Bull. Aust. Math. Soc. (First published online 2024), page 1 of 11*
doi:10.1017/S0004972724000923
*Provisional—final page numbers to be inserted when paper edition is published

LOCALLY NILPOTENT SUBGROUPS OF $\mathrm{GL}_n(D)$

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(Received 27 July 2024; accepted 18 August 2024)

Abstract

Let A be an F-central simple algebra of degree $m^2 = \prod_{i=1}^k p_i^{2\alpha_i}$ and G be a subgroup of the unit group of A such that $F[G] = A$. We prove that if G is a central product of two of its subgroups M and N, then $F[M] \otimes_F F[N] \cong F[G]$. Also, if G is locally nilpotent, then G is a central product of subgroups H_i, where $[F[H_i] : F] = p_i^{2\alpha_i}$, $A = F[G] \cong F[H_1] \otimes_F \cdots \otimes_F F[H_k]$ and $H_i/Z(G)$ is the Sylow p_i-subgroup of $G/Z(G)$ for each i with $1 \le i \le k$. Additionally, there is an element of order p_i in F for each i with $1 \le i \le k$.

2020 *Mathematics subject classification*: primary 16K20; secondary 15A30, 20H25.

Keywords and phrases: division ring, locally nilpotent subgroup, central product of groups, crossed product.

1. Introduction

The multiplicative group of a noncommutative division ring has been investigated in various papers by Amitsur [3], Herstein [13, 14], Hua [15, 16], Huzurbazar [17] and Scott [23, 24]. Given a noncommutative division ring D with centre $Z(D) = F$, the structure of the skew linear group $\mathrm{GL}_n(D)$ for $n \ge 1$ is generally unknown. A good account of the most important results concerning skew linear groups can be found in [25], as well as in [26] particularly for linear groups. For instance, it is shown in [12] that there is a close connection between the question of the existence of maximal subgroups in the multiplicative group of a finite-dimensional division algebra and Albert's conjecture concerning the cyclicity of division algebras of prime degree. In this direction, in [20], it is shown that when D is a central division F-algebra of prime degree p, then D is cyclic if and only if D^* contains a nonabelian soluble subgroup. Furthermore, a theorem of Albert (see [6, page 87]) asserts that D is cyclic if D^*/F^* contains an element of order p.

The structure of locally nilpotent subgroups of $\mathrm{GL}_n(D)$ is studied in many papers. The basic structure of locally nilpotent skew linear groups over a locally finite-dimensional division algebra was studied by Zaleeskii [30]. One important problem raised by Zaleeskii remains open, namely, is every locally nilpotent subgroup of $\mathrm{GL}_n(D)$ hypercentral. In [10], Garascuk proved a theorem that shows this question has a positive answer in the case where $[D : F] < \infty$. A treatment of such results which is both more elaborate and more refined may be found in [4, 25–29]. For example, it is shown in [29] that when H is a locally nilpotent normal subgroup of the absolutely irreducible skew linear group G, then H is centre-by-locally-finite and $G/C_G(H)$ is periodic. In special cases, the structure of maximal subgroups of $\mathrm{GL}_n(D)$ has been investigated (see [1, 2, 5, 7, 9]). For instance, it is shown in [1] that when D is a finite-dimensional division ring with infinite centre F and M is a locally nilpotent maximal subgroup of $\mathrm{GL}_n(D)$, then M is an abelian group. Also, by [25, Theorem 3.3.8], when D is an F-central locally finite-dimensional division algebra, every locally nilpotent subgroup of $\mathrm{GL}_n(D)$ is soluble.

Another important property of locally nilpotent subgroups arises in crossed product constructions. Let R be a ring, S a subring of R and G a group of units of R normalising S such that $R = S[G]$. Suppose that $N = S \cap G$ is a normal subgroup of G and $R = \oplus_{t\in T} tS$, where T is some transversal of N to G. Set $H = G/N$. We summarise this construction by saying that (R, S, G, H) is a crossed product. Sometimes, we say that R is a crossed product of S by H. Let O be the class of all groups H such that every crossed product of a division ring by H is an Ore domain. In [25, Remark 1.4.4], it is shown that the group ring EG is an Ore domain for any division ring E and any torsion-free locally nilpotent group G. In addition, any hyper torsion-free locally nilpotent group is in O.

