Proceedings of the Royal Society of Edinburgh, page 1 of 22
DOI:10.1017/prm.2024.66

On a class of self-similar sets which contain finitely many common points

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(Received 19 August 2023; accepted 3 May 2024)

For $\lambda \in (0, 1/2]$ let $K_\lambda \subset \mathbb{R}$ be a self-similar set generated by the iterated function system $\{\lambda x, \lambda x + 1 - \lambda\}$. Given $x \in (0, 1/2)$, let $\Lambda(x)$ be the set of $\lambda \in (0, 1/2]$ such that $x \in K_\lambda$. In this paper we show that $\Lambda(x)$ is a topological Cantor set having zero Lebesgue measure and full Hausdorff dimension. Furthermore, we show that for any $y_1, \ldots, y_p \in (0, 1/2)$ there exists a full Hausdorff dimensional set of $\lambda \in (0, 1/2]$ such that $y_1, \ldots, y_p \in K_\lambda$.

Keywords: Hausdorff dimension; thickness; self-similar set; Cantor set; intersection

2020 *Mathematics Subject Classification:* Primary: 28A78; Secondary: 28A80, 37B10, 11A63

1. Introduction

For $\lambda \in (0, 1/2]$ let K_λ be the *self-similar set* generated by the *iterated function system* (simply called, *IFS*) $\{f_{\lambda,d}(x) = \lambda x + d(1-\lambda) : d = 0, 1\}$. Then K_λ is the unique nonempty compact set satisfying (cf. [13])

$$K_\lambda = f_{\lambda,0}(K_\lambda) \cup f_{\lambda,1}(K_\lambda) = \left\{(1-\lambda)\sum_{n=1}^{\infty} i_n \lambda^{n-1} : i_n \in \{0,1\} \ \forall n \geqslant 1\right\}. \qquad (1.1)$$

Clearly, $1-\lambda$ is chosen so that the convex hull of K_λ is the unit interval $[0, 1]$ for all $\lambda \in (0, 1/2]$. Then 0 and 1 are common points of K_λ for all $\lambda \in (0, 1/2]$. For other

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$x \in (0,\, 1)$ it is natural to ask how likely the self-similar sets K_λ, $\lambda \in (0,\, 1/2]$ contain the common point x? Or even ask how likely the self-similar sets K_λ, $\lambda \in (0,\, 1/2]$ contain any given points $y_1, \ldots, y_p \in (0,\, 1)$? These questions are motivated by the work of Boes, Darst and Erdős [**4**], in which they considered a class of fat Cantor sets C_λ with positive Lebesgue measure. They showed that for a given point $x \in (0,\, 1)$ the set of parameters $\lambda \in (0,\, 1/2)$ such that $x \in C_\lambda$ is of first category.

Given $x \in [0,\, 1]$, let

$$\Lambda(x) := \{\lambda \in (0, 1/2] : x \in K_\lambda\}. \tag{1.2}$$

Then $\Lambda(x)$ consists of all $\lambda \in (0,\, 1/2]$ such that x is the common point of K_λ. Note that K_λ is symmetric, i.e., $x \in K_\lambda$ if and only if $1 - x \in K_\lambda$. Then $\Lambda(x) = \Lambda(1 - x)$ for any $x \in [0,\, 1]$. So, we only need to consider $x \in [0,\, 1/2]$. Note that $\Lambda(0) = (0,\, 1/2]$, and $\Lambda(1/2) = \{1/2\}$. So, it is interesting to study $\Lambda(x)$ for $x \in (0,\, 1/2)$.

Recall that a set $F \subset \mathbb{R}$ is called a *Cantor set* if it is a non-empty compact set containing neither interior nor isolated points. Our first result considers the topology of $\Lambda(x)$.

THEOREM 1.1. *For any* $x \in (0,\, 1/2)$ *the set* $\Lambda(x)$ *is a Cantor set with* $\min \Lambda(x) = x$ *and* $\max \Lambda(x) = 1/2$.

By theorem 1.1 it follows that $\Lambda(x)$ is a fractal set for any $x \in (0,\, 1/2)$. Our next result considers the Lebesgue measure and fractal dimension of $\Lambda(x)$.

THEOREM 1.2. *For any* $x \in (0,\, 1/2)$ *the set* $\Lambda(x)$ *is a Lebesgue null set of full Hausdorff dimension. Furthermore,*

$$\lim_{\delta \to 0^+} \dim_H(\Lambda(x) \cap (\lambda - \delta, \lambda + \delta)) = \frac{\log 2}{-\log \lambda} \quad \forall\ \lambda \in \Lambda(x),$$

where $\dim_H$ *denotes the Hausdorff dimension.*

In 1984, Mahler [**18**] proposed the problem on studying how well elements in the middle third Cantor set $K_{1/3}$ can be approximated by rational numbers in it, and by rational numbers outside of it. Some recent progress on this problem can be found in [**10**, **11**, **16**, **22**] and the references therein. On the other hand, this question also motivates the study of rational numbers in a fractal set (cf. [**23**, **26**]). As a corollary of theorem 1.2 we show that for Lebesgue almost every $\lambda \in (0,\, 1/2)$ the Cantor set K_λ contains only two rational numbers 0 and 1.

COROLLARY 1.3. *For Lebesgue almost every* $\lambda \in (0,\, 1/2]$ *the set* $K_\lambda \setminus \{0,\, 1\}$ *contains only irrational numbers.*

Proof. By theorem 1.2 it follows that $\Lambda(x)$ has zero Lebesgue measure for any $x \in (0,\, 1)$. But if $K_\lambda \setminus \{0,\, 1\}$ contains a rational number, then $\lambda \in \bigcup_{x \in \mathbb{Q} \cap (0,1)} \Lambda(x)$ which has zero Lebesgue measure. □

Given $y_1, \ldots, y_p \in (0,\, 1/2)$, by theorems 1.1 and 1.2 it follows that the intersection $\bigcap_{i=1}^{p} \Lambda(y_i)$ is small from the topological and Lebesgue measure perspectives.

https://doi.org/10.1017/prm.2024.66 Published online by Cambridge University Press

On the other hand, by using the *thickness* method introduced by Newhouse [**19**] we can show that $\bigcap_{i=1}^p \Lambda(y_i)$ contains a sequence of Cantor sets whose thickness can be arbitrarily large, and from this we conclude that the intersection $\bigcap_{i=1}^p \Lambda(y_i)$ has full Hausdorff dimension.

THEOREM 1.4. *For any points* $y_1, y_2, \ldots, y_p \in (0, 1/2)$ *we have*

$$\dim_H \bigcap_{i=1}^{p} \Lambda(y_i) = 1.$$

Recently, the first three authors studied in [**14**] analogous objects but with different family of self-similar sets (their self-similar sets have different convex hulls). Theorem 1.4 shows that the intersection of any finitely many $\Lambda(y_i)$ has full Hausdorff dimension, while in [**14**, Theorem 1.5] their method can only prove this result for the intersection of two associated sets.

The rest of the paper is organized as follows. In § 2 we prove theorem 1.1 for the topology of $\Lambda(x)$; and in § 3 we investigate the local Hausdorff dimension of $\Lambda(x)$ and prove theorem 1.2. In § 4 we consider the intersection $\bigcap_{i=1}^p \Lambda(y_i)$ and prove theorem 1.4; and in the final section we make some further remarks.

2. Topological properties of $\boldsymbol{\Lambda(x)}$

In this section we investigate the topology of $\Lambda(x)$, and prove theorem 1.1. Given $x \in (0, 1/2)$, note by (1.1) that for each $\lambda \in \Lambda(x) \setminus \{1/2\}$ there exists a unique sequence $(d_i) \in \{0, 1\}^{\mathbb{N}}$ such that $x = (1-\lambda)\sum_{n=1}^{\infty} d_n \lambda^{n-1}$. We will show that $\Lambda(x)$ is homeomorphic to a subset in the symbolic space $\{0, 1\}^{\mathbb{N}}$, and then the topological properties of $\Lambda(x)$ can be deduced by studying the corresponding symbolic set.

First we recall some terminology from symbolic dynamics (cf. [**17**]). Let $\{0, 1\}^{\mathbb{N}}$ be the set of all infinite sequences of zeros and ones. For a *word* we mean a finite string of zeros and ones. Let $\{0, 1\}^*$ be the set of all words over the alphabet $\{0, 1\}$ together with the empty word ϵ. For two words $\mathbf{c} = c_1 \ldots c_m$, $\mathbf{d} = d_1 \ldots d_n$ from $\{0, 1\}^*$ we write $\mathbf{cd} = c_1 \ldots c_m d_1 \ldots d_n$ for their concatenation. In particular, for $n \in \mathbb{N}$ we denote by $\mathbf{c}^n$ the n-fold concatenation of $\mathbf{c}$ with itself, and by $\mathbf{c}^\infty$ the periodic sequence with period block $\mathbf{c}$. Throughout the paper we will use lexicographical order '$\prec, \preccurlyeq, \succ$' or '$\succcurlyeq$' between sequences and words. For example, for two sequences $(c_i), (d_i) \in \{0, 1\}^{\mathbb{N}}$, we say $(c_i) \prec (d_i)$ if $c_1 < d_1$, or there exists $n \in \mathbb{N}$ such that $c_1 \ldots c_n = d_1 \ldots d_n$ and $c_{n+1} < d_{n+1}$. For two words $\mathbf{c}, \mathbf{d}$, we say $\mathbf{c} \prec \mathbf{d}$ if $\mathbf{c}0^\infty \prec \mathbf{d}0^\infty$.

Let $\lambda \in (0, 1/2]$. We define the coding map $\pi_\lambda : \{0, 1\}^{\mathbb{N}} \to K_\lambda$ by

$$\pi_\lambda((i_n)) = \lim_{n\to\infty} f_{\lambda,i_1} \circ f_{\lambda,i_2} \circ \cdots \circ f_{\lambda,i_n}(0) = (1-\lambda)\sum_{n=1}^{\infty} i_n \lambda^{n-1}. \tag{2.1}$$

If $\lambda \in (0, 1/2)$, then the IFS $\{f_{\lambda,d}(x) = \lambda x + d(1-\lambda) : d = 0, 1\}$ satisfies the strong separation condition, and thus the map π_λ is bijective. If $\lambda = 1/2$, then $\pi_{1/2}$ is bijective up to a countable set. The map π_λ defined in (2.1) naturally induces a function with two parameters:

$$\Pi : \{0,1\}^{\mathbb{N}} \times (0,1/2] \rightarrow [0,1]; \quad ((i_n),\lambda) \mapsto \pi_\lambda((i_n)). \tag{2.2}$$

Note that the symbolic space $\{0, 1\}^{\mathbb{N}}$ becomes a compact metric space under the metric

$$\rho((i_n),(j_n)) = 2^{-\inf\{n\geqslant 1: i_n \neq j_n\}}. \tag{2.3}$$

Equipped with the product topology on $\{0, 1\}^{\mathbb{N}} \times (0, 1/2]$ we show that Π is continuous.

