ON PROPERTIES POSSESSED BY SOLVABLE AND NILPOTENT GROUPS

CHRISTINE AYOUB [1]

(Received 5 June 1967)

The object of this note is to study two properties of groups, which we will denote by (*) and (**). The property (*) is possessed by solvable groups (and in fact, by groups which have a solvable invariant system) and the property (**) is possessed by nilpotent groups (and in fact, by groups which have a central system).

It is quite easy to show that if a group satisfies (*) locally, then it satisfies (*); this gives a short proof of Malcev's theorem that a locally solvable group cannot be simple unless it is cyclic of prime order. It should be remarked, however, that the proof given is simply an adaption of Malcev's proof — its only virtue is that it is short and easy.

Theorem 2 states that a finitely generated group G satisfying (*) and the minimum condition for normal subgroups is finite and solvable, and Theorem 3 studies the connection between property (*) and a property studied by Ore.

Theorem 5 states that if the group G — with hypercentre C — satisfies (**), then G/C satisfies (**); from this we deduce that if G satisfies (**) and the minimum condition for normal subgroups, G is hypercentral.

Notations

$[a, b] \quad = a^{-1}b^{-1}ab$.
$n(U) \quad$ = normalizer of the subgroup U in G.
$Z(G) \quad$ = centre of the group G.
$A \leqq B: = :$ A is a subgroup of B.
$A < B: = :$ A is a proper subgroup of B.
$A \lhd B: = :$ A is a normal subgroup of B.
$E \quad$ = trivial subgroup (consisting of the identity element).

Following Kurosh we call G an SI-group (SN-group) if it has an invariant (normal) system with abelian factors (see Kurosh [5, p. 171—73

[1] This paper was written while the author held an NSF Science Faculty Fellowship.

and p. 182]), and we call G a Z-group if it has a central system — see Kurosh [5, p. 218]. We say that G is a ZA-group if the upper central chain for G, possibly continued transfinitely, leads up to G — see Kurosh [5, p. 218—19]. (Baer calls such a group hypercentral and uses the equivalent definition that G is hypercentral if every epimorphic image ($\neq E$) has a non-trivial centre.) G is an SI^*-group if it has a solvable ascending invariant series (this is what Baer calls hyperabelian; again an equivalent definition is that the group G is hyperabelian if every epimorphic image ($\neq E$) has a non-trivial normal abelian subgroup).

The property (*)

DEFINITION 1. The group G satisfies (*) if: given elements a, b ($\neq 1, 1$) in G, there is a normal subgroup $C = C(a, b)$ of G such that $[a, b]$ is in C but not both a and b are in C.

REMARK. If G satisfies (*), and a, b ($\neq 1, 1$) are elements of G, we can define

$$C_{a,b} = \{\cap C | C \lhd G, [a, b] \in C \text{ and not both } a \text{ and } b \text{ are in } C\}.$$

Clearly $C_{a,b}$ is normal in G, $[a, b]$ is in $C_{a,b}$ but not both a and b are in $C_{a,b}$. $C_{a,b}$ is the unique smallest normal subgroup of G with these properties.

LEMMA 1. (i) *If S is a subgroup of the group G and if G satisfies* (*), *then S satisfies* (*).

(ii) *If N is a normal subgroup of the group G and if G satisfies* (*), *then given elements $a, b(\neq 1, 1)$ in N there exists a normal subgroup C of G such that $C < N$, $[a, b] \in C$ and not both a and b are in C.*

Thus if G has a local system each of whose subgroups satisfies (*), the finitely generated subgroups of G satisfy (*).

PROPOSITION 1. *If G is an SI-group, then G satisfies* (*). *In particular, if G is solvable, G satisfies* (*).

PROOF. Let Σ be an invariant system for G with abelian factors. Let a, b ($\neq 1, 1$) be any two elements of G and define

$$\bar{C} = \{\cap N | N \in \Sigma, a \text{ and } b \text{ both} \in N\}, \text{ and}$$
$$C = \{\cup K | K \in \Sigma, \text{not both } a \text{ and } b \in K\}.$$

Then $C < \bar{C}$ is a jump in Σ; hence $\bar{C}/C$ is abelian so that $[a, b] \in C$. Clearly C is a normal subgroup of G and not both a and b belong to C.