Let D be an F-central division algebra and G a subgroup of $\mathrm{GL}_n(D)$. The F-algebra of G, that is, the F-subalgebra generated by elements of G over F in $M_n(D)$ is denoted by $F[G]$. Further, G is absolutely irreducible if $F[G] = M_n(D)$. When $M_n(D)$ is a crossed product over a maximal subfield K, from [6, page 92], K/F is Galois and we can write $M_n(D) = \oplus_{\sigma\in \mathrm{Gal}(K/F)} Ke_\sigma$, where $e_\sigma \in \mathrm{GL}_n(D)$ and for each $x \in K$ and $\sigma \in \mathrm{Gal}(K/F)$, there exists $\sigma(x) \in K$ such that $e_\sigma x = \sigma(x)e_\sigma$. Several recent papers investigate the group theoretical properties which give useful tools to realise maximal Galois subfields of central simple algebras in terms of absolutely irreducible subgroups (see [1, 8, 9, 11, 18–20]).

We say a group G is a central product of two of its subgroups M and N if $G = MN$ and $M \subseteq C_G(N)$. In fact, a central product of two groups is a quotient group of $M \times N$. If F is a field and FG denotes the group algebra of G, then it is well known that $FM \otimes_F FN \cong F(M \times N)$. We prove a similar result for skew linear groups. Let A be an F-central simple algebra of degree $n^2 = \prod_{i=1}^k p_i^{2\alpha_i}$ and G be a subgroup of the unit group of A such that $F[G] = A$. We prove that if G is a central product of two of its subgroups M and N, then $F[M] \otimes_F F[N] \cong F[G]$. Also, if G is locally nilpotent, then G is a central product of subgroups H_i, where $[F[H_i] : F] = p_i^{2\alpha_i}$, $A = F[G] \cong F[H_1] \otimes_F \cdots \otimes_F F[H_k]$ and $H_i/Z(G)$ is the Sylow p_i-subgroup of $G/Z(G)$ for $1 \le i \le k$. Additionally, there is an element of order p_i in F for $1 \le i \le k$.

2. Notation and conventions

We recall here some of the notation that we will need throughout this article. Given a subset S and a subring K of a ring R, the subring generated by K and S is denoted by

$K[S]$. The unit group of R is written as R^*. For a group G and subset $S \subset G$, we denote by $Z(G)$ and $C_G(S)$ the centre and the centraliser of S in G and the same notation is applied for R. We use $N_G(S)$ for the normaliser of S in G and G' for the derived subgroup of G. A group G is a central product of its subgroups $H_1, \ldots, H_k$ if $G = H_1 \cdots H_k$ and $H_i \subseteq C_G(H_j)$ for each $i \neq j$.

Let F be a field, and A and B be two unital F-algebras. Let H be a subgroup of A^* and G be a subgroup of B^*. We define $H \otimes_F G$ by

$$H \otimes_F G = \{a \otimes b \mid a \in H, b \in G\}.$$

Note that $(a \otimes b)^{-1} = a^{-1} \otimes b^{-1}$, so it is easily checked that $H \otimes_F G$ is a subgroup of $(A \otimes B)^*$. Also, $F[H] \otimes_F F[G] = F[H \otimes_F G]$ in $A \otimes_F B$.

Given a division ring D with centre F and a subgroup G of $\mathrm{GL}_n(D)$, the space of column n-vectors $V = D^n$ over D is a G–D bimodule; G is called irreducible, completely reducible or reducible according to whether V is irreducible, completely reducible or reducible as a G–D bimodule.

An irreducible group G is said to be imprimitive if for some integer $m \geq 2$, there exist subspaces $V_1, \ldots, V_m$ of V such that $V = \oplus_{i=1}^{m} V_i$ and for any $g \in G$, the mapping $V_i \to gV_i$ is a permutation of the set $\{V_1, \ldots, V_m\}$; otherwise, G is called primitive.

The following important results on central simple algebras will be used later.

