THE LAPLACE TRANSFORM OF THE MODIFIED BESSEL FUNCTION $K(t^{\pm m}x)$ WHERE $m=1, 2, 3, \ldots, n$

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1. Introduction

In the present paper we determine the Laplace transforms of the modified Bessel function of the second kind $K_n(t^{\pm m}x)$, where m is any positive integer. The Laplace transforms are given in (2) and (4) below, p being the transform parameter and having positive real part.

The formulæ to be established are as follows (1)-(4):

$$\int_0^\infty e^{-t}t^{k-1}K_n(t^m x)dt = 2^{m-\frac{3}{2}}\pi^{m-\frac{1}{2}}$$

$$\sum_{v=0}^{2m-1}\left\{(-1)^v(2m)^{-\frac{1}{2}-v}\left(\frac{x^2}{4}\right)^{-\frac{k+v}{2m}}\frac{\Gamma\left(\frac{n}{2}+\frac{k+v}{2m}\right)\Gamma\left(-\frac{n}{2}+\frac{k+v}{2m}\right)}{\Gamma\left(\frac{v+1}{2m}\right)\Gamma\left(\frac{v+2}{2m}\right)\ldots\Gamma\left(\frac{v+2m}{2m}\right)}\right.$$

$$\left.{}_2F_{2m-1}\left[\begin{matrix}\frac{n}{2}+\frac{k+v}{2m}, -\frac{n}{2}+\frac{k+v}{2m}; \frac{1}{(2m)^{2m}x^2}\\ \frac{v+1}{2m}, \frac{v+2}{2m}, \ldots*\ldots, \frac{v+2m}{2m}\end{matrix}\right]\right\}, \qquad \ldots\ldots\ldots\ldots(1)$$

where $R(k \pm nm) > 0$ and x is taken for simplicity to be real and positive. When $m = 1$, x may be taken to be complex with real part greater than 1. The asterisk in the generalised hypergeometric function denotes that the factor $\frac{2m}{2m}$ in the parameters $\frac{v+1}{2m}, \frac{v+2}{2m}, \ldots, \frac{v+2m}{2m}$ is omitted; m is a positive integer.

$$\int_0^\infty e^{-pt}K_n(t^m x)dt = 2^{m-\frac{3}{2}}\pi^{m-\frac{1}{2}}p^{-1}$$

$$\sum_{v=0}^{2m-1}\left\{(-1)^v(2m)^{-\frac{1}{2}-v}\left(\frac{x^2}{4p^{2m}}\right)^{-\frac{v+1}{2m}}\frac{\Gamma\left(\frac{n}{2}+\frac{v+1}{2m}\right)\Gamma\left(-\frac{n}{2}+\frac{v+1}{2m}\right)}{\Gamma\left(\frac{v+1}{2m}\right)\Gamma\left(\frac{v+2}{2m}\right)\ldots\Gamma\left(\frac{v+2m}{2m}\right)}\right.$$

$$\left.{}_2F_{2m+1}\left[\begin{matrix}\frac{n}{2}+\frac{v+1}{2m}, -\frac{n}{2}+\frac{v+1}{2m}; \frac{4p^{2m}}{(2m)^{2m}x^2}\\ \frac{v+1}{2m}, \frac{v+2}{2m}, \ldots*\ldots, \frac{v+2m}{2m}\end{matrix}\right]\right\}, \qquad \ldots\ldots\ldots\ldots(2)$$

where m is a positive integer, $R(\pm mn+1)>0$ and $R(p)>0$; x is taken to be rea and positive and the asterisk has the same meaning as before.

$$\int_0^\infty e^{-\lambda}\lambda^{k-1}K_n\left(\frac{x}{\lambda^m}\right)d\lambda = 2^{-k-m-2}m^{k-\frac{1}{2}}\pi^{-m-\frac{1}{2}}$$

$$\sum_{i,-i}\frac{1}{i}E\left(1, \frac{n}{2}, -\frac{n}{2}, \frac{k}{2m}, \frac{k+1}{2m}, \ldots, \frac{k+2m-1}{2m} :: \frac{e^{i\pi}x^2}{4(2m)^{2m}}\right), \qquad \ldots\ldots\ldots\ldots(3)$$

where $R(x)>0$, $R(k)>0$ and m is any positive integer.

$$\int_0^\infty e^{-pt}K_n\left(\frac{x}{t^m}\right)dt = 2^{-m-1}m^{\frac{1}{2}}\pi^{-m-\frac{1}{2}}p^{-1}$$

$$\sum_{i,-i}\frac{1}{i}E\left(1, 1, \frac{n}{2}, -\frac{n}{2}, \frac{1}{2m}, \frac{2}{2m}, \ldots, \frac{2m-1}{2m} :: \frac{e^{i\pi}x^2p^{2m}}{4(2m)^{2m}}\right), \qquad \ldots\ldots\ldots\ldots\ldots(4)$$

where $R(p)>0$ m is any positive integer and x is real and positive. In (3) and (4) $\sum$ means that in the expression following it i is to be replaced by $-i$ and the two expressions added.