PROPOSITION 2. *Let G be a group and assume that for each pair of elements $a, b(\neq 1, 1)$ a normal subgroup $C_{a,b}$ can be chosen so that $[a, b] \in C_{a,b}$, but not both a and b are in $C_{a,b}$ and that in addition these subgroups can be chosen* *in such a way that for* $a, b(\neq 1, 1), c, d(\neq 1, 1)$ *in* G *either* $C_{a,b} \leqq C_{c,d}$ *or* $C_{c,d} \leqq C_{a,b}$ *(i.e. in such a way that the subgroups are linearly ordered). Then* G *is an SI-group.*

PROOF. Complete the system of normal subgroups $\{C_{a,b}\}$ to a system Σ. We show that if $K < L$ is a jump in Σ, then L/K is abelian. For suppose not; then there are elements a and b in L with $[a, b]$ not in K. Now if $L \leqq C_{a,b}$, a and b both lie in $C_{a,b}$, which is impossible. Hence $C_{a,b} < L$, which implies that $C_{a,b} \leqq K$. But then $[a, b] \in K$, a contradiction.

THEOREM 1. *If the group* G *satisfies* (*) *locally, then* G *satisfies* (*).

PROOF. Let Σ consist of all finitely generated subgroups of G. For A in Σ and $a, b(\neq 1, 1)$ in A let $C_{a,b}(A)$ be a fixed normal subgroup of A such that a and b are not both in $C_{a,b}(A)$ but $[a, b] \in C_{a,b}(A)$.

For $a, b(\neq 1, 1)$ in G and S a finite subset of G define

$$K_{a,b}(S) = \{\cap C_{a,b}(A) | A \in \Sigma, \{a, b, S\} \subseteqq A\}.$$

Clearly if $S_1 \subseteqq S_2$ are finite subsets of $G, K_{a,b}(S_1) \leqq K_{a,b}(S_2)$. Thus for arbitrary finite subsets S_1 and S_2 of $G, K_{a,b}(S_i) \leqq K_{a,b}(S_1 \cup S_2)$ for $i = 1, 2$.

Let $H_{a,b} = \{\cup K_{a,b}(S) | S$ a finite subset of $G\}$. It is clear that $H_{a,b}$ is a subgroup of G which contains $[a, b]$ but does not contain both a and b. It remains to verify that $H_{a,b}$ is normal in G. So let $c \in H_{a,b}$ and $d \in G$. Then $c \in K_{a,b}(S)$ for some finite subset S of G and we can assume that $d \in S$. Now $c \in C_{a,b}(A)$ for each A in Σ with $\{a, b, S\} \subseteqq A$. Hence by the normality of $C_{a,b}(A)$ in A, $d^{-1}cd$ is in $C_{a,b}(A)$ for each A in Σ with $\{a, b, S\} \subseteqq A$. Hence $d^{-1} cd \in K_{a,b}(S)$ and this implies that $d^{-1} cd \in H_{a,b}$.

COROLLARY 1. *If* G *is locally solvable and not cyclic of prime order, then* G *is not simple.*

As noted in the introduction the proof of Theorem 1 is just Malcev's proof adapted to the case considered. Malcev's Theorem states that if a group has the property SI locally then it is an SI-group. For a proof see Kurosh [5, p. 183—87].

DEFINITION 2. Let V be a maximal normal subgroup of the group U; then U/V is a tor of U.

LEMMA 2. *Let* G *be a group which satisfies* (*) *and the minimum condition for normal subgroups. Then if* K *is a normal subgroup of* G*, any tor of* K *is abelian.*

PROOF. Assume that the lemma is false and let U be a minimal normal subgroup of G with a non-abelian tor.[2] Hence there exists $V \triangleleft U$ such that

[2] i.e. U is a normal subgroup of G, has a non-abelian tor and is minimal with respect to this property.

