \in \mathbb{T}^k$. By [7, Corollary 4.22], when β is irrational, $(\mathbb{T}^k, T_\beta)$ is uniquely ergodic and the unique T-invariant Borel probability measure is $m^{\otimes k}$, where

$$m^{\otimes k} = m \times \cdots \times m \quad (k \text{ times}).$$

Let

$$C = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ \binom{1}{0} & \binom{1}{1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \binom{k}{0} & \binom{k}{1} & \cdots & \binom{k}{k} \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & k & \cdots & k^k \end{pmatrix}.$$

Let $\pi_k : \mathbb{T}^k \to \mathbb{T}$ be the kth coordinate projection. Then, when

$$(c_0, \ldots, c_k)^T = B^{-1} C (x_k, \ldots, x_0)^T, \tag{4.5}$$

https://doi.org/10.1017/etds.2025.10254 Published online by Cambridge University Press

we have that $k!\, c_k = x_0$ and for each $n \in \mathbb{N}$,

$$\pi_k(T_{x_0}^n(x_1, \ldots, x_k)) = (c_k n^k + \cdots + c_1 n + c_0) \mod 1. \tag{4.6}$$

Fix $f \in L^1(m)$. Let $\mathcal{B}(\mathbb{T}^k)$ be the Borel σ-algebra on $\mathbb{T}^k$. After applying Theorem 1.4 to $(\mathbb{T}^k, \mathcal{B}(\mathbb{T}^k), m^{\otimes k}, T_{k!\alpha})$ and $f \circ \pi_k$, we know that there is an $m^{\otimes k}$-null subset X_f of $\mathbb{T}^k$ such that for any $(x_1, \ldots, x_k) \notin X_f$, any uniquely ergodic topological dynamical system (Y, S) with the unique S-invariant Borel probability measure ν, any $g \in C(Y)$ and any $y \in Y$,

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} f(\pi_k(T_{k!\alpha}^n(x_1, \ldots, x_k)))g(S^{\Omega(n)}y) = \int_{\mathbb{T}} f\, dm \int_Y g\, d\nu.$$

Reference [8, Theorem 2.44.a] tells us that the image of any zero Lebesgue measure subset of $\mathbb{R}^k$ under an invertible linear map is still of zero Lebesgue measure. Combining this, (4.5) and (4.6), we can find a set $A_f \subset \mathbb{R}^k$ with zero Lebesgue measure such that for any $(c_0, \ldots, c_{k-1}) \notin A_f$, any uniquely ergodic topological dynamical system (Y, S) with the unique S-invariant Borel probability measure ν, any $g \in C(Y)$ and any $y \in Y$,

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} f(\alpha n^k + c_{k-1}n^{k-1} + \cdots + c_1 n + c_0)g(S^{\Omega(n)}y) = \int_{\mathbb{T}} f\, dm \int_Y g\, d\nu.$$

This finishes the proof. □

5. *Proofs of Theorems 1.9 and 1.10*

Before proving Theorems 1.9 and 1.10, let us introduce two results, which point out the characteristic factor behaviour of the ∞-step factors.

THEOREM 5.1. ([15, Theorem 1], [18, Theorem 3]) *Given $k \in \mathbb{N}$, let $P_1, \ldots, P_k \in \mathbb{Z}[n]$. Let T be an invertible measure-preserving transformation acting on the Lebesgue space $(X, \mathcal{X}, \mu)$. Fix $f_1, \ldots, f_k \in L^\infty(\mu)$. If there is some $1 \leq j \leq k$ such that $\mathbb{E}_\mu(f_j | \mathcal{Z}_\infty(T)) = 0$, then*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \prod_{i=1}^{k} f_i(T^{P_i(n)}x) = 0$$

in $L^2(\mu)$.

THEOREM 5.2. [11, Theorem 2.8] *Given $k \in \mathbb{N}$, let $P_1, \ldots, P_k \in \mathbb{Z}[n]$ and they are pairwise independent. Let $T_1, \ldots, T_k$ be a family of commuting invertible measure-preserving transformations acting on the Lebesgue space $(X, \mathcal{X}, \mu)$. Fix $f_1, \ldots, f_k \in L^\infty(\mu)$. If there is some $j \in \{1, \ldots, k\}$ such that $\mathbb{E}_\mu(f_j | \mathcal{Z}_\infty(T_j)) = 0$, then*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \prod_{i=1}^{k} f_i(T_i^{P_i(n)}x) = 0$$

in $L^2(\mu)$.