THEOREM 2.1 (Double centraliser theorem; [6, page 43]). *Let $B \subseteq A$ be simple rings such that $K := Z(A) = Z(B)$. Then, $A \cong B \otimes_K C_A(B)$ whenever $[B : K]$ is finite.*

THEOREM 2.2 (Centraliser theorem; [6, page 42]). *Let B be a simple subring of a simple ring A, $K := Z(A) \subseteq Z(B)$ and $n := [B : K]$ be finite. Then:*

(1) $C_A(B) \otimes_K M_n(K) \cong A \otimes_K B^{\mathrm{op}}$;
(2) *$C_A(B)$ is a simple ring;*
(3) $Z(C_A(B)) = Z(B)$;
(4) $C_A(C_A(B)) = B$;
(5) *if $L := Z(B)$ and $r := [L : K]$, then $A \otimes_K L \cong M_r(B) \otimes_L C_A(B)$;*
(6) *A is a free left (right) $C_A(B)$-module of unique rank n;*
(7) *if, in addition to the above assumptions, $m := [A : K]$ is also finite, then A is a free left (right) B-module of unique rank $m/n = [C_A(B) : K]$.*

THEOREM 2.3 [6, page 30]. *Let A, B be K-algebras, $K := Z(A) \subseteq Z(B)$ a field and either $[A : K]$ or $[B : K]$ finite. Then, $A \otimes_K B$ is a simple ring if and only if A and B are simple rings.*

3. Central products of skew linear groups and tensor products of central simple algebras

In this section, we prove a theorem which relates a central decomposition of an absolutely irreducible group G to the tensor product decomposition of $F[G]$.

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It is well known that every finite dimensional division algebra is isomorphic to a tensor product of division algebras of prime power degree [6, page 68]. Since each central simple algebra is isomorphic to some $M_n(D)$, we easily obtain the following result.

LEMMA 3.1. *Let A be an F-central simple algebra of degree $m^2 = \prod_{i=1}^{k} p_i^{2\alpha_i}$. Then, $A \cong A_1 \otimes_F \cdots \otimes_F A_k$, where A_i is a unique (up to isomorphism) F-central simple algebra of degree $p_i^{2\alpha_i}$.*

Additionally, we have the following easy lemma.

LEMMA 3.2. *Let A, B be two F-central simple algebras, and $M \leq A^*$ and $N \leq B^*$. Then, M and N are absolutely irreducible if and only if $M \otimes_F N$ is an absolutely irreducible subgroup of $A \otimes_F B$.*

LEMMA 3.3. *Let F be a field, A, B be two unital F-algebras and $a \in A, b \in B$. Then, $a \otimes b = 1 \otimes 1$ if and only if $a, b \in F$ and $ab = 1$.*

PROOF. First, if $a, b \in F$ and $ab = 1$, then $a \otimes b = ab \otimes 1 = 1 \otimes 1$.

Conversely, assume $a \otimes b = 1 \otimes 1$. It is clear that $a \neq 0$ and $b \neq 0$. First, assume that $a, b \notin F^*$. Then, $\{1, a\}$ is an F-linearly independent set in A and $\{1, b\}$ is an F-linearly independent set in B. By [6, Theorem 4.3], $\{a \otimes b, 1 \otimes 1\}$ is an F-linearly independent set in $A \otimes_F B$. Therefore, $a \otimes b \neq 1 \otimes 1$. Next, assume that $a \notin F^*$ and $b \in F^*$. Then, $ab \notin F^*$ and $\{1, ab\}$ is an F-linearly independent set in B. Thus, $\{1 \otimes ab, 1 \otimes 1\}$ is an F-linearly independent set in $A \otimes_F B$ and $a \otimes b = 1 \otimes ab \neq 1 \otimes 1$. When $b \notin F^*$ and $a \in F^*$, the proof is similar. We conclude that if $a \otimes b = 1 \otimes 1$, then $a, b \in F^*$. Now, we have $1 \otimes 1 = a \otimes b = ab \otimes 1 = ab(1 \otimes 1)$. Consequently, $ab = 1$, as we desired. □

The following result shows that any absolutely irreducible skew linear group can be viewed as an absolutely irreducible linear group.