The function appearing on the right of (3) and (4) is MacRobert's E-function whose definitions and properties are to be found in ((**1**), pp. 348-358).

These formulæ will be proved in section **3** by means of a subsidiary formula which will be proved in section **2**. The following formulæ are required in the proofs ((**2**), p. 77):

$$\int_0^\infty e^{-\lambda}\lambda^{k-1}E(p;\ \alpha_r:\ q;\ \rho_s:\ \lambda^{2m}z)d\lambda = \pi \operatorname{cosec}(k\pi)(2\pi)^{m-\frac{1}{2}}(2m)^{k-\frac{1}{2}}$$

$$E\left[\begin{matrix}\alpha_1, \alpha_2, \ldots, \alpha_p;\ e^{\pm 2im\pi}(2m)^{2m}z \\ 1-\frac{k}{2m}, 1-\frac{k+1}{2m}, \ldots, 1-\frac{k+2m-1}{2m}, \rho_1, \rho_2, \ldots, \rho_q\end{matrix}\right]$$

$$+2^{\frac{1}{2}-m}\pi^{\frac{1}{2}+m}\sum_{\nu=0}^{2m-1}\left\{(-1)^{\nu+1}(2m)^{-\frac{1}{2}-\nu}\operatorname{cosec}\left(\frac{k+\nu}{2m}\right)\pi\right.$$

$$\left.E\left[\begin{matrix}\alpha_1+\frac{k+\nu}{2m}, \ldots, \alpha_p+\frac{k+\nu}{2m}:\ e^{\pm 2im\pi}(2m)^{2m}z \\ \frac{\nu+1}{2m}, \ldots * \ldots, \frac{\nu+2m}{2m}, 1+\frac{k+\nu}{2m}, \rho_1+\frac{k+\nu}{2m}, \ldots, \rho_q+\frac{k+\nu}{2m}\end{matrix}\right]\right\}, \qquad \ldots\ldots(5)$$

where $p \geqq q+1$, m is a positive integer, $R(m\alpha_r+k)>0$, $r = 1, 2, 3, \ldots, p$, $|\operatorname{amp} z|<\pi$ and the asterisk denotes that the factor $\frac{2m}{2m}$ in the parameters $\frac{\nu+1}{2m}, \frac{\nu+2}{2m}, \ldots, \frac{\nu+2m}{2m}$ is omitted. For other values of p and q the formula holds if the integral is convergent. ((1), p. 406):

$$\int_0^\infty e^{-\lambda}\lambda^{k-1}E(p;\ \alpha_r:\ q;\ \rho_s:\ z/\lambda^{2m})d\lambda$$

$$(2\pi)^{\frac{1}{2}-m}(2m)^{k-\frac{1}{2}}E\left(p+2m;\ \alpha_r:\ q;\ \rho_s:\ \frac{z}{(2m)^{2m}}\right), \qquad \ldots\ldots\ldots\ldots(6)$$

where m is any positive integer, $R(k)>0$, $\alpha_{p+v}=\dfrac{k+v-1}{2m}$, $v=1, 2, 3, \ldots, 2m$.

((1), p. 352):

$$E(p;\ \alpha_r:\ q;\ \rho_s:\ z)=\sum_{r=1}^{p}\prod_{s=1}^{p}{}'\ \Gamma(\alpha_s-\alpha_r)\{\prod\Gamma(\rho_t-\alpha_r)\}^{-1}\Gamma(\alpha_r)$$

$$z^{\alpha_r}{}_{q+1}F_{p-1}\begin{bmatrix}\alpha_r,\ \alpha_r-\rho_1+1,\ \ldots,\ \alpha_r-\alpha_p+1:\ (-1)^{p-q}z\\ \alpha_r-\alpha_1+1,\ \alpha_r-\alpha_2+1,\ \ldots*\ldots,\ \alpha_r-\alpha_p+1\end{bmatrix}, \qquad \ldots\ldots\ldots\ldots(7)$$

where $p\geqq q+1$ and $r=1, 2, \ldots, p$. The prime in the product sign signifies that the factor for which s is equal to r is omitted.

$$E(p;\ \alpha_r:\ q;\ p_s:\ z)=\frac{\Gamma(\alpha_1)\ldots\Gamma(\alpha_p)}{\Gamma(\rho_1)\ldots\Gamma(\rho_q)}{}_pF_q\left(\begin{matrix}\alpha_1,\ \ldots,\ \alpha p;\ -\dfrac{1}{z}\\ \rho_1,\ \ldots,\ \rho_q\end{matrix}\right), \qquad \ldots\ldots\ldots(8)$$

where $p\leqq q$.

$$\Gamma(z)\Gamma(1-z)=\pi\ cosec\ \pi z. \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(9)$$

2. The subsidiary formula

The formula to be proved is

$$K_n(x)=\frac{1}{4\pi}\sum_{i,-i}\frac{1}{i}E\left(1,\frac{n}{2},\ -\frac{n}{2}::\frac{d^{i\pi}x^2}{4}\right), \qquad \ldots\ldots\ldots\ldots\ldots\ldots(10)$$

where $K_n(x)$ is the modified Bessel function of the second kind and $\sum_{i,-i}$ has the same meaning as in (4).