Now, we begin to prove Theorem 1.9.

https://doi.org/10.1017/etds.2025.10254 Published online by Cambridge University Press

Proof of Theorem 1.9. Without loss of generality, we can assume that $P_1, \ldots, P_k$ have zero constant terms. Let $X \to \mathcal{M}(X)$, $x \mapsto \mu_x$ be the ergodic decomposition of μ with respect to S. Fix 1-bounded $g_1, \ldots, g_{k+1} \in L^\infty(\mu)$. By [18, Theorem 1], it suffices to prove the following:

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} g_1(T_1^{P_1(n)}x)\cdots g_k(T_k^{P_k(n)}x)g_{k+1}(S^{\Omega(n)}x)$$

$$= \int_X g_{k+1}\, d\mu_x \lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} g_1(T_1^{P_1(n)}x)\cdots g_k(T_k^{P_k(n)}x) \tag{5.1}$$

in $L^2(\mu)$.

First, we show that the characteristic factor of the ergodic averages stated in the left side of (5.1) is $\mathcal{Z}_\infty(T_i)$, $1 \le i \le k$.

By [5, Proof of Lemma 2.15], we can assume that $|g_{k+1}| \equiv 1$. Let the set $D = \{(p, q) \in \mathbb{P}^2 : p \ne q$ and there are $1 \le i \le j \le k$, $x, y \in \mathbb{Z}\backslash\{0\}$ such that $xP_i(pn) + yP_j(qn) \equiv 0\}$. Then, by the assumption on $P_1, \ldots, P_k$, a simple calculation gives that D is empty or finite.

For each $n \in \mathbb{N}$, let

$$A(n) = T_1^{P_1(n)}g_1 \cdots T_k^{P_k(n)}g_k \cdot S^{\Omega(n)}g_{k+1}.$$

Fix two distinct $p, q \in \mathbb{P}$ such that $(p, q) \notin D$. Then,

$$\frac{1}{N}\sum_{n=1}^{N}\langle A(pn), A(qn)\rangle$$

$$= \frac{1}{N}\sum_{n=1}^{N}\int_X \prod_{i=1}^{k} g_i(T_i^{P_i(pn)}x)\cdot \prod_{i=1}^{k} \bar{g}_i(T_i^{P_i(qn)}x)\cdot(g_{k+1}\cdot \bar{g}_{k+1})(S^{\Omega(n)+1}x)\, d\mu(x)$$

$$= \frac{1}{N}\sum_{n=1}^{N}\int_X \prod_{i=1}^{k} g_i(T_i^{P_i(pn)}x)\cdot \prod_{i=1}^{k} \bar{g}_i(T_i^{P_i(qn)}x)\, d\mu(x). \tag{5.2}$$

Note that by the choices of p and q, the polynomial family

$$\{P_1(pn), \ldots, P_k(pn), P_1(qn), \ldots, P_k(qn)\}$$

is pairwise independent. By Theorem 5.2 and (5.2), we know that if there is some $1 \le j \le k$ such that $\mathbb{E}_\mu(g_j|\mathcal{Z}_\infty(T_j)) = 0$, then

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\langle A(pn), A(qn)\rangle = 0. \tag{5.3}$$

Combining (5.3) and Lemma 2.3, we have that if there is some $1 \le j \le k$ such that $\mathbb{E}_\mu(g_j|\mathcal{Z}_\infty(T_j)) = 0$, then

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N} A(n) = 0 \tag{5.4}$$

in $L^2(\mu)$. Equation (5.4) means that

$$\lim_{N\to\infty}\left|\frac{1}{N}\sum_{n=1}^{N} g_1(T_1^{P_1(n)}x)\cdots g_k(T_k^{P_k(n)}x)g_{k+1}(S^{\Omega(n)}x)\right.$$
$$\left.-\frac{1}{N}\sum_{n=1}^{N}\mathbb{E}_\mu(g_1|\mathcal{Z}_\infty(T_1))(T_1^{P_1(n)}x)\cdots\mathbb{E}_\mu(g_k|\mathcal{Z}_\infty(T_k))(T_k^{P_k(n)}x)g_{k+1}(S^{\Omega(n)}x)\right|=0 \tag{5.5}$$

in $L^2(\mu)$.