PROPOSITION 3.4. *Let F be a field and D be a finite dimensional F-central division algebra such that $[D : F] = n^2$. Let K be a maximal subfield of D and G be an absolutely irreducible subgroup of $\mathrm{GL}_m(D)$. Then, $M_m(D) \otimes K \cong M_{mn}(K)$ and $G \otimes_F 1$ is an absolutely irreducible subgroup of $U(M_m(D) \otimes_F K) \cong \mathrm{GL}_{nm}(K)$ isomorphic to G.*

PROOF. By [21, Propositions 13.5 and 13.3], there exists a maximal subfield K of D such that $[D : K] = [K : F] = n$ and $D \otimes_F K \cong M_n(K)$. Therefore, $M_m(D) \otimes_F K \cong M_m(F) \otimes_F (D \otimes_F K) \cong (M_m(F) \otimes_F M_n(F)) \otimes_F K \cong M_{mn}(K)$. Now, by Lemma 3.3, the map $\phi : G \to G \otimes_F 1$ given by $\phi(g) = g \otimes 1$ is an isomorphism. However, G is an absolutely irreducible subgroup of $\mathrm{GL}_m(D)$, so $F[G] = M_m(D)$. Also, $M_m(D) \otimes_F K = F[G] \otimes_F K = K[G \otimes_F K^*] \subseteq K[G \otimes_F 1] \subseteq M_m(D) \otimes_F K$. Consequently, $K[G \otimes_F 1] = M_m(D) \otimes_F K$. This means $G \otimes_F 1$ is an absolutely irreducible subgroup of $\mathrm{GL}_m(D) \otimes_F K^*$ isomorphic to G. In addition, G is isomorphic to an absolutely irreducible subgroup of $\mathrm{GL}_{nm}(K)$. □

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COROLLARY 3.5. *Let F be a field and D be a finite dimensional F-central division algebra. Assume that G is a subgroup of $\mathrm{GL}_m(D)$ such that $F[G]$ is a simple ring. Then, there exists an absolutely irreducible linear group H isomorphic to G.*

THEOREM 3.6 [25, page 7]. *Let F be a field, D a locally finite-dimensional division F-algebra and G a subgroup of $\mathrm{GL}_n(D)$. Set $R = F[G] \subseteq M_n(D)$.*

(1) *If G is completely reducible, then R is semisimple Artinian.*
(2) *If G is irreducible, then R is simple Artinian.*

Using Theorem 3.6, we obtain the following result.

COROLLARY 3.7. *Let F be a field and D be a finite dimensional F-central division algebra. If G is an irreducible subgroup of $\mathrm{GL}_m(D)$, then there exists an absolutely irreducible linear group H isomorphic to G.*

When F is a field, a subgroup G of $\mathrm{GL}_n(F)$ is said to be absolutely irreducible if it is an irreducible subgroup of $\mathrm{GL}_n(K)$ for any extension K of F. Hence, we obtain the following result.

COROLLARY 3.8. *Let F be a field and D be a finite dimensional F-central division algebra. If G is an irreducible subgroup of $\mathrm{GL}_m(D)$ such that either G is irreducible or $F[G]$ is a simple ring, then there exists an algebraically closed field Ω and an irreducible Ω-linear group H isomorphic to G.*

THEOREM 3.9 [25, page 8]. *Let F be a field, D a division F-algebra and G a subgroup of $\mathrm{GL}_n(D)$. Set $R = F[G] \subseteq M_n(D)$.*

(1) *If R is semiprime (for example, if R is semisimple Artinian), then G is isomorphic to a completely reducible subgroup of $\mathrm{GL}_n(D)$.*
(2) *If R is simple Artinian, then for some $m \leq n$, the group G is isomorphic to an irreducible subgroup of $\mathrm{GL}_m(D)$.*

Using Theorem 3.9, we obtain the following result.

COROLLARY 3.10. *Let F be a field and D be a finite dimensional F-central division algebra such that $[D : F] = n^2$. Let $A = M_m(D) \subseteq M_{n^2m}(F) = B$ be an F-central simple algebra. If G is a subgroup of $\mathrm{GL}_m(D)$ such that either G is irreducible or $F[G]$ is a simple ring, then for some $s \leq mn^2$, the group G is isomorphic to an irreducible subgroup of $\mathrm{GL}_s(F)$.*

THEOREM 3.11 [26, page 111]. *Let V be a finite dimensional linear space over a division ring D and G an irreducible subgroup of $\mathrm{GL}(V)$ which can be represented in the form $G = HF$, where H and F are elementwise permutable normal subgroups of G. Then, the irreducible components of $H(F)$ are pairwise equivalent.*

PROPOSITION 3.12. *Let F be a field and D be a finite dimensional F-central division algebra. Assume that G is an absolutely irreducible subgroup of $\mathrm{GL}_n(D)$. If $G = MN$ is a central product decomposition of G, then $F[M] \otimes_F F[N] \cong F[G]$ and under*

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this isomorphism, $M \otimes_F N \cong G$. Additionally, $F[M]$ and $F[N]$ are F-central division algebras.