LEMMA 2.1. *The function* Π *is continuous. Furthermore,*

(i) *for* $\lambda \in (0, 1/2]$ *the function* $\Pi(\cdot, \lambda)$ *is increasing with respect to the lexicographical order, and is strictly increasing if* $\lambda \in (0, 1/2)$*;*

(ii) *if* $0^\infty \prec (i_n) \preccurlyeq 01^\infty$*, then* $\Pi((i_n), \cdot)$ *has positive derivative in* $(0, 1/2)$*.*

Proof. First we prove the continuity of Π. For any two points $((i_n), \lambda_1), ((j_n), \lambda_2) \in \{0, 1\}^{\mathbb{N}} \times (0, 1/2]$ we have

$$|\Pi((j_n),\lambda_2) - \Pi((i_n),\lambda_1)| \leqslant |\Pi((j_n),\lambda_2) - \Pi((i_n),\lambda_2)| + |\Pi((i_n),\lambda_2) - \Pi((i_n),\lambda_1)|. \tag{2.4}$$

Note that if $\rho((j_n), (i_n)) \leqslant 2^{-m}$, then $|\Pi((j_n), \lambda_2) - \Pi((i_n), \lambda_2)| \leqslant \lambda_2^{m-1} \leqslant 2^{1-m}$. So the first term in (2.4) converges to zero as $\rho((j_n), (i_n)) \to 0$. Moreover, since the series $\Pi((i_n), \lambda) = (1-\lambda)\sum_{n=1}^{\infty} i_n\lambda^{n-1}$ with parameter λ converges uniformly in $(0, 1/2]$, the second term in (2.4) also converges to zero as $|\lambda_2 - \lambda_1| \to 0$. Therefore, Π is continuous.

For (i) let $\lambda \in (0, 1/2]$ and take two sequences $(i_n), (j_n) \in \{0, 1\}^{\mathbb{N}}$. Suppose $(i_n) \prec (j_n)$. Then there exists $m \in \mathbb{N}$ such that $i_1 \ldots i_{m-1} = j_1 \ldots j_{m-1}$ and $i_m < j_m$. This implies that

$$\begin{aligned}
\Pi((i_n),\lambda) = (1-\lambda)\sum_{n=1}^{\infty} i_n\lambda^{n-1} &\leqslant (1-\lambda)\left(\sum_{n=1}^{m} i_n\lambda^{n-1} + \sum_{n=m+1}^{\infty}\lambda^{n-1}\right)\\
&\leqslant (1-\lambda)\sum_{n=1}^{m} j_n\lambda^{n-1}\\
&\leqslant \Pi((j_n),\lambda),
\end{aligned}$$

where the second inequality follows from $\lambda/(1-\lambda) \leqslant 1$ for $\lambda \in (0, 1/2]$, and this inequality is strict if $\lambda \in (0, 1/2)$.

For (ii) let $(i_n) \in \{0, 1\}^{\mathbb{N}}$ with $0^\infty \prec (i_n) \preccurlyeq 01^\infty$. Then $i_1 = 0$. So for any $\lambda \in (0, 1/2)$ we have $\Pi((i_n), \lambda) = (1-\lambda)\sum_{n=2}^{\infty} i_n\lambda^{n-1}$. This implies that

$$\frac{\mathrm{d}\Pi((i_n),\lambda)}{\mathrm{d}\lambda} = \sum_{n=2}^{\infty} n\left(\frac{n-1}{n} - \lambda\right) i_n\lambda^{n-2} > 0, \tag{2.5}$$

where the strict inequality follows since $\lambda < 1/2$ and $(i_n) \succ 0^\infty$. This completes the proof. □

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Note that the map Π defined in (2.2) is surjective but not injective. Given $x \in [0,\, 1]$, for $\lambda \in (0,\, 1/2]$ we consider the horizontal fibre

$$\Gamma_x(\lambda) := \Pi^{-1}(x) \cap \left(\{0,1\}^{\mathbb{N}} \times \{\lambda\}\right) = \left\{((i_n),\lambda) \in \{0,1\}^{\mathbb{N}} \times (0,1/2] : (1-\lambda)\sum_{n=1}^{\infty} i_n \lambda^{n-1} = x\right\}.$$

Then $\Gamma_x(\lambda) \neq \emptyset$ if and only if $\lambda \in \Lambda(x)$. Furthermore, by lemma 2.1 (i) it follows that for any $\lambda \in \Lambda(x) \cap (0,\, 1/2)$ the fibre set $\Gamma_x(\lambda)$ consists of only one sequence; and if $\lambda = 1/2 \in \Lambda(x)$ then the set $\Gamma_x(1/2)$ consists of at most two sequences. This defines a map

$$\Psi_x : \Lambda(x) \to \{0,1\}^{\mathbb{N}}; \quad \lambda \mapsto \Psi_x(\lambda),$$

where $\Psi_x(\lambda)$ denotes the lexicographically largest sequence in $\Gamma_x(\lambda)$. The sequence $\Psi_x(\lambda)$ is also called the *greedy coding* of x in base λ.

Given $x \in (0,\, 1/2)$, we reserve the notation $(x_n) := \Psi_x(1/2)$ for the greedy coding of x in base $1/2$. Then (x_n) begins with 0 and does not end with 1^∞.

LEMMA 2.2. *For any $x \in (0,\, 1/2)$ the map $\Psi_x : \Lambda(x) \to \Omega(x)$ is a decreasing homeomorphism, where*

$$\Omega(x) := \left\{(i_n) \in \{0,1\}^{\mathbb{N}} : (x_n) \preccurlyeq (i_n) \preccurlyeq 01^\infty\right\}.$$

Proof. Let $x \in (0,\, 1/2)$. By lemma 2.1 it follows that Ψ_x is strictly decreasing. Observe that $x \notin K_\lambda$ for any $\lambda < x$. Then $\Lambda(x) \subset [x,\, 1/2]$. Note that $\Psi_x(x) = 01^\infty$ and $\Psi_x(1/2) = (x_n)$. Since Ψ_x is monotonically decreasing, we have

$$(x_n) \preccurlyeq \Psi_x(\lambda) \preccurlyeq 01^\infty \quad \forall\, \lambda \in \Lambda(x).$$

So, $\Psi_x(\Lambda(x)) \subset \Omega(x)$.

Next we show that $\Psi_x(\Lambda(x)) = \Omega(x)$. Let $(i_n) \in \Omega(x)$. Then by lemma 2.1 it follows that

$$\Pi\left((i_n), \frac{1}{2}\right) \geqslant \Pi\left((x_n), \frac{1}{2}\right) = x \quad \text{and} \quad \Pi((i_n),\lambda) \searrow 0 < x \quad \text{as } \lambda \searrow 0.$$

So, by the continuity of Π in lemma 2.1 there must exist $\lambda \in (0,\, 1/2]$ such that

$$\Pi((i_n),\lambda) = x. \tag{2.6}$$

If $\lambda \in (0,\, 1/2)$, then (2.6) gives that $\Psi_x(\lambda) = (i_n)$. If $\lambda = 1/2$, then by (2.6) and using $(i_n) \succcurlyeq (x_n)$ we still have $\Psi_x(\lambda) = (i_n)$. This proves $\Psi_x(\Lambda(x)) = \Omega(x)$. Hence, $\Psi_x : \Lambda(x) \to \Omega(x)$ is a decreasing bijection.

To completes the proof it remains to prove the continuity of Ψ_x and its inverse Ψ_x^{-1}. Since the proof for the continuity of Ψ_x^{-1} is similar, we only prove it for Ψ_x. Take $\lambda_* \in \Lambda(x)$. Suppose Ψ_x is not continuous at λ_*. Then there exists $N \in \mathbb{N}$ such that for any $\delta > 0$ we can find $\lambda \in \Lambda(x) \cap (\lambda_* - \delta,\, \lambda_* + \delta)$ such that

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$|\Psi_x(\lambda) - \Psi_x(\lambda_*)| \geqslant 2^{-N}$. Letting $\delta = 1/k$ with $k = 1, 2, \ldots$, we can find a sequence $(\lambda_k) \subset \Lambda(x)$ such that

$$\lim_{k\to\infty} \lambda_k = \lambda_* \quad \text{and} \quad |\Psi_x(\lambda_k) - \Psi_x(\lambda_*)| \geqslant 2^{-N} \quad \forall\, k \geqslant 1. \tag{2.7}$$

Write $\Psi_x(\lambda_k) = (i_n^{(k)})$ and $\Psi_x(\lambda_*) = (i_n^*)$. Then by (2.7) we have $i_1^{(k)} \ldots i_N^{(k)} \neq i_1^* \ldots i_N^*$ for all $k \geqslant 1$. Note that $(\{0, 1\}^{\mathbb{N}}, \rho)$ is a compact metric space, where ρ is defined in (2.3). So we can find a subsequence $\{k_j\} \subset \mathbb{N}$ such that the limit $\lim_{j\to\infty}(i_n^{(k_j)})$ exists, say (i'_n). Then $i'_1 \ldots i'_N \neq i_1^* \ldots i_N^*$. Observe that

$$\Pi((i_n^{(k_j)}), \lambda_{k_j}) = x = \Pi((i_n^*), \lambda_*) \quad \forall j \geqslant 1. \tag{2.8}$$

Letting $j \to \infty$ in (2.8), by (2.7) and lemma 2.1 it follows that

$$\Pi((i'_n), \lambda_*) = x = \Pi((i_n^*), \lambda_*). \tag{2.9}$$

If $\lambda_* = 1/2$, then $\lambda_{k_j} \leqslant \lambda_*$ for all $j \geqslant 1$. Since Ψ_x is decreasing, it follows that $\Psi_x(\lambda_{k_j}) \succcurlyeq \Psi_x(\lambda_*) = (i_n^*)$ for all $j \geqslant 1$, and thus $(i'_n) \succcurlyeq (i_n^*)$. Note that (i_n^*) is the greedy coding of x in base λ_*. Then by (2.9) it follows that $(i'_n) = (i_n^*)$, leading to a contradiction with $i'_1 \ldots i'_N \neq i_1^* \ldots i_N^*$. If $\lambda_* < 1/2$, then (2.9) gives that $(i'_n) = (i_n^*)$. This again leads to a contradiction. Therefore, Ψ_x is continuous at λ_*. Since $\lambda_* \in \Lambda(x)$ is arbitrary, Ψ_x is continuous in $\Lambda(x)$. This completes the proof. □