Next, we show that

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\mathbb{E}_\mu(g_1|\mathcal{Z}_\infty(T_1))(T_1^{P_1(n)}x)\cdots\mathbb{E}_\mu(g_k|\mathcal{Z}_\infty(T_k))(T_k^{P_k(n)}x)g_{k+1}(S^{\Omega(n)}x)$$
$$=\int_X g_{k+1}d\mu_x\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\mathbb{E}_\mu(g_1|\mathcal{Z}_\infty(T_1))(T_1^{P_1(n)}x)\cdots\mathbb{E}_\mu(g_k|\mathcal{Z}_\infty(T_k))(T_k^{P_k(n)}x) \tag{5.6}$$

in $L^2(\mu)$. By combining Theorem 5.2 and (5.5), we know that if we prove this, then we finish the proof of (5.1).

By Theorem 2.2 and [**16**, Theorem 14.15], for each $1\le i\le k$, there exists a sequence of functions $\{f_{i,j}\}_{j\ge1}$ such that the following hold:

(1) for each $1\le i\le k$, $j\ge1$ and μ-a.e. $x\in X$, $\{f_{i,j}(T_i^{P_i(n)}x)\}_{n\in\mathbb{Z}}$ is a nilsequence;
(2) for each $1\le i\le k$, $j\ge1$, $\|\mathbb{E}_\mu(g_i|\mathcal{Z}_\infty(T_i))-f_{i,j}\|_{L^{2k}(\mu)}\le 1/2^j$;
(3) for each $1\le i\le k$, $j\ge1$, $\|f_{i,j}\|_{L^\infty(\mu)}\le1$.

Then, there exists $X_0\in\mathcal{X}$ with $\mu(X_0)=1$ such that the following hold:

(1) for any $y\in X_0$, each $1\le i\le k$, $j\ge1$, and μ_y-a.e. $x\in X$, $\{f_{i,j}(T_i^{P_i(n)}x)\}_{n\in\mathbb{Z}}$ is a nilsequence;
(2) for any $y\in X_0$, $(X,\mathcal{X},\mu_y,S)$ is ergodic;
(3) for any $y\in X_0$, $\|g_{k+1}\|_{L^\infty(\mu_y)}\le1$;
(4) for any $y\in X_0$ and each $1\le i\le k$, $j\ge1$, $\|f_{i,j}\|_{L^\infty(\mu_y)}\le1$.

To prove (5.6), it suffices to show that for any $y\in X_0$ and each $j\ge1$,

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N} f_{1,j}(T_1^{P_1(n)}x)\cdots f_{k,j}(T_k^{P_k(n)}x)g_{k+1}(S^{\Omega(n)}x)$$
$$=\int_X g_{k+1}\,d\mu_y\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N} f_{1,j}(T_1^{P_1(n)}x)\cdots f_{k,j}(T_k^{P_k(n)}x) \tag{5.7}$$

in $L^2(\mu_y)$. To see this, let us do the following calculation:

$$\limsup_{N\to\infty}\left\|\frac{1}{N}\sum_{n=1}^{N}\prod_{i=1}^{k}\mathbb{E}_\mu(g_i|\mathcal{Z}_\infty(T_i))(T_i^{P_i(n)}x)\cdot\left(g_{k+1}(S^{\Omega(n)}x)-\int_X g_{k+1}\,d\mu_x\right)\right\|_{L^2(\mu)}$$

https://doi.org/10.1017/etds.2025.10254 Published online by Cambridge University Press

$$
\begin{aligned}
&\leq \limsup_{j\to\infty}\limsup_{N\to\infty}\left\|\frac{1}{N}\sum_{n=1}^{N}\prod_{i=1}^{k} f_{i,j}(T_i^{P_i(n)}x)\cdot\left(g_{k+1}(S^{\Omega(n)}x)-\int_X g_{k+1}\,d\mu_x\right)\right\|_{L^2(\mu)}\\
&\quad+2\limsup_{j\to\infty}\limsup_{N\to\infty}\left(\frac{1}{N}\sum_{n=1}^{N}\left\|\left(\prod_{i=1}^{k}T_i^{P_i(n)}f_{i,j}-\prod_{i=1}^{k}T_i^{P_i(n)}\mathbb{E}_\mu(g_i|\mathcal{Z}_\infty(T_i))\right)\right\|_{L^2(\mu)}^2\right)^{1/2}\\
&\leq \sup_{j\geq 1}\limsup_{N\to\infty}\left(\int_X\left|\frac{1}{N}\sum_{n=1}^{N}\prod_{i=1}^{k} f_{i,j}(T_i^{P_i(n)}x)\cdot\left(g_{k+1}(S^{\Omega(n)}x)-\int_X g_{k+1}\,d\mu_x\right)\right|^2 d\mu(x)\right)^{1/2}\\
&\leq \sup_{j\geq 1}\left\|\limsup_{N\to\infty}\left\|\frac{1}{N}\sum_{n=1}^{N}\prod_{i=1}^{k} f_{i,j}(T_i^{P_i(n)}x)\cdot\left(g_{k+1}(S^{\Omega(n)}x)-\int_X g_{k+1}\,d\mu_y\right)\right\|_{L_x^2(\mu_y)}^2\right\|_{L_y^1(\mu)}^{1/2}\\
&=0,
\end{aligned}
$$