PROOF. By [25, Theorem 1.2.1], G is irreducible. Using [25, Theorem 1.1.7] and Theorem 3.11, we conclude that M is a homogeneous completely irreducible subgroup. So Theorem 3.11 implies $D^n \cong V^m$, where V is an irreducible $M - D$ bimodule. Hence, $F[N] \subseteq A = C_{M_n(D)}(M) = \mathrm{End}_{M-D}(D^n) \cong M_m(E)$, where $E = \mathrm{End}_{M-D}(V)$ is a division ring by Schur's lemma. Note that $F[N] \otimes F[M] \leq A \otimes_F C_{M_n(D)}(A)$. Hence, by the centraliser theorem, $[F[M] : F]FN] : F] \leq [A : F][C_{M_n(D)}(A) : F] = n^2[D : F]$. Furthermore, $F[M], F[N] \subseteq F[G]$ implies that there is a surjective homomorphism f from $F[N] \otimes_F F[M]$ onto $F[G] = M_n(D)$ such that $f(a \otimes b) = ab$ for each $a \in M, b \in N$. So $F[M] \otimes_F F[N] \cong F[G]$ by dimension counting. It is clear that $\overline{f}$, the restriction of f to $M \otimes_F N$, is a surjective homomorphism on G. If $\overline{f}(a \otimes b) = ab = 1$, then $a = b^{-1} \in M \cap N \subseteq Z(G) \subseteq F$. Hence, $a \otimes b = b^{-1} \otimes b = 1 \otimes b^{-1}b = 1 \otimes 1$. So, $\ker(\overline{f})$ is trivial and $\overline{f}$ is an isomorphism from $M \otimes_F N$ to G. Consequently, $F[M]$ and $F[N]$ are F-central division algebras by Theorem 2.3. □

The following example shows that the above result is not true in semisimple rings.

EXAMPLE 3.13. Let $A = F \times F$, $G = \{(1, 1), (1, -1), (-1, 1), (-1, -1)\}$, $M = \{(1, 1), (1, -1)\}, N = \{(1, 1), (-1, 1)\}$. Then, G is a central product of M and N. However, $[F[M] \otimes_F F[N] : F] = 4$. So $F[M] \otimes_F F[N] \ncong F[G] = A$.

Next we introduce some notation from [26]. Let V be a finite dimensional linear space over a division ring D and G a completely irreducible subgroup of $\mathrm{GL}(V)$. Let $D^n = V = L_1 \oplus \cdots \oplus L_r$ and suppose that L_i is a G-invariant G-irreducible subspace of V for $1 \leq i \leq r$. We determine the irreducible components of G, that is, the irreducible representations d_i of the form

$$d_i : G \to \mathrm{GL}(L_i), \quad g \to g \mid L_i, \quad i = 1, \ldots, r.$$

By [26, Lemma 13.1], the irreducible components d_i and d_j of G are equivalent if and only if there exists a module isomorphism $\Psi : L_i \to L_j$ such that for any $y \in G$,

$$d_j(y) = \Psi d_1(y)\Psi^{-1}.$$

In addition, these representations are equivalent if and only if the modules L_i and L_j have respective bases B_1 and B_2 such that for any $y \in G$, the matrix of the endomorphism $d_i(y)$ in B_1 is the same as that of $d_j(y)$ in B_2. This observation gives the following result.

LEMMA 3.14. *Let G be a completely irreducible subgroup of $\mathrm{GL}_n(D)$ such that the irreducible components of G are pairwise equivalent. Let r be the degree of an irreducible component of G and $n = rs$. Then, there is an isomorphism f with $f : M_n(D) \longrightarrow M_r(D) \otimes_F M_s(F)$ and an irreducible subgroup H of $\mathrm{GL}_r(D)$ such that $f(G) = H \otimes \{1\}$.*

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4. Locally nilpotent subgroups of $\mathrm{GL}_n(D)$

In this section, we prove that every absolutely irreducible locally nilpotent subgroup of $\mathrm{GL}_n(D)$ is a central product of some of its subgroups which gives a decomposition of $M_n(D)$ as a tensor product of central simple algebras of prime power degree. First, we recall the following general results which play a key role in proving our main theorems.