U/V is simple non-abelian. Thus there exist elements a and b in U such that $[a, b] \notin V$. Let C be a normal subgroup of G such that $C < U$, $[a, b] \in C$ and not both a and $b \in C$. Then $V \leqq VC \leqq U$ and $V \neq VC$ since $[a, b] \in C$ but $[a, b] \notin V$. Hence by the maximality of V, $U = VC$.

Now $U/V = VC/V \cong C/V \cap C$. Thus C is a normal subgroup of G with a non-abelian tor and $C < U$. This contradicts the minimality of U.

THEOREM 2. *Let G be a finitely generated group which satisfies* (*) *and the minimum condition for normal subgroups. Then G is a finite, solvable group.*

PROOF. Let K be a normal subgroup of G and assume K is minimal such that G/K is finite and solvable. Assume $K \neq E$. Then since K is finitely generated, it possesses a maximal normal subgroup M. By Lemma 2, K/M is abelian and hence cyclic of prime order. Let $\bar{M} = \{\cap M^x | x \in G\}$. Since M is of finite index in G, $\bar{M}$ is also of finite index in G. Furthermore, $\bar{M}$ is normal in G and $G/\bar{M}$ is solvable since $K/\bar{M}$ is solvable. But $\bar{M} < K$ so that the minimality of K is contradicted. Hence $K = E$ and G is finite and solvable.

COROLLARY 2. *Let G be a group which satisfies* (*) *and the minimum condition for subgroups U such that $n(U) > U$. Then G is locally finite and locally solvable. Furthermore, G is an SI^*-group.*

PROOF. If H is a finitely generated subgroup of G, H satisfies (*) and the minimum condition for normal subgroups. Hence H is finite and solvable. In particular, H is an SI-group. By the local theorem for SI-groups, G is an SI-group and by the minimum condition for normal subgroups, G is an SI^*-group.

We now consider a property which Kurosh denotes by (**Q**), and a somewhat weaker one which will be denoted by (**Q′**). The property (**Q**) was introduced by Ore (see Kurosh [5, p. 181] and Ore [7, p. 251, Theorem 9]).

DEFINITION 3. The subgroup A of G is almost normal in G if there exists a normal subgroup N of G such that $G = AN$ and $A \cap N \lhd G$.

DEFINITION 4. The group G satisfies (**Q**) if $A < B \leqq G$, and A maximal in B, implies that A is almost normal in B.

DEFINITION 5. The group G satisfies (**Q′**) if $A < B \leqq G$, and A maximal in B, implies that either $A \lhd B$, or there exists a proper normal subgroup N of B such that $B = AN$.

It is clear that if G satisfies (**Q**), it satisfies (**Q′**).

THEOREM 3. (i) *If the group G satisfies* (*), *it satisfies* (**Q′**).

(ii) *If the group G satisfies* (*) *and the minimum condition for subgroups U such that $n(U) > U$, then G satisfies* (**Q**).

(iii) *If the group G satisfies* ($\boldsymbol{Q'}$) *and the minimum condition for subgroups, then G satisfies* (*).

Proof. (i): Let $A < B \leqq G$ with A maximal in B. If A is not normal in B, let a and b be elements of B with $[a, b]$ not in A. By (*) there is a subgroup $C \lhd B$ which does not contain both a and b but which contains $[a, b]$. Then $A \leqq AC \leqq B$ but $C \nleqq A$. Hence by the maximality of A, $AC = B$.