where the last equality comes from (5.7).

Now, we verify (5.7). Fix $y \in X_0$ and $j \geq 1$. Then, we get an ergodic measure-preserving system $(X, \mathcal{X}, \mu_y, S)$. By [**14**, Theorem 15.27], there exists a uniquely ergodic topological dynamical system (Y, R) with the unique R-invariant Borel probability measure ν such that $(X, \mathcal{X}, \mu_y, S)$ and $(Y, \mathcal{B}(Y), \nu, R)$ are isomorphic via the invertible measure-preserving transformation $\pi : X \to Y$, where $\mathcal{B}(Y)$ is the Borel σ-algebra on Y. Then, we can find $X_1 \in \mathcal{X}$ with $\mu_y(X_1) = 1$ and a sequence of functions $\{h_i\}_{i\geq 1}$ in $C(Y)$ such that the following hold:

(1) for each $i \geq 1$, $\|h_i \circ \pi - g_{k+1}\|_{L^2(\mu_y)} \leq 1/2^i$;

(2) for any $x \in X_1$, $1 \leq t \leq k$, $\{f_{t,j}(T_t^{P_t(n)}x)\}_{n\in\mathbb{Z}}$ is a nilsequence;

(3) for any $x \in X_1$ and each $n \in \mathbb{Z}$, $\pi(S^n x) = R^n \pi(x)$;

(4) for any $y \in Y$ and each $i \geq 1$, $|h_i(y)| \leq 1$.

Note that the product of finitely many nilsequences is still a nilsequence. Then, by Theorem 2.1, we know that for any $x \in X_1$, $i \geq 1$,

$$
\begin{aligned}
&\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N} f_{1,j}(T_1^{P_1(n)}x)\cdots f_{k,j}(T_k^{P_k(n)}x)h_i\circ\pi(S^{\Omega(n)}x)\\
&\quad=\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N} f_{1,j}(T_1^{P_1(n)}x)\cdots f_{k,j}(T_k^{P_k(n)}x)h_i(R^{\Omega(n)}\pi(x))\\
&\quad=\int_Y h_i d\nu\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N} f_{1,j}(T_1^{P_1(n)}x)\cdots f_{k,j}(T_k^{P_k(n)}x)\\
&\quad=\int_Y h_i\circ\pi\, d\mu_y\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N} f_{1,j}(T_1^{P_1(n)}x)\cdots f_{k,j}(T_k^{P_k(n)}x).
\end{aligned}
\tag{5.8}
$$

Based on (5.8), by a standard approximation argument, we know that (5.7) exists in $L^2(\mu_y)$. This finishes the whole proof. □

The proof of Theorem 1.10 is similar that of Theorem 1.9. The only difference between them is that we should use Theorem 5.1 in the proof of Theorem 1.10 instead of Theorem 5.2.

https://doi.org/10.1017/etds.2025.10254 Published online by Cambridge University Press

Let $X \to \mathcal{M}(X), x \mapsto \mu_x$ be the ergodic decomposition of μ with respect to S. Once one finishes the proof of Theorem 1.10, the following equality will appear:

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} g_1(T^{P_1(n)}x) \cdots g_k(T^{P_k(n)}x) g_{k+1}(S^{\Omega(n)}x)$$
$$= \int_X g_{k+1} d\mu_x \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} g_1(T^{P_1(n)}x) \cdots g_k(T^{P_k(n)}x) \tag{5.9}$$

in $L^2(\mu)$, where $g_1, \ldots, g_{k+1} \in L^\infty(\mu)$.

6. *Proofs of Propositions 1.11–1.13*

First, let us recall Furstenberg's correspondence principle.