To prove (10) expand each E function by means of (7) and combine the two resulting expressions by omitting common terms; then applying (9) the right side of (10) becomes

$$\frac{\pi}{2\sin n\pi}\left[\{\Gamma(1-n)\}^{-1}\left(\frac{x}{2}\right)^{-n}{}_0F_1\left(;\ 1-n;\ \frac{x^2}{4}\right)\right.$$
$$\left.-\{\Gamma(1+n)\}^{-1}\left(\frac{x}{2}\right)^{n}{}_0F_1\left(;\ 1+n;\ \frac{x^2}{4}\right)\right]=K_n(x),$$

and the formula is proved.

3. Proofs

In (5) take $q=0$, $p=3$ with $\alpha_1=1$, $\alpha_2=\frac{1}{2}n$, $\alpha_3=-\frac{1}{2}n$, write $\dfrac{e^{i\pi}x^2}{4}$

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for z, then $\frac{e^{-i\pi}x^2}{4}$ for z, apply (5) twice making use of (10) and get

$$\int_0^\infty e^{-t}t^{k-1}\sum_{i,-i}\frac{1}{i}E\left(1,\frac{n}{2},-\frac{n}{2}::\frac{e^{i\pi}t^{2m}x^2}{4}\right)dt=\pi\operatorname{cosec}k\pi(2\pi)^{m-\frac{1}{2}}(2m)^{k-\frac{1}{2}}$$

$$\sum_{i,-i}\frac{1}{i}E\left[\begin{matrix}1,\frac{n}{2},-\frac{n}{2}:e^{\pm(2im\pi)}(2m)^{2m}\frac{e^{i\pi}x^2}{4}\\ 1-\frac{k}{2m},1-\frac{k+1}{2m},\ldots,1-\frac{k+2m-1}{2m}\end{matrix}\right];$$

$$+2^{\frac{1}{2}-m}\pi^{\frac{1}{2}+m}\sum_{\nu=0}^{2m-1}\sum_{i,-i}\left\{\frac{1}{i}(-1)^{\nu+1}\frac{(2m)^{-\frac{1}{2}-\nu}2^{2m-1}}{\sin\left(\frac{k+\nu}{2m}\right)\pi}\left(\frac{e^{i\pi}x^2}{4}\right)^{-\frac{k+\nu}{2m}}\right.$$

$$\left.E\left[\begin{matrix}1+\frac{k+\nu}{2m},\frac{n}{2}+\frac{k+\nu}{2m},-\frac{n}{2}+\frac{k+\nu}{2m}:e^{2im\pi}(2m)^{2m}\frac{e^{i\pi}x^2}{4}\\ \frac{\nu+1}{2m},\ldots*\ldots,\frac{\nu+2m}{2m},1+\frac{k+\nu}{2m}\end{matrix}\right]\right\},$$

where $m=1, 2, 3, \ldots$. Now change each E-function to a generalised hypergeometric function by means of (8), noting that the first two series cancel, apply (9) and (10) and so obtain (1).

(2) can be deduced from (1) by writing pt for t and taking $k=1$, then writing $\frac{x}{pm}$ for x.

Proofs of (3) *and* (4): To prove (3), apply formula (6) with $\frac{e^{\pm i\pi}x^2}{4}$ for z, taking $p=3$, $q=0$ with $\alpha_1=1$, $\alpha_2=\frac{1}{2}n$, $\alpha_2=-\frac{1}{2}n$. Combine the two results using (10) and so obtain (3).

Formula (4) can easily be deduced from (3) by taking $k=1$ and writing xp^m for x. It may be noted that the first two series which are obtained by expanding any of the two E functions appearing on the right of (4) by means of (7) cease to exist because $\alpha_1=\alpha_2=1$. These two series are replaced (see (3), p. 30) by

$$z\sum_{\lambda=0}^{\infty}\frac{\Gamma\left(\frac{n}{2}-\lambda-1\right)\Gamma\left(-\frac{n}{2}-\lambda-1\right)\prod_{\nu=1}^{2m-1}\Gamma\left(\frac{\nu}{2m}-\lambda-1\right)}{\lambda!}\Delta_\lambda z^\lambda,$$

where

$$\Delta_\lambda=\psi(\lambda)-\log z+\sum_{\nu=1}^{2m-1}\psi\left(\frac{\nu}{2m}-\lambda-2\right)+\psi\left(\frac{n}{2}-\lambda-2\right)+\psi\left(-\frac{n}{2}-\lambda-2\right).$$

Here

$$\psi(z) = \frac{d}{dz}\log\Gamma(z+1)$$

and

$$z = \frac{e^{\pm i\pi}x^2}{4(2m)^{2m}}.$$

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