Proof of theorem 1.1. Let $x \in (0, 1/2)$. By lemma 2.2 it follows that $\min \Lambda(x) = \Psi_x^{-1}(01^\infty) = x$ and $\max \Lambda(x) = \Psi_x^{-1}((x_n)) = 1/2$. Observe that $\Omega(x)$ is a Cantor set under the metric ρ defined in (2.3), which means that $\Omega(x)$ is a non-empty compact, perfect and totally disconnected set under ρ. Then by lemma 2.2 we conclude that $\Lambda(x)$ is also a Cantor set. □

3. Lebesgue measure and Hausdorff dimension of $\Lambda(x)$

In this section we will prove theorem 1.2, which states that for any $x \in (0, 1/2)$ the set $\Lambda(x)$ is a Lebesgue null set of full Hausdorff dimension. The key ingredient in our proof of theorem 1.2 is proposition 3.1 (see below), which indicates that the local Hausdorff dimension of $\Lambda(x)$ at some $\lambda_0 \in \Lambda(x)$ is equal to the Hausdorff dimension of the self-similar set K_{λ_0}. This property on the interplay between the 'parameter space' (in this case, $\Lambda(x)$) and the 'dynamical space' (in our case K_λ) was first observed by Douady [**6**] in the context of dynamics of real quadratic polynomials. A similar result was proved by Tiozzo [**24**], who considers for $c \in \mathbb{R}$ the set of angles of external rays which 'land' on the real slice of the Mandelbrot set to the right of c (parameter space) and the set of external angles which land on the real slice of the Julia set of the map $z \mapsto z^2 + c$ (dynamical space), showing that these two sets have the same Hausdorff dimension. Some other similar results in different settings can be found in [**3**, **5**, **15**, **25**].

PROPOSITION 3.1. *Let* $x \in (0, 1/2)$. *Then for any* $\lambda \in \Lambda(x)$ *we have*

$$\lim_{\delta\to 0^+} \dim_H(\Lambda(x) \cap (\lambda - \delta, \lambda + \delta)) = \dim_H K_\lambda = -\frac{\log 2}{\log \lambda}. \tag{3.1}$$

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The second equality in (3.1) is obvious, since for any $\lambda \in \Lambda(x)$ the self-similar set K_λ is generated by the IFS $\{\lambda x, \lambda x + (1-\lambda)\}$ satisfying the open set condition (cf. [**13**]). So it suffices to prove the first equality in (3.1).

LEMMA 3.2. *Let* $x \in (0, 1/2)$. *Then for any* $\lambda \in (x, 1/2)$ *we have*

$$\dim_H(\Lambda(x) \cap [x, \lambda]) \leqslant \dim_H K_\lambda.$$

Proof. Let $\lambda \in (x, 1/2)$. Note by lemma 2.2 that $\pi_\lambda \circ \Psi_x : \Lambda(x) \cap [x, \lambda] \to K_\lambda$ is injective. By [**7**, Proposition 3.3] we only need to prove that the inverse map $(\pi_\lambda \circ \Psi_x)^{-1}$ is Lipschitz. In other words, it suffices to prove that for any $\lambda_1, \lambda_2 \in \Lambda(x) \cap [x, \lambda]$ we have

$$|\pi_\lambda(\Psi_x(\lambda_1)) - \pi_\lambda(\Psi_x(\lambda_2))| \geqslant C|\lambda_1 - \lambda_2|, \tag{3.2}$$

where $C > 0$ is a constant independent of λ_1 and λ_2.

Take $\lambda_1, \lambda_2 \in \Lambda(x) \cap [x, \lambda]$ with $\lambda_1 < \lambda_2$, and write $\Psi_x(\lambda_1) = (i_n)$, $\Psi_x(\lambda_2) = (j_n)$. By lemma 2.2 we have $i_1 = j_1 = 0$ and $(i_n) \succ (j_n)$. Then there exists $m \geqslant 2$ such that $i_1 \dots i_{m-1} = j_1 \dots j_{m-1}$ and $i_m > j_m$. Note that

$$(1-\lambda_1)\sum_{n=2}^{\infty} i_n \lambda_1^{n-1} = x = (1-\lambda_2)\sum_{n=2}^{\infty} j_n \lambda_2^{n-1}.$$

Then

$$\begin{aligned}\frac{x(1-\lambda_1-\lambda_2)}{\lambda_1\lambda_2(1-\lambda_1)(1-\lambda_2)}(\lambda_2-\lambda_1) &= \frac{x}{\lambda_1(1-\lambda_1)} - \frac{x}{\lambda_2(1-\lambda_2)} \\ &= \sum_{n=2}^{\infty} i_n\lambda_1^{n-2} - \sum_{n=2}^{\infty} j_n\lambda_2^{n-2} \\ &\leqslant \sum_{n=2}^{m-1} i_n\lambda_1^{n-2} + \sum_{n=m}^{\infty} \lambda_1^{n-2} - \sum_{n=2}^{m-1} i_n\lambda_2^{n-2} \\ &\leqslant \sum_{n=m}^{\infty} \lambda_1^{n-2} = \frac{\lambda_1^{m-2}}{1-\lambda_1},\end{aligned}$$

where the first inequality follows by $i_1 \dots i_{m-1} = j_1 \dots j_{m-1}$, and the second inequality follows by $\lambda_1 < \lambda_2$. This, together with $\lambda_1 < \lambda_2 \leqslant \lambda$, implies that

$$\lambda^m \geqslant \lambda_1^{m-1}\lambda_2 \geqslant \frac{x(1-\lambda_1-\lambda_2)}{1-\lambda_2}(\lambda_2-\lambda_1) \geqslant x(1-2\lambda)(\lambda_2-\lambda_1).$$

Therefore,

$$\begin{aligned}|\pi_\lambda(\Psi_x(\lambda_1)) - \pi_\lambda(\Psi_x(\lambda_2))| &= (1-\lambda)\sum_{n=1}^{\infty} i_n\lambda^{n-1} - (1-\lambda)\sum_{n=1}^{\infty} j_n\lambda^{n-1} \\ &\geqslant (1-\lambda)\left(\lambda^{m-1} - \sum_{n=m+1}^{\infty}\lambda^{n-1}\right) \\ &= (1-2\lambda)\lambda^{m-1} \geqslant C|\lambda_2-\lambda_1|,\end{aligned}$$

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where $C = x(1-2\lambda)^2/\lambda > 0$ (since $\lambda < 1/2$). This proves (3.2), and then completes the proof. □

Lemma 3.3. *Let* $x \in (0, 1/2)$. *If* $\lambda \in \Lambda(x) \setminus \{1/2\}$ *such that* $\Psi_x(\lambda)$ *does not end with* 0^∞, *then for any* $\delta > 0$,

$$\dim_H(\Lambda(x) \cap [\lambda, \lambda + \delta]) \geqslant \dim_H K_\lambda.$$

Proof. Let $\lambda \in \Lambda(x) \setminus \{1/2\}$ such that $(c_n) = \Psi_x(\lambda)$ does not end with 0^∞. Take $\delta > 0$. We will construct a sequence of subsets in $\Lambda(x) \cap [\lambda, \lambda + \delta]$ whose Hausdorff dimension can be arbitrarily close to $\dim_H K_\lambda$.

Since $\lambda < 1/2$, by lemma 2.2 we have $(c_n) \succ \Psi_x(1/2) = (x_n)$. Then there exists $n_0 \geqslant 2$ such that $c_1 \ldots c_{n_0-1} = x_1 \ldots x_{n_0-1}$ and $c_{n_0} > x_{n_0}$. Since (c_n) does not end with 0^∞, we can find an increasing sequence $\{n_k\} \subset \mathbb{N}$ such that $n_0 < n_1 < n_2 < \cdots$, and $c_{n_k} = 1$ for all $k \geqslant 1$. Now for $k \geqslant 1$, we define

$$\Omega_{\lambda,k} := \left\{c_1 \ldots c_{n_k-1} 0 i_1 i_2 \ldots : \ i_{n+1} \ldots i_{n+k} \neq 0^k \ \forall n \geqslant 0\right\}. \tag{3.3}$$

Note by lemma 2.2 that $\Psi_x(\Lambda(x) \cap [\lambda, 1/2]) = \{(i_n) : (x_n) \preceq (i_n) \preceq (c_n)\}$. Then by using $c_{n_0} > x_{n_0}$ and $c_{n_k} = 1$ it follows that

$$\Omega_{\lambda,k} \subset \Psi_x(\Lambda(x) \cap [\lambda, 1/2]) \quad \text{for all } k \geqslant 1. \tag{3.4}$$

Since $\delta > 0$, by (3.3), (3.4) and lemma 2.2 there exists $N \in \mathbb{N}$ such that

$$\Lambda(x) \cap [\lambda, \lambda + \delta] \supset \Psi_x^{-1}(\Omega_{\lambda,k}) \quad \forall k \geqslant N.$$

So, to finish the proof it suffices to prove that

$$\lim_{k \to \infty} \dim_H \Psi_x^{-1}(\Omega_{\lambda,k}) \geqslant \dim_H K_\lambda. \tag{3.5}$$

Take $k \geqslant N$, and consider the map $\pi_\lambda \circ \Psi_x : \Psi_x^{-1}(\Omega_{\lambda,k}) \to \pi_\lambda(\Omega_{\lambda,k})$. Let $\lambda_1, \lambda_2 \in \Psi_x^{-1}(\Omega_{\lambda,k})$ with $\lambda_1 < \lambda_2$, and write $\Psi_x(\lambda_1) = (i_n)$, $\Psi_x(\lambda_2) = (j_n)$. Then $(i_n), (j_n) \in \Omega_{\lambda,k}$. Since $\lambda_1 < \lambda_2$, by lemma 2.2 we have $(i_n) \succ (j_n)$. So there exists $m > n_k$ such that $i_1 \ldots i_{m-1} = j_1 \ldots j_{m-1}$ and $i_m > j_m$. Note that $i_m i_{m+1} \ldots$ does not contain k consecutive zeros. Then

$$x = (1-\lambda_1) \sum_{n=1}^{\infty} i_n \lambda_1^{n-1} > (1-\lambda_2) \sum_{n=1}^{\infty} i_n \lambda_1^{n-1} > (1-\lambda_2) \left(\sum_{n=1}^{m} i_n \lambda_1^{n-1} + \lambda_1^{m+k-1} \right). \tag{3.6}$$

On the other hand,

$$x = (1-\lambda_2) \sum_{n=1}^{\infty} j_n \lambda_2^{n-1} \leqslant (1-\lambda_2) \sum_{n=1}^{m} i_n \lambda_2^{n-1}. \tag{3.7}$$

Note that $\lambda_1, \lambda_2 \in \Lambda(x) \cap [\lambda, 1/2]$. Then by (3.6) and (3.7) it follows that

$$\begin{aligned} \lambda^{m+k-1} \leqslant \lambda_1^{m+k-1} &< \sum_{n=1}^{m} i_n (\lambda_2^{n-1} - \lambda_1^{n-1}) \\ &< \sum_{n=1}^{\infty} (\lambda_2^{n-1} - \lambda_1^{n-1}) = \frac{1}{1-\lambda_2} - \frac{1}{1-\lambda_1} = \frac{\lambda_2 - \lambda_1}{(1-\lambda_1)(1-\lambda_2)}. \end{aligned}$$

This implies that

$$\begin{aligned}|\pi_\lambda(\Psi_x(\lambda_1)) - \pi_\lambda(\Psi_x(\lambda_2))| &= (1-\lambda)\sum_{n=1}^{\infty} i_n\lambda^{n-1} - (1-\lambda)\sum_{n=1}^{\infty} j_n\lambda^{n-1} \\ &\leqslant (1-\lambda)\sum_{n=m}^{\infty}\lambda^{n-1} \\ &= \lambda^{m-1} < \frac{\lambda_2-\lambda_1}{\lambda^k(1-\lambda_1)(1-\lambda_2)} \leqslant \frac{4}{\lambda^k}(\lambda_2-\lambda_1),\end{aligned} \tag{3.8}$$

where the last inequality follows by $\lambda_1,\ \lambda_2 \leqslant 1/2$.