THEOREM 6.1. (See [12, Theorem 1.1], [10, §2.1]) *Let $k \in \mathbb{N}$ and $E \subset \mathbb{Z}^k$. There exists a Lebesgue space $(X, \mathcal{X}, \mu)$, commuting invertible measure-preserving transformations $T_1, \ldots, T_k : X \to X$, and $A \in \mathcal{X}$ with $\mu(A) = d^*(E)$ such that*

$$d^*(E \cap (E - d_1) \cap \cdots \cap (E - d_m)) \geq \mu\bigg(A \cap \prod_{i=1}^{k} T_i^{-d_{1,i}} A \cdots \cap \prod_{i=1}^{k} T_i^{-d_{m,i}} A\bigg)$$

for all $m \in \mathbb{N}$ and all $d_1 = (d_{1,1}, \ldots, d_{1,k}), \ldots, d_m = (d_{m,1}, \ldots, d_{m,k}) \in \mathbb{Z}^k$.

Based on Theorem 6.1, Proposition 1.11 can be deduced from the following result.

PROPOSITION 6.2. *Let $P_1, \ldots, P_k$ be pairwise independent polynomials with integer coefficients and zero constant terms, and any one of them is not of the form an^d. Let $S, T_1, \ldots, T_k$ be a family of invertible measure-preserving transformations acting on the Lebesgue space $(X, \mathcal{X}, \mu)$ and $T_1, \ldots, T_k$ are commuting. Then, for every $A \in \mathcal{X}$ with $\mu(A) > 0$, there exists a positive constant c, depending only on $\mu(A)$ and $P_1, \ldots, P_k$, such that*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \mu(A \cap T_1^{-P_1(n)} A \cap \cdots \cap T_k^{-P_k(n)} A \cap S^{-\Omega(n)} A) \geq c.$$

Likely, Propositions 1.12 and 1.13 can be deduced from a similar result.

PROPOSITION 6.3. *Let $P_1, \ldots, P_k \in \mathbb{Z}[n]$ with zero constant terms. Let S, T be two invertible measure-preserving transformations acting on the Lebesgue space $(X, \mathcal{X}, \mu)$. Then, for every $A \in \mathcal{X}$ with $\mu(A) > 0$, there exists a positive constant c, depending only on $\mu(A)$ and $P_1, \ldots, P_k$, such that*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \mu(A \cap T^{-P_1(n)} A \cap \cdots \cap T^{-P_k(n)} A \cap S^{-\Omega(n)} A) \geq c.$$

Before proving the above propositions, we introduce a quantitative version of the polynomial Szemerédi theorem.

THEOREM 6.4. [10, Theorem 4.1] *Let $P_1, \ldots, P_k \in \mathbb{Z}[n]$ with zero constant terms. Let $T_1, \ldots, T_k$ be a family of commuting invertible measure-preserving transformations acting on the Lebesgue space $(X, \mathcal{X}, \mu)$. Then, for every $A \in \mathcal{X}$ with $\mu(A) > 0$, there exists a positive constant c, depending only on $\mu(A)$ and $P_1, \ldots, P_k$, such that*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \mu(A \cap T_1^{-P_1(n)} A \cap \cdots \cap T_k^{-P_k(n)} A) \geq c.$$

Now, we are about to prove Proposition 6.2.

Proof of Proposition 6.2. Fix $\delta \in (0, 1)$. Fix a Lebesgue space $(X, \mathcal{X}, \mu)$, a family of commuting invertible measure-preserving transformations $T_1, \ldots, T_k$ acting on it and $A \in \mathcal{X}$ with $\mu(A) = \delta$. Fix an invertible measure-preserving transformation S acting on $(X, \mathcal{X}, \mu)$. By Theorem 6.4, there exists a constant $c(\delta) \in (0, 1)$, depending only on δ and $P_1, \ldots, P_k$, such that

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \mu(A \cap T_1^{-P_1(n)} A \cap \cdots \cap T_k^{-P_k(n)} A) \geq c(\delta). \tag{6.1}$$

Then, by (6.1) and Theorem 5.2,

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \int_X 1_A(x) \cdot \prod_{j=1}^{k} \mathbb{E}_\mu(1_A | \mathcal{Z}_\infty(T_j))(T_j^{P_j(n)} x)\, d\mu(x) \geq c(\delta). \tag{6.2}$$