(ii): By Corollary 2, G is an SI^*-group and from this fact it follows that G satisfies ($\boldsymbol{Q}$) (see Kurosh [5, p. 183]). However, it is easy to give a proof which does not use the local theorem for SI-groups (which is needed for Corollary 2): Let $A < B \leqq G$ with A maximal in B. Since the normal subgroups of B satisfy the minimum condition, we can choose a minimal subgroup K such that $K \lhd B$ and $B = AK$. Now $A \cap K \lhd A$; if $A \cap K \lhd K$, then $A \cap K \lhd B$. So assume that $A \cap K$ is not normal in K and let a and b be elements K such that $[a, b] \notin A \cap K$. By (*) there exists a subgroup C of K such that $C \lhd B$, $[a, b] \in C$, but not both a and b are in C. Hence $A < AC \leqq B$ since $[a, b] \notin A$. Thus $B = AC$ and the minimality of K is contradicted.

(iii): Assume that the group G satisfies the hypotheses of (iii) but does not satisfy (*). Let H be a minimal subgroup of G which does not satisfy (*). If H is not finitely generated, all the finitely generated subgroups of H satisfy (*); but this implies that H satisfies (*) by Theorem 1. Hence H is finitely generated.

If H contains a maximal subgroup M which is normal, then H/M is cyclic of prime order. Hence M is finitely generated and satisfies (*) by the minimality of H. Therefore, by Theorem 2, M is (finite and) solvable. But this implies that H is solvable so that by Proposition 1, H satisfies (*) — a contradiction.

So assume that every maximal subgroup of H is not normal and let A be a maximal subgroup of H. Then by ($\boldsymbol{Q'}$) there is a proper normal subgroup N of H such that $H = NA$. Let M be a maximal normal subgroup of H containing N. Then $H = MA$. H/M is simple and non-abelian. Also $H/M = MA/M \cong A/M \cap A$ so that A has a non-abelian tor. But A satisfies (*) since it is a proper subgroup of H, and hence by Lemma 2, any tor of A is abelian. Thus we have a contradiction and the theorem is proved.

Corollary 3. *Let G be a group which satisfies the minimum condition for subgroups. Then the following are equivalent*:

(1) *G is solvable.*
(2) *G satisfies* (*).
(3) *G satisfies* ($\boldsymbol{Q}$).
(4) *G satisfies* ($\boldsymbol{Q'}$).

https://doi.org/10.1017/S1446788700005796 Published online by Cambridge University Press

PROOF. By Proposition 1, (1) implies (2). (2) implies (3) by Theorem 3 (ii). Clearly (3) implies (4). So assume (4). Then by Theorem 3 (iii) G satisfies (*). Hence by Corollary 2, G is an SI^*-group. Therefore, by a theorem of Cernikov, G is solvable (see Kurosh [5, p. 191]).

REMARK: Since submitting this paper it has been drawn to my attention that Baer has two papers to appear shortly ([1] and [2]) in which he considers among other things the properties (Q) and (Q'). The main theorem of [1] gives a number of criteria for a group G to be artinian and solvable. One of these is:

(a) Abelian subgroups of G are artinian.
(V) (b) If F is a finitely generated subgroup of G, then
(b′) the normal subgroups of F satisfy the minimum condition
and (b″) if S is a maximal subgroup of F, S is almost normal in F.

This criterion implies that if G is artinian, then G is solvable if, and only if G satisfies (Q). But, of course, it is a much stronger result.

In the same spirit we could prove: G is artinian and solvable if, and only if

(a) Abelian subgroups of G are artinian.
(b) If F is a finitely generated subgroup of G, then
(b′) the normal subgroups of F satisfy the minimum condition
and (b‴) F satisfies (*).

This follows from our Theorem 2 and the theorem of Cernikov (see [4]) which states: Let G be locally finite and locally solvable. Then if abelian subgroups of G are artinian, G is artinian and solvable.

In Baer's paper 'Normalizatorreiche Gruppen' there is another proof of the fact that an artinian group G is solvable if, and only if it satisfies (Q') (see [2] Hilfsatz 3.6).

The property (**)

DEFINITION 6. The group G satisfies (**) if: given an element $a(\neq 1)$ in G, there is a normal subgroup $N = N(a)$ of G such that $[a, x] \in N \ \forall x \in G$ but $a \notin N$.