So, by (3.3) and (3.8) it follows that

$$\begin{aligned}\dim_H \Psi_x^{-1}(\Omega_{\lambda,k}) &\geqslant \dim_H \pi_\lambda(\Omega_{\lambda,k}) \\ &= \dim_H \pi_\lambda\left(\{(i_n) : i_{n+1}\dots i_{n+k} \neq 0^k\ \forall n \geqslant 0\}\right) \\ &\geqslant \dim_H \pi_\lambda\left(\left\{(i_n) \in \{0,1\}^{\mathbb{N}} : i_n = 1 \text{ for all } n \equiv 0 (\mathrm{mod}\ k)\right\}\right) \\ &= -\frac{(k-1)\log 2}{k\log\lambda} \quad \to \frac{\log 2}{-\log\lambda} = \dim_H K_\lambda\end{aligned}$$

as $k \to \infty$. This proves (3.5), and then completes the proof. □

Proof of proposition 3.1. Take $\lambda \in \Lambda(x)$. Note that $\Lambda(x) \subset [x,\ 1/2]$ and $x,\ 1/2 \in \Lambda(x)$. We will prove (3.1) in the following two cases.

Case I. $\lambda \in \Lambda(x) \cap [x,\ 1/2)$. Then by lemma 3.2 it follows that for any $\delta \in (0,\ 1/2-\lambda)$,

$$\begin{aligned}\dim_H(\Lambda(x) \cap (\lambda-\delta, \lambda+\delta)) &\leqslant \dim_H(\Lambda(x) \cap [x, \lambda+\delta]) \\ &\leqslant \dim_H K_{\lambda+\delta} = \frac{\log 2}{-\log(\lambda+\delta)}.\end{aligned}$$

This implies that

$$\lim_{\delta\to 0^+} \dim_H(\Lambda(x) \cap (\lambda-\delta, \lambda+\delta)) \leqslant \frac{\log 2}{-\log\lambda} = \dim_H K_\lambda. \tag{3.9}$$

On the other hand, take $\delta > 0$. Note by theorem 1.1 that $\Lambda(x)$ is a Cantor set, and $\lambda \in \Lambda(x)$. Then we can find a sequence $\{\lambda_k\}$ in $\Lambda(x) \cap (\lambda-\delta,\ \lambda+\delta)$ such that each $\Psi_x(\lambda_k)$ does not end with 0^∞, and $\lambda_k \to \lambda$ as $k \to \infty$. Therefore, by lemma 3.3 it follows that

$$\begin{aligned}\dim_H(\Lambda(x) \cap (\lambda-\delta, \lambda+\delta)) &\geqslant \dim_H(\Lambda(x) \cap [\lambda_k, \lambda+\delta]) \\ &\geqslant \dim_H K_{\lambda_k} = \frac{\log 2}{-\log\lambda_k} \quad \to \frac{\log 2}{-\log\lambda} = \dim_H K_\lambda\end{aligned}$$

as $k \to \infty$. This, together with (3.9), proves (3.1).

Case II. $\lambda = 1/2$. The proof is similar to that for the second part of Case I. Let $\delta > 0$. Since $\Lambda(x)$ is a Cantor set and $\max\Lambda(x) = 1/2$, there exists a sequence $\{\lambda_k\}$

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in $\Lambda(x) \cap (1/2 - \delta,\, 1/2)$ such that each $\Psi_x(\lambda_k)$ does not end with 0^∞, and $\lambda_k \nearrow 1/2$ as $k \to \infty$. Then by lemma 3.3 it follows that

$$\begin{aligned}\dim_H(\Lambda(x) \cap (1/2 - \delta, 1/2 + \delta)) &\geqslant \dim_H(\Lambda(x) \cap [\lambda_k, \lambda_{k+1}]) \\ &\geqslant \dim_H K_{\lambda_k} = \frac{\log 2}{-\log \lambda_k} \to 1 = \dim_H K_{1/2},\end{aligned}$$

proving (3.1). □

As a direct consequence of proposition 3.1 we have the following result of $\Lambda(x)$.

Corollary 3.4. *Let $x \in (0,\, 1/2)$. Then for any open interval $I \subset \mathbb{R}$ with $\Lambda(x) \cap I \neq \emptyset$ we have*

$$\dim_H(\Lambda(x) \cap I) = \sup_{\lambda \in \Lambda(x) \cap I} \dim_H K_\lambda.$$

Proof of theorem 1.2. By corollary 3.4 it follows that

$$\dim_H \Lambda(x) = \dim_H(\Lambda(x) \cap (x, 1/2)) = \sup_{\lambda \in \Lambda(x) \cap (x, 1/2)} \dim_H K_\lambda = 1.$$

Furthermore, for any $n \in \mathbb{N}$ the Hausdorff dimension of $\Lambda_n(x) := \Lambda(x) \cap [x,\, 1/2 - 1/n]$ is strictly smaller than one, and thus each $\Lambda_n(x)$ has zero Lebesgue measure. Since $\Lambda(x) \setminus \{1/2\} = \bigcup_{n=1}^\infty \Lambda_n(x)$, the set $\Lambda(x)$ also has zero Lebesgue measure. This together with proposition 3.1 completes the proof. □

4. Hausdorff dimension of the intersection $\bigcap_{i=1}^p \Lambda(y_i)$

Given finitely many numbers $y_1,\, y_2,\, \ldots,\, y_p \in (0,\, 1/2)$, we will show in this section that the intersection $\bigcap_{i=1}^p \Lambda(y_i)$ has full Hausdorff dimension (see theorem 1.4). Note by theorem 1.1 that each set $\Lambda(y_i)$ is a Cantor set. We will construct in each $\Lambda(y_i)$ a sequence of Cantor subsets $C_\ell(y_i)$, $\ell \geqslant 1$, such that each $C_\ell(y_i)$ has the same maximum point $1/2$, and the thickness of $C_\ell(y_i)$ tends to infinity as $\ell \to \infty$. Then by using a result from Hunt, Kan and Yorke [**12**] (see lemma 4.2 below) we conclude that the intersection $\bigcap_{i=1}^p \Lambda(y_i)$ contains a sequence of Cantor subsets whose thickness tends to infinity. This, together with lemma 4.1 (see below), implies that $\bigcap_{i=1}^p \Lambda(y_i)$ has full Hausdorff dimension.

4.1. Thickness of a Cantor set in $\mathbb{R}$

First we recall the *thickness* of a Cantor set in $\mathbb{R}$ from Newhouse [**19**] (see [**1**] for some recent progress). Let E be a Cantor set in $\mathbb{R}$ with its convex hull $conv(E) = E_0$. Then the complement $E_0 \setminus E = \bigcup_{n=1}^\infty V_n$ is the union of countably many disjoint open intervals. The sequence $\mathscr{V} = (V_1,\, V_2,\, \ldots)$ is called a *defining sequence* for E. If moreover $|V_1| \geqslant |V_2| \geqslant |V_3| \geqslant \cdots$, where $|V|$ denotes the diameter of a set $V \subset \mathbb{R}$, then we call $\mathscr{V}$ an *ordered defining sequence* for E. Let $E_n := E_0 \setminus \bigcup_{k=1}^n V_k$. Then E_n is the union of finitely many disjoint closed intervals. So, for any $n \geqslant 1$, the open interval V_n is contained in some connected component of E_{n-1}, say E_{n-1}^*. Then the set $E_{n-1}^* \setminus V_n$ is the union of two closed intervals $L_{\mathscr{V}}(V_n)$ and $R_{\mathscr{V}}(V_n)$, where we always assume that $L_{\mathscr{V}}(V_n)$ lies to the left of $R_{\mathscr{V}}(V_n)$. We emphasize that both intervals $L_{\mathscr{V}}(V_n)$ and $R_{\mathscr{V}}(V_n)$ have positive length, since otherwise E will contain isolated points which is impossible. Then the thickness of E with respect to the defining sequence $\mathscr{V}$ is defined by

$$\tau_{\mathscr{V}}(E) := \inf_{n \geqslant 1} \min \left\{ \frac{|L_{\mathscr{V}}(V_n)|}{|V_n|}, \frac{|R_{\mathscr{V}}(V_n)|}{|V_n|} \right\}, \tag{4.1}$$

and the *thickness* of E is defined by

$$\tau(E) := \sup_{\mathscr{V}} \tau_{\mathscr{V}}(E), \tag{4.2}$$

where the supremum is taken over all defining sequences $\mathscr{V}$ for E. It was shown in [**27**] that $\tau(E) = \tau_{\mathscr{V}}(E)$ for every ordered defining sequence $\mathscr{V}$ for E.

The following lower bound for the Hausdorff dimension of a Cantor set in $\mathbb{R}$ in terms of thickness was proven by Newhouse [**20**] (see also [**21**, P. 77]).

LEMMA 4.1 [**20**, P. 107]. *If E is a Cantor set in $\mathbb{R}$, then*

$$\dim_H E \geqslant \frac{\log 2}{\log\left(2 + \frac{1}{\tau(E)}\right)}.$$

Two Cantor sets in $\mathbb{R}$ are called *interleaved* if neither set lies in the closure of a gap of the other. The following result for the intersection of two interleaved Cantor sets was shown by Hunt, Kan and Yorke [**12**].