By Theorem 2.2 and [16, Theorem 14.15], for each $1 \leq i \leq k$, there exists a sequence of functions $\{\phi_{i,j}\}_{j\geq 1}$ such that the following hold:

(1) for any $1 \leq i \leq k$, $j \geq 1$ and μ-a.e. $x \in X$, $\{\phi_{i,j}(T_i^{P_i(n)} x)\}_{n\in\mathbb{Z}}$ is a nilsequence;
(2) for each $1 \leq i \leq k$, $j \geq 1$, $\|\mathbb{E}_\mu(1_A|\mathcal{Z}_\infty(T_i)) - \phi_{i,j}\|_{L^{2k}(\mu)} \leq 1/16^{jk}$;
(3) for each $1 \leq i \leq k$, $j \geq 1$, $0 \leq \phi_{i,j} \leq 1$.

Then, there exists a sufficiently large j_0 such that

$$\lim_{N\to\infty} \bigg\| \frac{1}{N} \sum_{n=1}^{N} \bigg(\prod_{i=1}^{k} \mathbb{E}_\mu(1_A|\mathcal{Z}_\infty(T_i))(T_i^{P_i(n)} x) - \prod_{i=1}^{k} \phi_{i,j_0}(T_i^{P_i(n)} x) \bigg) \bigg\|_{L^2(\mu)} < c(\delta)^3/32. \tag{6.3}$$

Let $X \to \mathcal{M}(X)$, $y \mapsto \mu_y$ be the ergodic decomposition of μ with respect to S. Note that the product of finitely many nilsequences is still a nilsequence. Then, we can define $G : X \to [0, 1]$ by putting

$$G(y) = \int_X 1_A(x) \cdot \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \prod_{i=1}^{k} \phi_{i,j_0}(T_i^{P_i(n)} x)\, d\mu_y(x)$$

for μ-a.e. $y \in X$. Clearly,

$$\int_X G(y)\, d\mu(y) = \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \int_X 1_A(x) \cdot \prod_{i=1}^{k} \phi_{i,j_0}(T_i^{P_i(n)} x)\, d\mu(x).$$

So, by (6.3) and (6.2),

$$\int_X G(y)\, d\mu(y) \geq 7c(\delta)/8.$$

So,

$$\begin{aligned}
7c(\delta)/8 &\leq \int_X G(y)\, d\mu(y) \\
&= \int_{\{y\in X: G(y)>c(\delta)/2\}} G(y)\, d\mu(y) + \int_{\{y\in X: G(y)\leq c(\delta)/2\}} G(y)\, d\mu(y) \\
&\leq \mu(\{y \in X : G(y) > c(\delta)/2\}) + c(\delta)/2. \qquad (6.4)
\end{aligned}$$

By (6.4),

$$\mu(E := \{y \in X : G(y) > c(\delta)/2\}) \geq 3c(\delta)/8. \qquad (6.5)$$

By Theorem 1.9, we know that the limit

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \mu(A \cap T_1^{-P_1(n)} A \cap \cdots \cap T_k^{-P_k(n)} A \cap S^{-\Omega(n)} A)$$

exists.