REMARK. If G satisfies (**) and $a(\neq 1)$ is an element of G, we can define $N_a = \{\cap N | N \lhd G, a \notin N \text{ and } [a, x] \in N \ \forall x \in G\}$ then $N_a \lhd G$, $a \notin N_a$ and $[a, x] \in N_a$. N_a is the unique smallest normal subgroup of G with these properties.

As in the case of (*) we have:

LEMMA 3. (i) *If S is a subgroup of the group G, and if G satisfies* (**), *then S satisfies* (**).

https://doi.org/10.1017/S1446788700005796 Published online by Cambridge University Press

(ii) *If K is a normal subgroup of the group G, and if G satisfies* (**), *then given an element $a(\neq 1)$ in K there exists a normal subgroup N of G such that $N < K$, $a \notin N$ but $[a, x] \in N \; \forall x \in G$.*

It is clear that (**) implies (*). For if $a, b(\neq 1, 1)$ are elements of the group G, then if $a \neq 1$ we can find a normal subgroup N of G such that $a \notin N$ but $[a, x] \in N$ for all $x \in G$. Thus $[a, b] \in N$ but not both a and b are in N. If $a = 1$, $b \neq 1$ and we interchange the rôles of a and b.

Proposition 3. *If G is a Z-group, then G satisfies* (**). *In particular, if G is nilpotent, G satisfies* (**).

Proof. Let Σ be a central system for G. Let $a(\neq 1)$ be an element of G and define

$$\bar{N} = \{\cap K | K \in \Sigma, a \in K\}$$
$$N = \{\cup L | L \in \Sigma, a \notin L\}$$

Then $N < \bar{N}$ is a jump in Σ; hence $\bar{N}/N \leqq Z(G/N)$ and this implies that $[a, x] \in N \; \forall x \in G$.

Proposition 4. *Let G be a group and assume that for each element $a(\neq 1)$ a normal subgroup N_a can be chosen so that $a \notin N_a$ but $[a, x] \in N_a \; \forall x \in G$ and that in addition these subgroups are linearly ordered. Then G is a Z-group.*

The proof of this proposition is quite similar to the proof of Proposition 2 and will be omitted.

Theorem 4. *If the group G satisfies* (**) *locally, then G satisfies* (**).

Proof. Let Σ consist of all finitely generated subgroups of G. For H in Σ and $a(\neq 1)$ in H let $N_a(H)$ be a fixed normal subgroup of H such that $a \notin N_a(H)$ but $[a, x) \in N_a(H) \; \forall x \in H$.

For $a(\neq 1)$ in G and S a finite subset of G containing a, define $K_a(S) = \{\cap N_a(H) | H \in \Sigma, S \subseteqq H\}$. Let $K_a = \{\cup K_a(S) | S$ a finite subset of G containing $a\}$. It is easy to verify that K_a is a normal subgroup of G such that $a \notin K_a$ but $[a, x] \in K_a \; \forall x \in G$.

Lemma 4. *Let G be a group which satisfies* (**) *and Z a subgroup of the centre of G. Then G/Z satisfies* (**).

Proof. Let $a \in G \setminus Z$ and let N_a be the minimal normal subgroup of G such that $a \notin N_a$ but $[a, x] \in N_a$, $\forall x \in G$. Then $ZN_a/Z \lhd G/Z$ and $[Za, Zx] \in ZN_a/Z$, $\forall x \in G$. We have to verify that $Za \notin ZN_a/Z$.

So suppose that $a \in ZN_a$. Then we can write: $a = zc$, where $z \in Z$ and $c \in N_a$.

Now let N_c be a normal subgroup of G such that $c \notin N_c$, but $[c, x] \in N_c \; \forall x \in G$. Then $N_c \cap N_a \lhd G$ and $N_c \cap N_a < N_a$ since $c \notin N_c$ but $c \in N_a$. Clearly $a \notin N_c \cap N_a$ since $a \notin N_a$. For any element $x \in G$, we have:

https://doi.org/10.1017/S1446788700005796 Published online by Cambridge University Press

$$[a, x] = [zc, x] = [z, x]^c[c, x]$$
$$= [c, x] \text{ since } z \text{ is a central element.}$$

Hence $[a, x] \in N_a \cap N_c$, and the minimality of N_a is contradicted. Thus $a \notin ZN_a$ so that $Za \notin ZN_a/Z$.