LEMMA 4.2 [**12**, Theorem 1]. *There exists a function $\varphi : (1+\sqrt{2}, \infty) \to (0, \infty)$ such that for all interleaved Cantor sets E and F in $\mathbb{R}$ with $\tau(E), \tau(F) \geqslant t > 1+\sqrt{2}$, there exists a Cantor subset $K \subset E \cap F$ with $\tau(K) \geqslant \varphi(t)$.*

REMARK 4.3.

(i) In [**12**, P. 882] the authors pointed out that when t is sufficiently large, $\varphi(t)$ is of order $\sqrt{t}$. So, lemma 4.2 implies that if the thicknesses of two interleaved Cantor sets E and F in $\mathbb{R}$ are sufficiently large, then the thickness of the resulting Cantor set $K \subset E \cap F$ is also very large.

(ii) It is clear that if two Cantor sets E and F in $\mathbb{R}$ have the same maximum point ξ, then they are interleaved. Furthermore, if the maximum point ξ is also an accumulation point of $E \cap F$, then from the proof of [**12**, Theorem 1] (see also [**12**, P. 887]) it follows that the resulting Cantor set $K \subset E \cap F$ in lemma 4.2 can be required to have the same maximum point ξ.

We first construct a sequence of disjoint Cantor subsets of $\Lambda(x)$ whose thickness tends to infinity. Then by the following lemma, we can construct a sequence of Cantor subsets of $\Lambda(x)$ with the same maximum point, whose thickness also tends to infinity. These Cantor sets will be used to show that the intersection has full Hausdorff dimension.

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Lemma 4.4. *Let* $\{F_k\}_{k=1}^{\infty}$ *be a sequence of Cantor sets with* $\alpha_k = \min F_k$ *and* $\beta_k = \max F_k$. *Suppose that*

(i) $\alpha_1 < \beta_1 < \alpha_2 < \beta_2 < \ldots < \alpha_k < \beta_k < \ldots$, *and* $\beta := \lim_{k\to\infty} \beta_k$*;*

(ii) $\lim_{k\to\infty} \tau(F_k) = +\infty$*;*

(iii)

$$\lim_{k\to\infty} \frac{\beta_k - \alpha_k}{\alpha_{k+1} - \beta_k} = +\infty, \quad \lim_{k\to\infty} \frac{\beta - \alpha_{k+1}}{\alpha_{k+1} - \beta_k} = +\infty.$$

Then

$$\lim_{\ell\to\infty} \tau\left(\bigcup_{k=\ell}^{\infty} F_k \cup \{\beta\}\right) = +\infty.$$

Proof. For $k \geqslant 1$, let $\mathscr{V}_k = \{V_{k,j}\}_{j=1}^{\infty}$ be an ordered defining sequence of the Cantor set F_k, i.e.,

$$[\alpha_k, \beta_k] \setminus F_k = \bigcup_{j=1}^{\infty} V_{k,j} \tag{4.3}$$

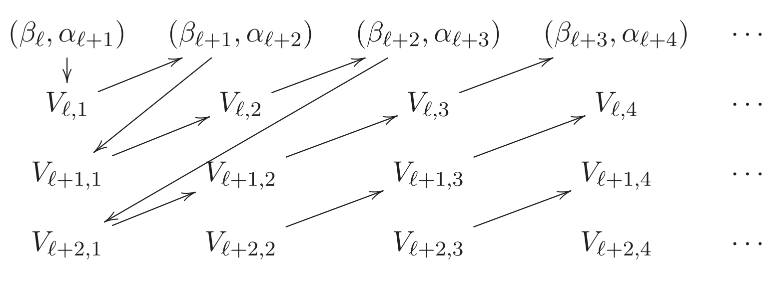
with $|V_{k,1}| \geqslant |V_{k,2}| \geqslant |V_{k,3}| \geqslant \ldots$. For $\ell \geqslant 1$, write

$$C_\ell := \bigcup_{k=\ell}^{\infty} F_k \cup \{\beta\}.$$

Note by (i) that each C_ℓ is a Cantor set, and the convex hull of C_ℓ is $[\alpha_\ell, \beta]$. Moreover, we have

$$[\alpha_\ell, \beta] \setminus C_\ell = \bigcup_{k=\ell}^{\infty} (\beta_k, \alpha_{k+1}) \ \cup \ \bigcup_{k=\ell}^{\infty} \bigcup_{j=1}^{\infty} V_{k,j}.$$

To estimate the thickness of C_ℓ, we enumerate the open intervals (β_k, α_{k+1}) and $V_{j,k}$ with $k \geqslant \ell$, $j \geqslant 1$ in the following way:



(4.4)

This means that we first remove from $[\alpha_\ell, 1/2]$ the open interval $(\beta_\ell, \alpha_{\ell+1})$, and next remove $V_{\ell,1}$, and then $(\beta_{\ell+1}, \alpha_{\ell+2})$, $V_{\ell+1,1}$, $V_{\ell,2}$, $(\beta_{\ell+2}, \alpha_{\ell+3})$, and so on.

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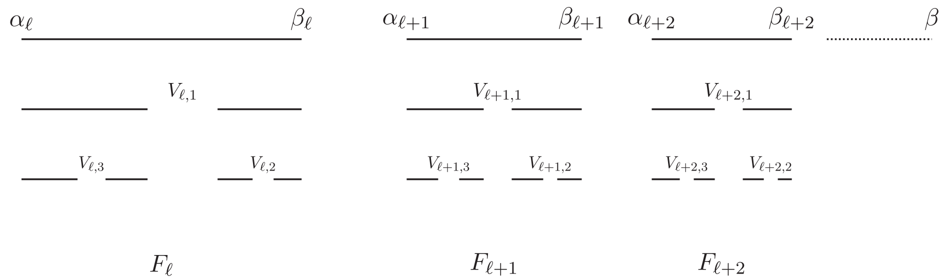

Figure 1. A defining sequence $\mathscr{W}_\ell = \{(\beta_k,\, \alpha_{k+1}),\, V_{k,j} : k \geqslant \ell,\, j \geqslant 1\}$ for the Cantor set $C_\ell = \bigcup_{k=\ell}^{\infty} F_k \cup \{\beta\}$, and for each $k \geqslant \ell$ a defining sequence $\mathscr{V}_k = \{V_{k,j}\}_{j=1}^{\infty}$ for the Cantor set F_k; see (4.3) and (4.4) for more explanation.

Thus, (4.4) gives a defining sequence $\mathscr{W}_\ell = \{(\beta_k,\, \alpha_{k+1}),\, V_{k,j} : k \geqslant \ell,\, j \geqslant 1\}$ for C_ℓ (see Fig. 1).

For the defining sequence $\mathscr{W}_\ell$ for C_ℓ, by (4.4) we have that

$$L_{\mathscr{W}_\ell}\big((\beta_k, \alpha_{k+1})\big) = [\alpha_k, \beta_k], \quad R_{\mathscr{W}_\ell}\big((\beta_k, \alpha_{k+1})\big) = [\alpha_{k+1}, \beta] \quad \text{for any } k \geqslant \ell;$$

and

$$L_{\mathscr{W}_\ell}(V_{k,j}) = L_{\mathscr{V}_k}(V_{k,j}), \quad R_{\mathscr{W}_\ell}(V_{k,j}) = R_{\mathscr{V}_k}(V_{k,j}) \quad \text{for any } k \geqslant \ell, j \geqslant 1.$$

Note that $\tau_{\mathscr{V}_k}(F_k) = \tau(F_k)$. Then by (4.1) it follows that

$$\begin{aligned} \tau_{\mathscr{W}_\ell}(C_\ell) &= \inf_{k \geqslant \ell, j \geqslant 1} \min\left\{\frac{\beta_k - \alpha_k}{\alpha_{k+1} - \beta_k}, \frac{\beta - \alpha_{k+1}}{\alpha_{k+1} - \beta_k}, \frac{|L_{\mathscr{V}_k}(V_{k,j})|}{|V_{k,j}|}, \frac{|R_{\mathscr{V}_k}(V_{k,j})|}{|V_{k,j}|}\right\} \\ &= \inf_{k \geqslant \ell} \min\left\{\tau(F_k), \frac{\beta_k - \alpha_k}{\alpha_{k+1} - \beta_k}, \frac{\beta - \alpha_{k+1}}{\alpha_{k+1} - \beta_k}\right\}. \end{aligned}$$

Note that $\tau(C_\ell) \geqslant \tau_{\mathscr{W}_\ell}(C_\ell)$. We conclude by (ii) and (iii) that $\lim_{\ell \to \infty} \tau(C_\ell) = +\infty$. □

4.2. Construction of Cantor subsets of $\Lambda(x)$

Let $x \in (0,\, 1/2)$. We will construct a sequence of Cantor subsets $\{F_k(x)\}_{k=1}^{\infty}$ of $\Lambda(x)$ satisfying the assumptions (i)–(iii) in lemma 4.4. Note by lemma 2.2 that

$$\Psi_x(\Lambda(x)) = \Omega(x) = \{(i_n) : (x_n) \preccurlyeq (i_n) \preccurlyeq 01^\infty\},$$

where $(x_n) = \Psi_x(1/2)$. Moreover, Ψ_x is a decreasing homeomorphism from $\Lambda(x)$ to $\Omega(x)$. Based on this, our strategy to construct these Cantor subsets $\{F_k(x)\}_{k=1}^{\infty}$ in $\Lambda(x)$ is to construct a sequence of Cantor subsets $\{\Omega_k(x)\}_{k=1}^{\infty}$ in the symbolic space $\Omega(x)$.

Note by the definition of $(x_n) = \Psi_x(1/2)$ that $x_1 = 0$ and (x_n) does not end with 1^∞. Denote by $\{n_k\}$ the set of all indices $n > 1$ such that $x_n = 0$. Then $x_{n_k} = 0$ for

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any $k \geqslant 1$, and $x_n = 1$ for any $n_k < n < n_{k+1}$. For $k \geqslant 1$, we define

$$\Omega_k(x) := \{(i_n) : x_1 \ldots x_{n_k-1}10^\infty \preccurlyeq (i_n) \preccurlyeq x_1 \ldots x_{n_k-1}1^\infty\}. \tag{4.5}$$

Then $\Omega_k(x) \subset \Omega(x)$, which implies that

$$F_k(x) := \Psi_x^{-1}(\Omega_k(x)) \subset \Lambda(x). \tag{4.6}$$

Note by (4.5) that for each $k \geqslant 1$ the set $(\Omega_k(x), \rho)$ is a topological Cantor set, where ρ is the metric defined in (2.3). By lemma 2.2 it follows that each $F_k(x)$ is a Cantor subset of $\Lambda(x)$.