Next, we begin to estimate the uniform lower bound,

$$\begin{aligned}
&\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \mu(A \cap T_1^{-P_1(n)} A \cap \cdots \cap T_k^{-P_k(n)} A \cap S^{-\Omega(n)} A) \\
&= \lim_{N\to\infty} \int_X 1_A(x) \cdot \bigg(\frac{1}{N} \sum_{n=1}^{N} 1_A(T_1^{P_1(n)} x) \cdots 1_A(T_k^{P_k(n)} x) \cdot 1_A(S^{\Omega(n)} x) \bigg)\, d\mu(x) \\
&= \lim_{N\to\infty} \int_X 1_A(x) \cdot \mu_x(A) \cdot \bigg(\frac{1}{N} \sum_{n=1}^{N} \prod_{j=1}^{k} 1_A(T_j^{P_j(n)} x) \bigg)\, d\mu(x) \ \ ((5.1)) \\
&= \lim_{N\to\infty} \int_X 1_A(x) \cdot \mu_x(A) \cdot \bigg(\frac{1}{N} \sum_{n=1}^{N} \prod_{j=1}^{k} \mathbb{E}_\mu(1_A | \mathcal{Z}_\infty(T_j))(T_j^{P_j(n)} x) \bigg)\, d\mu(x) \ \ (\text{Theorem } 5.2) \\
&\geq \lim_{N\to\infty} \bigg| \frac{1}{N} \sum_{n=1}^{N} \int_X 1_A(x) \cdot \mu_x(A) \cdot \prod_{i=1}^{k} \phi_{i,j_0}(T_i^{P_i(n)} x)\, d\mu(x) \bigg| - \lim_{N\to\infty} \bigg| \int_X 1_A(x) \cdot \mu_x(A) \\
&\quad \times \frac{1}{N} \sum_{n=1}^{N} \bigg(\prod_{i=1}^{k} \mathbb{E}_\mu(1_A | \mathcal{Z}_\infty(T_i))(T_i^{P_i(n)} x) - \prod_{i=1}^{k} \phi_{i,j_0}(T_i^{P_i(n)} x) \bigg)\, d\mu(x) \bigg| \\
&\geq \lim_{N\to\infty} \bigg| \frac{1}{N} \sum_{n=1}^{N} \int_X 1_A(x) \cdot \mu_x(A) \cdot \prod_{i=1}^{k} \phi_{i,j_0}(T_i^{P_i(n)} x)\, d\mu(x) \bigg| - c(\delta)^3/32 \ \ ((6.3)) \\
&= \int_X \mu_y(A) \int_X 1_A(x) \cdot \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \prod_{i=1}^{k} \phi_{i,j_0}(T_i^{P_i(n)} x)\, d\mu_y(x)\, d\mu(y) - c(\delta)^3/32
\end{aligned}$$

https://doi.org/10.1017/etds.2025.10254 Published online by Cambridge University Press

$$= \int_X \mu_y(A)G(y)\, d\mu(y) - c(\delta)^3/32$$
$$\geq \int_E G(y)^2 d\mu(y) - c(\delta)^3/32 \text{ (by the fact that } G(y) \leq \mu_y(A))$$
$$\geq \frac{1}{16} c(\delta)^3. \ ((6.5))$$

This finishes the whole proof. □

The proof of Proposition 6.3 is similar to that of Proposition 6.2. The only difference is that we should use Theorems 1.10, 5.1 and (5.9) in the proof of Proposition 6.3 instead of Theorems 1.9, 5.2 and (5.1).

7. *Some questions*

7.1. *On Proposition 1.11.* Due to those restrictions for polynomials in Theorem 1.9, we cannot give an answer to the following question.

Question 7.1. Fix $k \geq 2$ and $A \subset \mathbb{N}^{k+1}$ with positive upper Banach density. Are there $a \in \mathbb{N}^{k+1}$, $d \in \mathbb{N}$ such that

$$a, a + d\vec{e}_1, \ldots, a + kd\vec{e}_k, a + \Omega(d)\vec{e}_{k+1} \in A\ ?$$

So, we ask a special case of the above question here.

Question 7.2. Fix $k \geq 2$. Is it true that for any finite colouring of $\mathbb{N}^{k+1}$, there are $a \in \mathbb{N}^{k+1}$, $d \in \mathbb{N}$ such that the set

$$\{a, a + d\vec{e}_1, \ldots, a + kd\vec{e}_k, a + \Omega(d)\vec{e}_{k+1}\}$$

is monochromatic?

7.2. *On recurrence times.* As a direct result of Theorem 6.1 and Proposition 6.3, we know that for any $\delta > 0$, there is $c(\delta) > 0$, depending only on δ, such that for any $E \subset \mathbb{N}$ with $d^*(E) = \delta$, then

$$\liminf_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} d^*(E \cap (E - \Omega(n))) \geq c(\delta). \tag{7.1}$$

In [9], by combining some number theory results and a quantitative version of the Roth theorem, Frantzikinakis, Host and Kra proved that for any $E \subset \mathbb{N}$ with positive upper Banach density, there are infinitely many n in $\mathbb{P} - 1(\mathbb{P} + 1)$ such that

$$d^*(E \cap (E - n) \cap (E - 2n)) > 0.$$

Later, Wooley and Ziegler [21] extended it to general polynomials with zero constant terms. Based on these results and (7.1), we expect a similar result for $\Omega(n)$ here.

https://doi.org/10.1017/etds.2025.10254 Published online by Cambridge University Press

Question 7.3. Fix $E \subset \mathbb{N}$ with positive upper Banach density. Are there infinitely many n in $\mathbb{P} - 1(\mathbb{P} + 1)$ such that

$$\Omega(n) \in (E - E)?$$

If Question 7.3 has a positive answer, then by letting E be the arithmetic progressions with infinite length, we have that for each $k \in \mathbb{N}$, there are infinitely many n in $\mathbb{P} - 1(\mathbb{P} + 1)$ such that $k|\Omega(n)$.