THEOREM 5. *If the group G satisfies* (**) *and if C is the hypercentre of G, then G/C satisfies* (**).

PROOF. We define the ascending central chain of G by $Z_0 = E$, $Z_1 = Z(G), \cdots, Z_{\alpha+1}/Z_\alpha = Z(G/Z_\alpha)$ and $Z_\alpha = \{\cup Z_\beta | \beta < \alpha\}$ for α a limit ordinal. Then there is an ordinal ν such that $Z_\nu = Z_{\nu+1}$. $C = Z_\nu$ is the hypercentre of G.

We prove by transfinite induction that each G/Z_α satisfies (**). Clearly G/Z_0 satisfies (**).

CASE 1. $\alpha = \beta+1$, and G/Z_β satisfies (**). Then

$$G/Z_\alpha \cong \frac{G/Z_\beta}{Z_{\beta+1}/Z_\beta} = \frac{G/Z_\beta}{Z(G/Z_\beta)}$$

satisfies (**) by Lemma 4.

CASE 2. α is a limit ordinal, and G/Z_β satisfies (**) for $\beta < \alpha$. Thus if $a \in G \setminus Z_\beta$, there exists $U/Z_\beta \lhd G/Z_\beta$ such that $a \notin U$ but $[a, x] \in U$ for all x in G. Hence $Z_\beta \leqq U \lhd G, a \notin U$ but $[a, x] \in U$ for all x in G. Let

$$V_\beta(a) = \{\cap U | Z_\beta \leqq U \lhd G, a \notin U, [a, x] \in U \ \forall x \in G\}.$$

Then $Z_\beta \leqq V_\beta(a) \rhd G, a \notin V_\beta(a)$ and $[a, x] \in V_\beta(a) \ \forall x \in G$, and $V_\beta(a)$ is the unique minimal subgroup of G with these properties.

We verify that if $\beta \leqq \gamma < \alpha$ and $a \in G \setminus Z_\gamma$, then $V_\beta(a) \leqq V_\gamma(a)$. For $Z_\beta \leqq Z_\gamma \leqq V_\gamma(a)$, $a \notin V_\gamma(a)$ and $[a, x] \in V_\gamma(a) \ \forall x \in G$. Hence by the minimality of $V_\beta(a)$, $V_\beta(a) \leqq V_\gamma(a)$.

Now let $a \in G \setminus Z_\alpha$. Then $a \in G \setminus Z_\beta$ for all $\beta < \alpha$. Define $V(a) = \{\cup V_\beta(a) | \beta < \alpha\}$. Since the $V_\beta(a)$ are linearly ordered and normal, $V(a)$ is a normal subgroup of G; $a \notin V(a)$ but $[a, x] \in V(a) \ \forall x \in G$. Also, $Z_\beta \leqq V_\beta(a)$ for $\beta < \alpha \Rightarrow Z_\alpha = \cup Z_\beta \leqq \cup V_\beta(a) = V(a)$.

Hence G/Z_α satisfies (**) in this case also.

LEMMA 5. *If the group $G(\neq E)$ satisfies* (**) *and the minimum condition for normal subgroups then for $E < H \lhd G, H \cap Z(G) \neq E$.*

PROOF. Let N be a minimal normal subgroup of G contained in H. If $N \nleqq Z(G)$, there are elements $a \in N$ and $x \in G$ such that $[a, x] \neq 1$. By (**) and Lemma 3 (ii) we can find a normal subgroup N_a of G such that $N_a < N, a \notin N_a$ but $[a, y] \in N_a \ \forall y \in G$. Hence $1 \neq [a, x] \in N_a$ so that $E < N_a < N$ contrary to the minimality of N. Thus $N \leqq H \cap Z(G)$.