THEOREM 4.1 [26, page 216]. *Let F be an arbitrary field and G be an absolutely irreducible locally nilpotent subgroup of $\mathrm{GL}_n(F)$. Then, $G/Z(G)$ is periodic and $\pi(G/Z(G)) = \pi(n)$.*

THEOREM 4.2 [29]. *Let H be a locally nilpotent normal subgroup of the absolutely irreducible skew linear group G. Then, H is centre-by-locally finite and $G/C_G(H)$ is periodic.*

THEOREM 4.3 [22, page 342]. *Let G be a locally nilpotent group. Then, the elements of finite order in G form a fully invariant subgroup T (the torsion subgroup of G) such that G/T is torsion and T is a direct product of p-groups.*

THEOREM 4.4 [5]. *Let N be a normal subgroup in a primitive subgroup M of $\mathrm{GL}_n(D)$. Then:*

(1) *$F[N]$ is a prime ring;*
(2) *$C_{M_n(D)}(N)$ is a simple Artinian ring;*
(3) *if $C_{M_n(D)}(N)$ is a division ring, then N is irreducible.*

THEOREM 4.5 [18]. *Let D be a finite dimensional F-central division algebra. Then, $M_m(D)$ is a crossed product over a maximal subfield if and only if there exists an absolutely irreducible subgroup G of $M_m(D)$ and a normal abelian subgroup A of G such that $C_G(A) = A$ and $F[A]$ contains no zero divisor.*

THEOREM 4.6. *Let $A = M_n(D)$ be an F-central simple algebra of degree $m^2 = \prod_{i=1}^k p_i^{2\alpha_i}$ and G be an absolutely irreducible locally nilpotent subgroup A^*. Then:*

(1) *$G/Z(G)$ is locally finite and $\pi(G/Z(G)) = \pi(m)$;*
(2) *$G/Z(G)$ is a p-group for some prime p if and only if m is a pth power.*

PROOF. (1) By Theorem 4.2, G is centre-by-locally finite. Let K be a maximal subfield of D. By Proposition 3.4, G is isomorphic to an absolutely irreducible subgroup of $\mathrm{GL}_m(K)$. Now, Theorem 4.1 asserts that $\pi(G/Z(G)) = \pi(m)$.

(2) This statement is clear from item (1). □

COROLLARY 4.7. *Let $A = M_n(D)$ be an F-central simple algebra of degree $m^2 = \prod_{i=1}^k p_i^{2\alpha_i}$ and G be an absolutely irreducible locally nilpotent subgroup of A^*. Then:*

(1) *$G/Z(G)$ is locally finite and $\pi(G/Z(G)) = \pi(m^2/[C_{M_n(D)} : F]) \subseteq \pi(m)$;*
(2) *if $G/Z(G)$ is a p-group for some prime p, then $[F[G] : F]$ is a pth power;*
(3) *if m is a pth power for some prime p, then $G/Z(G)$ is a p-group.*

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PROOF. By Theorem 3.6, $F[G]$ is a simple ring. From the centraliser theorem, $[F[G] : F][C_{M_n(D)} : F] = m^2$. The reminder of the proof is similar to the proof of Theorem 4.6. □

Now we are ready to prove the main theorem of this article.

THEOREM 4.8. *Let $A = M_n(D)$ be an F-central simple algebra of degree $m^2 = \prod_{i=1}^{k} p_i^{2\alpha_i}$ and G be an absolutely irreducible locally nilpotent subgroup A^*. Then:*

(1) *$G/Z(G)$ is the internal direct product of $H_1/Z(G), \dots, H_k/Z(G)$, where $H_i/Z(G)$ is the Sylow p_i-subgroup of $G/Z(G)$;*
(2) *G is the central product of $H_1, \dots, H_k$;*
(3) *$A = F[G] \cong F[H_1] \otimes_F \cdots \otimes_F F[H_k]$ and $G \cong H_1 \otimes_F \cdots \otimes_F H_k$ under this isomorphism and, for each i, $A_i = F[H_i]$ is an F-central simple algebra and $[F[H_i] : F] = {p_i}^{2\alpha_i}$.*

PROOF. (1) The statement follows from Theorems 4.3 and 4.6.