Write $\alpha_k = \min F_k(x)$ and $\beta_k = \max F_k(x)$. Clearly, α_k and β_k depend on x. For simplicity we will suppress this dependence in our notation if no confusion arises. Since Ψ_x is decreasing by lemma 2.2, by (4.5) and (4.6) we have

$$\alpha_k = \Psi_x^{-1}(x_1x_2 \ldots x_{n_k-1}1^\infty), \quad \beta_k = \Psi_x^{-1}(x_1x_2 \ldots x_{n_k-1}10^\infty).$$

Note that $x_1x_2 \ldots x_{n_{k+1}-1}1^\infty = x_1x_2 \ldots x_{n_k-1}01^\infty \prec x_1x_2 \ldots x_{n_k-1}10^\infty$ for all $k \geqslant 1$. Thus,

$$\alpha_1 < \beta_1 < \alpha_2 < \beta_2 < \ldots < \alpha_k < \beta_k < \ldots.$$

Furthermore, the sequence $x_1x_2 \ldots x_{n_k-1}10^\infty$ decreases to $(x_n) = \Psi_x(1/2)$ as $k \to \infty$, again by lemma 2.2 we obtain that $\beta_k \nearrow 1/2$ as $k \to \infty$. Therefore, the sequence $\{F_k(x)\}_{k=1}^\infty$ of disjoint Cantor sets satisfies the assumption (i) of lemma 4.4.

In the following we show that the sequence $\{F_k(x)\}_{k=1}^\infty$ also satisfies the assumption (ii) of lemma 4.4.

Proposition 4.5. *For any $x \in (0, 1/2)$ we have $\lim_{k\to\infty} \tau(F_k(x)) = +\infty$.*

In view of the definition of thickness, we first describe a defining sequence for the Cantor set $F_k(x)$. Clearly, $[\alpha_k, \beta_k]$ is the convex hull of $F_k(x)$. By lemma 2.2 it follows that

$$[\alpha_k, \beta_k] \setminus F_k(x) = \bigcup_{\omega \in \{0,1\}^*} V_{k,\omega},$$

where

$$V_{k,\omega} := \left(\Psi_x^{-1}(x_1x_2 \ldots x_{n_k-1}1\ \omega\ 10^\infty), \Psi_x^{-1}(x_1x_2 \ldots x_{n_k-1}1\ \omega\ 01^\infty)\right).$$

We enumerate these open intervals $V_{k,\omega}$, $\omega \in \{0, 1\}^*$ according first to the length of ω and then to the lexicographical order of ω:

$$\begin{aligned} &V_{k,\epsilon}; \quad V_{k,0}, V_{k,1}; \quad V_{k,00}, V_{k,01}, V_{k,10}, V_{k,11}; \\ &V_{k,000}, V_{k,001}, V_{k,010}, V_{k,011}, V_{k,100}, V_{k,101}, V_{k,110}, V_{k,111}; \quad \ldots, \end{aligned} \tag{4.7}$$

where ϵ is the empty word. Thus, (4.7) gives a defining sequence $\mathscr{V}_k = \{V_{k,\omega} : \omega \in \{0, 1\}^*\}$ for $F_k(x)$, and moreover, we have

$$\begin{aligned} L_{\mathscr{V}_k}(V_{k,\omega}) &= \left[\Psi_x^{-1}(x_1x_2 \ldots x_{n_k-1}1\ \omega\ 1^\infty), \Psi_x^{-1}(x_1x_2 \ldots x_{n_k-1}1\ \omega\ 10^\infty)\right], \\ R_{\mathscr{V}_k}(V_{k,\omega}) &= \left[\Psi_x^{-1}(x_1x_2 \ldots x_{n_k-1}1\ \omega\ 01^\infty), \Psi_x^{-1}(x_1x_2 \ldots x_{n_k-1}1\ \omega\ 0^\infty)\right]. \end{aligned} \tag{4.8}$$

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Note by (4.8) that the endpoints of $L_{\mathscr{V}_k}(V_{k,\omega})$ and $R_{\mathscr{V}_k}(V_{k,\omega})$ have codings ending with $\omega 1^\infty$, $\omega 0^\infty$, $\omega 10^\infty$, $\omega 01^\infty$, respectively. In view of (4.1), to prove $\lim_{k\to\infty}\tau(F_k(x))$ we need the following inequalities.

LEMMA 4.6. *Let $x \in (0, 1/2)\setminus\{1/4\}$ with $(x_n) = \Psi_x(1/2)$, and let $m \geqslant 3$ such that $x_m = 1$.*

(i) *If $\lambda_1, \lambda_2 \in \Lambda(x)$ satisfy $\Psi_x(\lambda_1) = j_1 j_2 \dots j_q 1^\infty$ and $\Psi_x(\lambda_2) = j_1 j_2 \dots j_q 0^\infty$, then*

$$\lambda_2 - \lambda_1 \geqslant \frac{1}{4}\lambda_2^q.$$

(ii) *If $\lambda_3, \lambda_4 \in \Lambda(x)$ satisfy $\Psi_x(\lambda_3) = x_1 \dots x_m j_1 \dots j_q 10^\infty$ and $\Psi_x(\lambda_4) = x_1 \dots x_m j_1 \dots j_q 01^\infty$, then*

$$\lambda_4 - \lambda_3 \leqslant \min\left\{2(1-2\lambda_3)\lambda_3^{q+2}, 2(1-2\lambda_4)\frac{\lambda_4^{m+q}}{\lambda_3^{m-2}}\right\}.$$

Before giving the proof we emphasize that for $x = 1/4$ we have $(x_n) = \Psi_x(1/2) = 010^\infty$. So we can not find $x_m = 1$ for $m \geqslant 3$, which plays an essential role in the proof of lemma 4.6 (ii).

Proof. Since $x \in (0, 1/2)\setminus\{1/4\}$, the expansion $(x_n) = \Psi_x(1/2)$ satisfies $x_m = 1$ for some $m \geqslant 3$. For (i), let $\lambda_1, \lambda_2 \in \Lambda(x)$ with $\Psi_x(\lambda_1) = j_1 j_2 \dots j_q 1^\infty$ and $\Psi_x(\lambda_2) = j_1 j_2 \dots j_q 0^\infty$. Then by lemma 2.2 we have $\lambda_1 < \lambda_2$. Note that $x = \pi_{\lambda_1}(j_1 j_2 \dots j_q 1^\infty) = \pi_{\lambda_2}(j_1 j_2 \dots j_q 0^\infty)$. Then

$$\begin{aligned}
\lambda_2^q = \pi_{\lambda_2}(0^q 1^\infty) &= \pi_{\lambda_2}(j_1 j_2 \dots j_q 1^\infty) - \pi_{\lambda_2}(j_1 j_2 \dots j_q 0^\infty)\\
&= \pi_{\lambda_2}(j_1 j_2 \dots j_q 1^\infty) - \pi_{\lambda_1}(j_1 j_2 \dots j_q 1^\infty)\\
&\leqslant 4(\lambda_2 - \lambda_1),
\end{aligned}$$

where the inequality follows by lemma 2.1 (ii) since for any $(i_n) \in \Omega(x)$ we have

$$\frac{\mathrm{d}\pi_\lambda((i_n))}{\mathrm{d}\lambda} = \sum_{n=2}^{\infty} n\left(\frac{n-1}{n} - \lambda\right) i_n \lambda^{n-2} \leqslant \sum_{n=2}^{\infty}\frac{n-1}{2^{n-2}} = 4.$$

This proves (i).

For (ii) let $\lambda_3, \lambda_4 \in \Lambda(x)$ such that

$$\Psi_x(\lambda_3) = x_1 \dots x_m j_1 \dots j_q 10^\infty \quad \text{and} \quad \Psi_x(\lambda_4) = x_1 \dots x_m j_1 \dots j_q 01^\infty,$$

where $x_m = 1$ with $m \geqslant 3$. Then by lemma 2.2 we have $\lambda_3 < \lambda_4$. Note that $x = \pi_{\lambda_3}(x_1 \dots x_m j_1 j_2 \dots j_q 10^\infty) = \pi_{\lambda_4}(x_1 \dots x_m j_1 j_2 \dots j_q 01^\infty)$. Then

$$\begin{aligned}
&\pi_{\lambda_3}(0^{m+q}10^\infty) - \pi_{\lambda_4}(0^{m+q+1}1^\infty)\\
&\quad = \pi_{\lambda_4}(x_1 \dots x_m j_1 j_2 \dots j_q 0^\infty) - \pi_{\lambda_3}(x_1 \dots x_m j_1 j_2 \dots j_q 0^\infty)\\
&\quad \geqslant \frac{1}{2}\lambda_3^{m-2}(\lambda_4 - \lambda_3),
\end{aligned} \tag{4.9}$$

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where the last inequality follows by lemma 2.1 (ii) since by using $x_m = 1$ with $m \geqslant 3$ and (2.5) we have

$$\frac{\mathrm{d}\pi_\lambda(x_1 \dots x_m j_1 j_2 \dots j_q 0^\infty)}{\mathrm{d}\lambda} \geqslant m\left(\frac{m-1}{m} - \lambda\right) x_m \lambda^{m-2} \geqslant \frac{1}{2}\lambda^{m-2}.$$

Therefore, by (4.9) and using $\lambda_3 < \lambda_4$ it follows that

$$\begin{aligned}\lambda_4 - \lambda_3 &\leqslant \frac{2}{\lambda_3^{m-2}}\left((1-\lambda_3)\lambda_3^{m+q} - \lambda_4^{m+q+1}\right)\\ &\leqslant \frac{2}{\lambda_3^{m-2}}\left((1-\lambda_3)\lambda_3^{m+q} - \lambda_3^{m+q+1}\right) = 2(1-2\lambda_3)\lambda_3^{q+2},\end{aligned}$$

and

$$\begin{aligned}\lambda_4 - \lambda_3 &\leqslant \frac{2}{\lambda_3^{m-2}}\left((1-\lambda_3)\lambda_3^{m+q} - \lambda_4^{m+q+1}\right)\\ &\leqslant \frac{2}{\lambda_3^{m-2}}\left((1-\lambda_4)\lambda_4^{m+q} - \lambda_4^{m+q+1}\right) = 2(1-2\lambda_4)\frac{\lambda_4^{m+q}}{\lambda_3^{m-2}}.\end{aligned}$$

This completes the proof. □

When $x = 1/4$ we prove similar inequalities by using different estimation.