Acknowledgements. R.X. is supported by National Natural Science Foundation of China (123B2007,12371196). The initial ideas of the paper first arose in an online seminar held during the winter of 2023. The author's thanks go to Zhengxing Lian for organizing this seminar and the referee for useful remarks and suggestions.

REFERENCES

[1] V. Bergelson and A. Leibman. Polynomial extensions of van der Waerden's and Szemerédi's theorems. *J. Amer. Math. Soc.*, **9**(3) (1996), 725–753.

[2] V. Bergelson and F. K. Richter. Dynamical generalizations of the prime number theorem and disjointness of additive and multiplicative semigroup actions. *Duke Math. J.* **171**(15) (2022), 3133–3200.

[3] J. Bourgain. Double recurrence and almost sure convergence. *J. Reine Angew. Math.* **404** (1990), 140–161.

[4] J. Bourgain, P. Sarnak and T. Ziegler. Disjointness of Moebius from horocycle flows. *From Fourier Analysis and Number Theory to Radon Transforms and Geometry. In Memory of Leon Ehrenpreis*. Ed. H. M. Farkas, R. C. Gunning, M. I. Knopp and B. A. Taylor. Springer, Berlin, 2013, pp. 67–83.

[5] D. Charamaras. Mean value theorems in multiplicative systems and joint ergodicity of additive and multiplicative actions. *Trans. Amer. Math. Soc.* **378**(3) (2025), 1883–1937.

[6] H. Daboussi. Fonctions multiplicatives presque périodiques B. *Astérisque* **24**(25) (1975), 321–324.

[7] M. Einsiedler and T. Ward. *Ergodic Theory. With a View Towards Number Theory (Graduate Texts in Mathematics, 259)*. Springer, London, 2011.

[8] G. Folland. *Real Analysis. Modern Techniques and Their Applications (Pure and Applied Mathematics: A Wiley Series of Texts, Monographs and Tracts)*, 2nd edn. Wiley, New York, NY, 1999.

[9] N. Frantzikinakis, B. Host and B. Kra. Multiple recurrence and convergence for sequences related to the prime numbers. *J. Reine Angew. Math.* **611** (2007), 131–144.

[10] N. Frantzikinakis, B. Host and B. Kra. The polynomial multidimensional Szemerédi theorem along shifted primes. *Israel J. Math.* **194** (2013), 331–348.

[11] N. Frantzikinakis and B. Kuca. Joint ergodicity for commuting transformations and applications to polynomial sequences. *Invent. Math.* **239**(2) (2025), 621–706.

[12] H. Furstenberg. Ergodic behavior of diagonal measures and a theorem of Szemeredi on arithmetic progressions. *J. Anal. Math.* **31** (1977), 204–256.

[13] H. Furstenberg. *Recurrence in Ergodic Theory and Combinatorial Number Theory (Porter Lectures, 14)*. Princeton University Press, Princeton, NJ, 1981.

[14] E. Glasner. *Ergodic Theory via Joinings (Mathematical Surveys and Monographs, 101)*. American Mathematical Society (AMS), Providence, RI, 2003.

[15] B. Host and B. Kra. Convergence of polynomial ergodic averages. *Israel J. Math.* **149** (2005), 1–19.

[16] B. Host and B. Kra. *Nilpotent Structures in Ergodic Theory (Mathematical Surveys and Monographs, 236)*. American Mathematical Society (AMS), Providence, RI, 2018.

[17] I. Kátai. A remark on a theorem of H. Daboussi. *Acta Math. Hungar.* **47** (1986), 223–225.

[18] A. Leibman. Convergence of multiple ergodic averages along polynomials of several variables. *Israel J. Math.* **146** (2005), 303–315.

[19] K. Loyd. A dynamical approach to the asymptotic behavior of the sequence $\Omega(n)$. *Ergod. Th. & Dynam. Sys.* **43**(11) (2023), 3685–3706.

[20] M. N. Walsh. Norm convergence of nilpotent ergodic averages. *Ann. of Math. (2)* **175**(3) (2012), 1667–1688.

[21] T. Wooley and T. Ziegler. Multiple recurrence and convergence along the primes. *Amer. J. Math.* **134**(6) (2012), 1705–1732.