(2) Let $i \neq j$ and take $a \in H_i, b \in H_j$. Then, $ab = \lambda ba$ with $\lambda \in Z(G) \subseteq F^*$. Now, $a^{{p_i}^{\gamma}} \in F^*$ and $b^{{p_j}^{\delta}} \in F^*$, so $\lambda^{{p_i}^{\gamma}} = \lambda^{{p_j}^{\delta}} = 1$, which gives $\lambda = 1$ and $ab = ba$. So, $H_i \subseteq C_G(H_j)$ and G is the central product of $H_1, \dots, H_k$.

(3) This statement follows from Proposition 3.12 and induction on k. □

COROLLARY 4.9. *Keep the notation and assumptions of Theorem 4.8. If $n = 1$ and $F[H_i] = D_i$, then $D \cong D_1 \otimes_F \cdots \otimes_F D_k$, where $i(D_i) = {p_i}^{\alpha_i}$.*

Using [19, Theorem 2.4], we have the following proposition.

PROPOSITION 4.10. *Keep the notation and assumptions of Theorem 4.8. Then, $F[G] = M_n(D)$ is a crossed product over a maximal subfield K if and only if for each i, $F[H_i]$ is a crossed product over a maximal subfield K_i. In addition, under these circumstances, $K \cong K_1 \otimes_F \cdots \otimes_F K_k$ and $\mathrm{Gal}(K/F) \cong \mathrm{Gal}(K_1/F) \times \cdots \times \mathrm{Gal}(K_k/F)$.*

THEOREM 4.11. *Let D be an F-central finite dimensional division algebra. Assume that G be a primitive absolutely irreducible locally nilpotent subgroup of $\mathrm{GL}_n(D)$. Then, $M_n(D)$ is a crossed product over a maximal subfield K. With the notation and assumptions of Theorem 4.8:*

(1) *there exists an abelian normal subgroup S of G such that G/S and $\mathrm{Gal}(K/F)$ are finite nilpotent groups and $\mathrm{Gal}(K/F) \cong N_{\mathrm{GL}_n(D)}(K^*)/K^* \cong G/S$;*
(2) *for each i, there exists an abelian subgroup A_i of H_i such that $F[H_i]$ is a crossed product over a maximal subfield K_i and, in addition, H_i/A_i and $Gal(K_i/F)$ are finite nilpotent groups and $\mathrm{Gal}(K_i/F) \cong N_{F[H_i]^*}(K_i^*)/K_i^* \cong H_i/A_i$;*
(3) *$S \cong A_1 \otimes_F \cdots \otimes_F A_k$, $K \cong K_1 \otimes_F \cdots \otimes_F K_k$ and $S = A_1 \cdots A_k$.*

PROOF. By [25, Theorem 3.3.8], G is soluble. Now, using [26, Theorem 6, page 135], G contains a maximal abelian normal subgroup, say S, such that $|G/S| < \infty$. By Theorem 4.4, $K = F[S]$ is a field and by a result in [10], G is hypercentral. Hence, by an exercise from [22, page 354], we conclude that every maximal abelian normal

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subgroup of G is self-centralising. Now, using Theorem 4.5, we conclude that $M_n(D)$ is a crossed product over a maximal subfield K. By a result of [6, page 92], K/F is Galois and we can write $M_n(D) = \oplus_{\sigma\in \mathrm{Gal}(K/F)} Ke_\sigma$, where $e_\sigma \in \mathrm{GL}_n(D)$ and for each $x \in K$ and $\sigma \in \mathrm{Gal}(K/F)$, there exists $\sigma(x) \in K$ such that $e_\sigma x = \sigma(x)e_\sigma$. So, $e_\sigma \in N_{\mathrm{GL}_n(D)}(K^*)$. Now, using the Skolem–Noether theorem [6, page 39] and the fact that $C_{M_n(D)}(K) = K$, we obtain $\mathrm{Gal}(K/F) \cong N_{\mathrm{GL}_n(D)}(K^*)/K^*$. However, consider the homomorphism $\sigma : G \to \mathrm{Gal}(K/F)$ given by $\sigma(x) = f_x$, where $f_x(k) = xkx^{-1}$ for $k \in K$. Clearly, $\ker(\sigma) = C_G(K)$. Since $S \subseteq C_G(K) \subseteq C_G(S) = S$, we have $C_G(K) = S$. Choose an element $a \in \mathrm{Fix}(\mathrm{Im}\,\sigma)$. For any $x \in G$, we have $f_x(a) = a$ and hence $xa = ax$. This shows that $\mathrm{Fix}(\mathrm{Im}\,\sigma) \subseteq C_K(G) \subseteq C_{M_n(D)}(G) = F$. Hence, $F = \mathrm{Fix}(\mathrm{Im}\,\sigma)$ and σ is surjective. Therefore, $\mathrm{Gal}(K/F) \cong G/S$, as we claimed.