LEMMA 4.7. *Let $x = 1/4$. Then $(x_n) = \Psi_x(1/2) = 010^\infty$.*

(i) *If $\lambda_1, \lambda_2 \in \Lambda(x)$ satisfy $\Psi_x(\lambda_1) = 010^m j_1 j_2 \dots j_q 1^\infty$ and $\Psi_x(\lambda_2) = 010^m j_1 j_2 \dots j_q 0^\infty$, then*

$$\lambda_2 - \lambda_1 \geqslant \frac{\lambda_2^{m+2+q}}{1 - 2\lambda_1 + (m+3)2^{-m}}.$$

(ii) *If $\lambda_3, \lambda_4 \in \Lambda(x)$ satisfy $\Psi_x(\lambda_3) = 01 j_1 j_2 \dots j_q 10^\infty$ and $\Psi_x(\lambda_4) = 01 j_1 j_2 \dots j_q 01^\infty$, then*

$$\lambda_4 - \lambda_3 \leqslant \lambda_3^{2+q}.$$

Proof. For (i) we note by lemma 2.2 that $\lambda_1 < \lambda_2$. Since $x = \pi_{\lambda_1}(010^m j_1 j_2 \dots j_q 1^\infty) = \pi_{\lambda_2}(010^m j_1 j_2 \dots j_q 0^\infty)$, we have

$$\begin{aligned}\lambda_2^{m+2+q} = \pi_{\lambda_2}(0^{m+2+q}1^\infty) &= \pi_{\lambda_2}(010^m j_1 j_2 \dots j_q 1^\infty) - \pi_{\lambda_2}(010^m j_1 j_2 \dots j_q 0^\infty)\\ &= \pi_{\lambda_2}(010^m j_1 j_2 \dots j_q 1^\infty) - \pi_{\lambda_1}(010^m j_1 j_2 \dots j_q 1^\infty)\\ &\leqslant \left(1 - 2\lambda_1 + (m+3)2^{-m}\right)(\lambda_2 - \lambda_1),\end{aligned}$$

where the inequality follows by lemma 2.1 (ii) since by (2.5) we have

$$\begin{aligned}\frac{\mathrm{d}\pi_\lambda(010^m j_1 j_2 \dots j_q 1^\infty)}{\mathrm{d}\lambda} &\leqslant (1-2\lambda) + \sum_{n=m+3}^{\infty} n\left(\frac{n-1}{n} - \lambda\right)\lambda^{n-2}\\ &\leqslant 1 - 2\lambda + \sum_{n=m+3}^{\infty}\frac{n-1}{2^{n-2}} = 1 - 2\lambda + (m+3)2^{-m}.\end{aligned}$$

This proves (i).

For (ii), note by lemma 2.2 that $\lambda_3 < \lambda_4$. Then by using $x = \pi_{\lambda_3}(01j_1j_2 \ldots j_q10^\infty) = \pi_{\lambda_4}(01j_1j_2 \ldots j_q01^\infty)$ it follows that we have

$$\begin{aligned}\pi_{\lambda_3}(0^{2+q}10^\infty) - \pi_{\lambda_4}(0^{3+q}1^\infty) &= \pi_{\lambda_4}(01j_1j_2 \ldots j_q0^\infty) - \pi_{\lambda_3}(01j_1j_2 \ldots j_q0^\infty)\\ &\geqslant \pi_{\lambda_4}(010^\infty) - \pi_{\lambda_3}(010^\infty)\\ &= (1-\lambda_3-\lambda_4)(\lambda_4-\lambda_3).\end{aligned}$$

This implies that

$$\begin{aligned}\lambda_4 - \lambda_3 &\leqslant \frac{1}{1-\lambda_3-\lambda_4}\left((1-\lambda_3)\lambda_3^{2+q} - \lambda_4^{3+q}\right)\\ &\leqslant \frac{1}{1-\lambda_3-\lambda_4}\left((1-\lambda_3)\lambda_3^{2+q} - \lambda_4\lambda_3^{2+q}\right) = \lambda_3^{2+q},\end{aligned}$$

as desired. □

Proof of proposition 4.5. Write $\Psi_x(1/2) = (x_n)$. Let $\{n_k\}$ be the enumeration of all indices $n > 1$ such that $x_n = 0$. Recall from (4.7) the defining sequence $\mathscr{V}_k = \{V_{k,\omega} : \omega \in \{0, 1\}^*\}$ for $F_k(x)$. It follows from (4.1) that

$$\tau_{\mathscr{V}_k}(F_k(x)) = \inf\left\{\frac{|L_{\mathscr{V}_k}(V_{k,\omega})|}{|V_{k,\omega}|}, \frac{|R_{\mathscr{V}_k}(V_{k,\omega})|}{|V_{k,\omega}|} : \omega \in \{0,1\}^*\right\}.$$

Note by (4.8) that

$$\begin{aligned}L_{\mathscr{V}_k}(V_{k,\omega}) &= \left[\Psi_x^{-1}(x_1x_2\ldots x_{n_k-1}1\ \omega\ 1^\infty), \Psi_x^{-1}(x_1x_2\ldots x_{n_k-1}1\ \omega\ 10^\infty)\right]\\ &=: \left[\gamma_{\omega,1}, \gamma_{\omega,2}\right],\\ R_{\mathscr{V}_k}(V_{k,\omega}) &= \left[\Psi_x^{-1}(x_1x_2\ldots x_{n_k-1}1\ \omega\ 01^\infty), \Psi_x^{-1}(x_1x_2\ldots x_{n_k-1}1\ \omega\ 0^\infty)\right]\\ &=: \left[\gamma_{\omega,3}, \gamma_{\omega,4}\right].\end{aligned}$$

Then

$$V_{k,\omega} = \left(\Psi_x^{-1}(x_1x_2\ldots x_{n_k-1}1\ \omega\ 10^\infty), \Psi_x^{-1}(x_1x_2\ldots x_{n_k-1}1\ \omega\ 01^\infty)\right) = \left(\gamma_{\omega,2}, \gamma_{\omega,3}\right).$$

Observe by lemma 2.2 that $\alpha_k \leqslant \gamma_{\omega,1} < \gamma_{\omega,2} < \gamma_{\omega,3} < \gamma_{\omega,4}$, where $\alpha_k = \min F_k(x)$.

Case (A). $x \in (0, 1/2) \setminus \{1/4\}$**.** Then there exists an integer $m \geqslant 3$ such that $x_m = 1$. Take a sufficiently large k so that $n_k > m$. By lemma 4.6 it follows that

$$\gamma_{\omega,2} - \gamma_{\omega,1} \geqslant \frac{1}{4}\gamma_{\omega,2}^{n_k+q+1}, \quad \gamma_{\omega,4} - \gamma_{\omega,3} \geqslant \frac{1}{4}\gamma_{\omega,4}^{n_k+q+1},$$

and

$$\gamma_{\omega,3} - \gamma_{\omega,2} \leqslant 2(1-2\gamma_{\omega,2})\gamma_{\omega,2}^{n_k+q-m+2},$$

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where q is the length of the word ω. So, for any $\omega \in \{0, 1\}^*$ we obtain that

$$\frac{|L_{\mathscr{V}_k}(V_{k,\omega})|}{|V_{k,\omega}|} = \frac{\gamma_{\omega,2} - \gamma_{\omega,1}}{\gamma_{\omega,3} - \gamma_{\omega,2}} \geqslant \frac{\gamma_{\omega,2}^{m-1}}{8(1 - 2\gamma_{\omega,2})} \geqslant \frac{\alpha_k^{m-1}}{8(1 - 2\alpha_k)},$$
$$\frac{|R_{\mathscr{V}_k}(V_{k,\omega})|}{|V_{k,\omega}|} = \frac{\gamma_{\omega,4} - \gamma_{\omega,3}}{\gamma_{\omega,3} - \gamma_{\omega,2}} \geqslant \frac{\gamma_{\omega,4}^{m-1}}{8(1 - 2\gamma_{\omega,2})} \geqslant \frac{\alpha_k^{m-1}}{8(1 - 2\alpha_k)}.$$

Thus, for sufficiently large k we have

$$\tau(F_k(x)) \geqslant \tau_{\mathscr{V}_k}(F_k(x)) \geqslant \frac{\alpha_k^{m-1}}{8(1 - 2\alpha_k)}.$$

Note that $\alpha_k \nearrow 1/2$ as $k \to \infty$ because $\beta_k \nearrow 1/2$. Therefore, $\tau(F_k(x)) \to +\infty$ as $k \to \infty$.

Case (B). $x = 1/4$. Then $(x_n) = \Phi_x(1/2) = 010^\infty$, which gives $x_1 \ldots x_{n_k - 1} = 010^{n_k - 3}$ for all $k \geqslant 1$. By lemma 4.7 it follows that

$$\gamma_{\omega,2} - \gamma_{\omega,1} \geqslant \frac{\gamma_{\omega,2}^{n_k+q+1}}{1 - 2\gamma_{\omega,1} + n_k 2^{3-n_k}}, \quad \gamma_{\omega,4} - \gamma_{\omega,3} \geqslant \frac{\gamma_{\omega,4}^{n_k+q+1}}{1 - 2\gamma_{\omega,3} + n_k 2^{3-n_k}},$$

and

$$\gamma_{\omega,3} - \gamma_{\omega,2} \leqslant \gamma_{\omega,2}^{n_k+q},$$

where q is the length of the word ω. This implies that for all $\omega \in \{0, 1\}^*$,

$$\frac{|L_{\mathscr{V}_k}(V_{k,\omega})|}{|V_{k,\omega}|} = \frac{\gamma_{\omega,2} - \gamma_{\omega,1}}{\gamma_{\omega,3} - \gamma_{\omega,2}} \geqslant \frac{\gamma_{\omega,2}}{1 - 2\gamma_{\omega,1} + n_k 2^{3-n_k}} \geqslant \frac{\alpha_k}{1 - 2\alpha_k + n_k 2^{3-n_k}},$$
$$\frac{|R_{\mathscr{V}_k}(V_{k,\omega})|}{|V_{k,\omega}|} = \frac{\gamma_{\omega,4} - \gamma_{\omega,3}}{\gamma_{\omega,3} - \gamma_{\omega,2}} \geqslant \frac{\gamma_{\omega,4}}{1 - 2\gamma_{\omega,3} + n_k 2^{3-n_k}} \geqslant \frac{\alpha_k}{1 - 2\alpha_k + n_k 2^{3-n_k}}.$$

Thus,

$$\tau(F_k(x)) \geqslant \tau_{\mathscr{V}_k}(F_k(x)) \geqslant \frac{\alpha_k}{1 - 2\alpha_k + n_k 2^{3-n_k}}.$$

Note that $\alpha_k \nearrow 1/2$ and $n_k \to +\infty$ as $k \to \infty$. Therefore, $\tau(F_k(x)) \to +\infty$ as $k \to \infty$, completing the proof. □

4.3. Hausdorff dimension of $\bigcap_{i=1}^{p} \Lambda(y_i)$

For any $\ell \geqslant 1$, we define

$$C_\ell(x) := \bigcup_{k=\ell}^{\infty} F_k(x) \cup \{1/2\}. \tag{4.10}$$

Note that $\mathrm{conv}(F_k(x)) = [\alpha_k, \beta_k]$ for all $k \geqslant 1$, and

$$\alpha_1 < \beta_1 < \alpha_2 < \beta_2 < \cdots < \alpha_k < \beta_k < \cdots, \quad \text{and} \quad \lim_{k\to\infty} \beta_k = 1/2.$$

Since $F_k(x) \subset \Lambda(x)$ for all $k \geqslant 1$ and $1/2 \in \Lambda(x)$, it follows that each $C_\ell(x)$ is a Cantor subset of $\Lambda(x)$.