THEOREM 6. *If the group G satisfies* (**) *and the minimum condition for normal subgroups, then G is a ZA-group.*

PROOF. Let C be the hypercentre of G. If $C \neq G$, G/C satisfies (**) by Theorem 5. Hence since G/C satisfies the minimum condition for normal subgroups, $Z(G/C) \neq E$ (provided $G \neq C$) by Lemma 5. But this is impossible. Hence $G = C$ and G is a ZA-group.

COROLLARY 4. *If G is a finitely generated group satisfying* (**) *and the minimum condition for normal subgroups, then G is finite and nilpotent.*

PROOF. By Theorem 2, G is finite and by Theorem 6, G is a ZA-group. Hence G is finite and nilpotent.

Finally we recall two further conditions which may be imposed on groups:

DEFINITION 7. The group G is an N-group if the normalizer condition holds in G, i.e. if every proper subgroup of G is distinct from its normalizer.

DEFINITION 8. A group G is an $\tilde{N}$-group if in every subgroup B of G every maximal subgroup A is normal.

THEOREM 7. *Let G be a group satisfying the minimum condition for subgroups. Then the following are equivalent*:

(1) *G is a ZA-group.*
(2) *G is an N-group.*
(3) *G is an $\tilde{N}$-group.*
(4) *G satisfies* (**).
(5) *G is locally finite and locally nilpotent.*

PROOF. It is well-known that (1) implies (2) (see e.g. Kurosh p. 215 and p. 219). A group G is an N-group if and only if through each subgroup of G there passes an ascending normal series, while G is an $\tilde{N}$-group if for every subgroup of G there is some normal system passing through it (see Kurosh pp. 220—21). Hence (3) follows from (2).

Now assume that G is an $\tilde{N}$-group which does not satisfy (**), and let H be a minimal subgroup of G which does not satisfy (**). By Theorem 4, H is finitely generated. Let M be a maximal subgroup of H. Then M is normal in H, and hence of finite index. Thus M is a finitely generated subgroup of G which satisfies (**). By Corollary 4, M is finite. But this implies that H is finite and a finite $\tilde{N}$-group is nilpotent (see Kurosh p. 216). Hence by Proposition 3, H satisfies (**) — contrary to the choice of H. Therefore, (3) implies (4).

(5) follows from (4) by Corollary 4. Finally if G satisfies (5), it is a Z-group and hence a ZA-group since it satisfies the minimum condition for subgroups.

Remark. It should be noted that the (equivalent) conditions in Theorem 7 do not imply that G is nilpotent. For example, let A be a group of type p^∞ and let B be cyclic of order p. Then $G = A \text{ wr } B$ (the wreath product of A and B) is solvable and satisfies the minimum condition. Any finitely generated subgroup H of G is solvable and satisfies the minimum condition. Hence H is a finite p-group and so nilpotent. Therefore, G is locally nilpotent. But G is not nilpotent since A is not bounded (see Baumslag [3, § 3]).

References

[1] Reinhold Baer, Soluble Artinian Groups (to appear).

[2] Reinhold Baer, Normalizatorreiche Gruppen (to appear).

[3] Gilbert Baumslag, 'Wreath Products and p-Groups', *Proc. Cambridge Philos. Soc.* **55** (1959), 224–31.

[4] S. N. Cernikov, *'Über lokal auflösbare Gruppen, die der Minimalbedingung genügen'*, *Mat. Sbornik* 28 (0000), 119–29.

[5] A. G. Kurosh, *The Theory of Groups II* (English Translation, New York, 1956).

[6] D. H. McLain, 'On Locally Nilpotent Groups', *Proc. Cambridge Philos. Soc.* **52** (1956), 5–11.

[7] O. Ore, 'Contributions to the theory of groups of finite order', *Duke Math. J.* **5** (1939), 431–460.

Pennsylvania State University
University Park, Pennsylvania
and
University of Frankfurt
Frankfurt am Main, Germany

https://doi.org/10.1017/S1446788700005796 Published online by Cambridge University Press