The proof is completed by using Theorem 4.8 and Proposition 4.10. □

We can immediately deduce the following theorem.

THEOREM 4.12. *Let D be an F-central finite dimensional division algebra such that $[D : F] = i(D)^2 = \prod_{i=1}^k p_i^{2\alpha_i}$. If D^* contains an absolutely irreducible locally nilpotent subgroup G, then D is a crossed product over a maximal subfield K. With the notation and assumptions of Theorems 4.8 and 4.11, $D \cong D_1 \otimes_F \cdots \otimes_F D_k$, where $F[H_i] = D_i$ and D_i is a crossed product over a maximal subfield K_i.*

PROPOSITION 4.13. *Let $A = M_n(D)$ be an F-central simple algebra of degree $m^2 = \prod_{i=1}^k p_i^{2\alpha_i}$ and G be an absolutely irreducible locally nilpotent subgroup A^*. Then, there is an element of order p_i in F for $1 \le i \le k$.*

PROOF. Keep the notation and assumptions of Theorem 4.8, so that $[F[H_i] : F] = p_i^{2\alpha_i}$. Since $F[H_i]$ is a central simple algebra, $F[H_i] \cong M_{p_i^{\beta_i}}(D_i)$, where D_i is an F-central division algebra of degree a power of p_i. Assume that K_i is a maximal subfield of D_i. By [26, Theorem 27.6] and Proposition 3.4, K_i contains an element b, say, of order p_i. Now, $[F(b) : F] \le p_i - 1$ and $[F(b) : F] \mid [K_i : F]$. However, $[K_i : F]$ is a power of p_i, which implies $[F[b] : F] = 1$, that is, $b \in F$. □

PROPOSITION 4.14. *Let D be an F-central finite dimensional division algebra and suppose that for $p \in \pi(n)$, there is an element of order p in F, when $n > 1$. Then, $\mathrm{GL}_n(D)$ contains a finite irreducible nonabelian nilpotent subgroup G such that $F[G] = M_n(F) \subseteq M_n(D)$.*

PROOF. By [26, Theorem 27.6], there exists a finite nilpotent subgroup G of $\mathrm{GL}_n(F)$ such that $F[G] = M_n(F) \subseteq M_n(D)$. We show that G is an irreducible subgroup of $\mathrm{GL}_n(D)$. In contrast, assume that G is reducible in $\mathrm{GL}_n(D)$. By [25, Theorem 1.1.1], there exists a matrix $P \in \mathrm{GL}_n(D)$ such that

$$P(F[G])P^{-1} \subseteq \begin{bmatrix} M_r(D) & B \\ 0_{(n-s)\times r} & M_{n-s}(D) \end{bmatrix}.$$

This means that we can define a homomorphism from $M_n(F)$ to $M_r(D)$. However, $M_n(F)$ is a simple ring. Hence, this map is an injection. This contradicts

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[25, Theorem 1.1.9], which asserts that the matrix ring $M_r(D)$ contains at most r nonzero pairwise orthogonal idempotents. □

EXAMPLE 4.15. The multiplicative group of the real quaternion division algebra contains the quaternion group which is an absolutely irreducible 2-group. By [8, Corollary 3.5], if D is a noncommutative finite dimensional F-central division algebra and D^* contains an absolutely irreducible finite p-subgroup for some prime p, then D is a nilpotent crossed product with $[D : F] = 2^m$ for some $m \in \mathbb{N}$.

Acknowledgements

The first author thanks the Research Council of the Farhangian University for support. The second author is indebted to the Research Council of University of Jiroft for support.

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https://doi.org/10.1017/S0004972724000923 Published online by Cambridge University Press