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Proposition 4.8. *For any $x \in (0, 1/2)$ we have $\lim_{\ell\to\infty} \tau(C_\ell(x)) = +\infty$.*

Proof. Take $x \in (0, 1/2)$. Write $\Psi_x(1/2) = (x_n)$. Let $\{n_k\}$ be the enumeration of all indices $n > 1$ such that $x_n = 0$. By lemma 4.4 and proposition 4.5, it remains to prove that

$$\lim_{k\to\infty} \frac{\beta_k - \alpha_k}{\alpha_{k+1} - \beta_k} = +\infty \quad \text{and} \quad \lim_{k\to\infty} \frac{1/2 - \alpha_{k+1}}{\alpha_{k+1} - \beta_k} = +\infty, \tag{4.11}$$

where

$$\alpha_k = \Psi_x^{-1}(x_1x_2\ldots x_{n_k-1}1^\infty), \quad \beta_k = \Psi_x^{-1}(x_1x_2\ldots x_{n_k-1}10^\infty), \tag{4.12}$$

and

$$\alpha_{k+1} = \Psi_x^{-1}(x_1\ldots x_{n_k-1}x_{n_k}\ldots x_{n_{k+1}-1}1^\infty) = \Psi_x^{-1}(x_1x_2\ldots x_{n_k-1}01^\infty). \tag{4.13}$$

Case (A). $x \in (0, 1/2) \setminus \{1/4\}$. There exists $m \geqslant 3$ such that $x_m = 1$. For sufficiently large k, we have $n_k > m$. By (4.12), (4.13) and lemma 4.6 it follows that

$$\beta_k - \alpha_k \geqslant \frac{1}{4}\beta_k^{n_k}, \quad \alpha_{k+1} - \beta_k \leqslant \min\left\{2(1-2\beta_k)\beta_k^{n_k-m+1}, 2(1-2\alpha_{k+1})\frac{\alpha_{k+1}^{n_k-1}}{\beta_k^{m-2}}\right\},$$

which implies

$$\frac{\beta_k - \alpha_k}{\alpha_{k+1} - \beta_k} \geqslant \frac{\beta_k^{m-1}}{8(1-2\beta_k)} \quad \text{and} \quad \frac{1/2 - \alpha_{k+1}}{\alpha_{k+1} - \beta_k} \geqslant \frac{\beta_k^{m-2}}{4\alpha_{k+1}^{n_k-1}}.$$

Note that $\alpha_k \nearrow 1/2$, $\beta_k \nearrow 1/2$ and $n_k \to \infty$ as $k \to \infty$. We obtain (4.11) by letting $k \to \infty$.

Case (B). $x = 1/4$. Then $(x_n) = \Phi_x(1/2) = 010^\infty$. By (4.12), (4.13) and lemma 4.7 it follows that

$$\beta_k - \alpha_k \geqslant \frac{\beta_k^{n_k}}{1 - 2\alpha_k + n_k 2^{3-n_k}}, \quad \alpha_{k+1} - \beta_k \leqslant \beta_k^{n_k-1}. \tag{4.14}$$

This implies

$$\frac{\beta_k - \alpha_k}{\alpha_{k+1} - \beta_k} \geqslant \frac{\beta_k}{1 - 2\alpha_k + n_k 2^{3-n_k}}. \tag{4.15}$$

On the other hand, note by (4.13) that

$$\frac{1}{4} = x = \pi_{\alpha_{k+1}}(x_1x_2\ldots x_{n_k-1}01^\infty) = \pi_{\alpha_{k+1}}(010^{n_k-2}1^\infty) = (1-\alpha_{k+1})\alpha_{k+1} + \alpha_{k+1}^{n_k}.$$

This implies that

$$\left(\frac{1}{2} - \alpha_{k+1}\right)^2 = \alpha_{k+1}^{n_k}, \quad \text{and thus} \quad \frac{1}{2} - \alpha_{k+1} = \alpha_{k+1}^{n_k/2}. \tag{4.16}$$

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Note by (4.14) that $\alpha_{k+1} - \beta_k \leqslant \beta_k^{n_k-1} < \alpha_{k+1}^{n_k-1}$. This, together with (4.16), implies that

$$\frac{1/2 - \alpha_{k+1}}{\alpha_{k+1} - \beta_k} \geqslant \frac{1}{\alpha_{k+1}^{n_k/2-1}}. \tag{4.17}$$

Note that $\alpha_k \nearrow 1/2$, $\beta_k \nearrow 1/2$ and $n_k \to +\infty$ as $k \to \infty$. Letting $k \to \infty$ in (4.15) and (4.17) we obtain (4.11), completing the proof. □

Proof of theorem 1.4. Let $y_1, \ldots, y_p \in (0, 1/2)$. Then by (4.2), (4.10) and proposition 4.8 it follows that each $\Lambda(y_i)$ contains a sequence of Cantor subsets $C_\ell(y_i)$, $\ell \geqslant 1$ such that $\max C_\ell(y_i) = 1/2$ for all $\ell \geqslant 1$, and the thickness $\tau(C_\ell(y_i)) \to +\infty$ as $\ell \to \infty$. So, by lemma 4.2 and remark 4.3 (i) it follows that for sufficiently large ℓ and for any $i, j \in \{1, 2, \ldots, p\}$ the intersection $C_\ell(y_i) \cap C_\ell(y_j)$ contains a Cantor subset $C_\ell(y_i, y_j)$ such that

$$\tau(C_\ell(y_i, y_j)) \to +\infty \quad \text{as } \ell \to \infty. \tag{4.18}$$

Note that for any $k \in \{1, 2, \ldots, p\}$ we have $\min C_\ell(y_k) \nearrow 1/2 = \max C_\ell(y_k)$ as $\ell \to \infty$. Furthermore, $C_\ell(y_k) \supset C_{\ell+1}(y_k)$ for any $\ell \geqslant 1$. Then by (4.18) it follows that the maximum point $1/2$ is an accumulation point of $C_\ell(y_i) \cap C_\ell(y_j)$. So, by remark 4.3 (ii) we can require that the resulting Cantor set $C_\ell(y_i, y_j) \subset C_\ell(y_i) \cap C_\ell(y_j)$ has the maximum point $1/2$ for sufficiently large ℓ and any $i, j \in \{1, 2, \ldots, p\}$.

Proceeding this argument for all $y_1, y_2, \ldots, y_p$ we obtain that for sufficiently large ℓ the intersection $\bigcap_{i=1}^p C_\ell(y_i)$ contains a Cantor subset $C_\ell(y_1, \ldots, y_p)$ such that $\max C_\ell(y_1, \ldots, y_p) = 1/2$, and the thickness $\tau(C_\ell(y_1, \ldots, y_p)) \to +\infty$ as $\ell \to \infty$. Therefore, by lemma 4.1 it follows that

$$\begin{aligned}\dim_H \bigcap_{i=1}^p \Lambda(y_i) &\geqslant \dim_H \bigcap_{i=1}^p C_\ell(y_i) \geqslant \dim_H C_\ell(y_1, \ldots, y_p) \\ &\geqslant \frac{\log 2}{\log(2 + \frac{1}{\tau(C_\ell(y_1, \ldots, y_p))})} \to 1,\end{aligned}$$

as $\ell \to \infty$. This completes the proof. □

At the end of this section we remark that in the proof of theorem 1.4 we construct a sequence of Cantor subsets $C_\ell(y_1, \ldots, y_p)$, $\ell \geqslant 1$ in the intersection $\bigcap_{i=1}^p \Lambda(y_i)$ such that the thickness $\tau(C_\ell(y_1, \ldots, y_p)) \to +\infty$ as $\ell \to \infty$. By a recent work of Yavicoli [**28**, remark of Theorem 4] it follows that the intersection $\bigcap_{i=1}^p \Lambda(y_i)$ contains arbitrarily long arithmetic progression.

5. Final remarks

At the end of this paper we point out that our results theorem 1.1–1.4 can be extended to higher dimensions. To illustrate this we give two examples.

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EXAMPLE 5.1. For $\lambda \in (0, 1/2]$ let K_λ be the self-similar set defined in (1.1). Then for $n \in \mathbb{N}$ the product set $\bigotimes_{i=1}^n K_\lambda$ is also a self-similar set in $\mathbb{R}^n$. For $\mathbf{a} = (a_1, \ldots, a_n) \in (0, 1/2)^n$ let

$$\Lambda(\mathbf{a}) := \left\{ \lambda \in (0, 1/2] : \mathbf{a} \in \bigotimes_{i=1}^{n} K_\lambda \right\}$$

be the set of parameters $\lambda \in (0, 1/2]$ such that the n-dimensional self-similar set $\bigotimes_{i=1}^n K_\lambda$ contains the given point $\mathbf{a}$. It is clear that $\Lambda(\mathbf{a}) = \bigcap_{i=1}^n \Lambda(a_i)$, where for $x \in (0, 1/2)$ the set $\Lambda(x)$ is defined as in (1.2). So, by theorems 1.2 and 1.4 it follows that $\Lambda(\mathbf{a})$ has zero Lebesgue measure and full Hausdorff dimension for any $\mathbf{a} \in (0, 1/2)^n$.

EXAMPLE 5.2. Let $n \in \mathbb{N}$ and let $\lambda_i \in (0, 1/2]$ for all $1 \leqslant i \leqslant n$. Then $\bigotimes_{i=1}^n K_{\lambda_i} \subset \mathbb{R}^n$ is a self-affine set generated by the IFS $\{(\lambda_1 x_1, \lambda_2 x_2, \ldots, \lambda_n x_n) + \mathbf{i} : \mathbf{i} \in \bigotimes_{i=1}^n \{0, 1 - \lambda_i\}\}$. For any $\mathbf{b} = (b_1, \ldots, b_n) \in (0, 1/2)^n$ let

$$\Lambda'(\mathbf{b}) := \left\{ (\lambda_1, \lambda_2, \ldots, \lambda_n) \in (0, 1/2]^n : \mathbf{b} \in \bigotimes_{i=1}^{n} K_{\lambda_i} \right\}.$$

Then $(\lambda_1, \lambda_2, \ldots, \lambda_n) \in \Lambda'(\mathbf{b})$ if and only if $b_i \in K_{\lambda_i}$ for all $1 \leqslant i \leqslant n$, which is also equivalent to $\lambda_i \in \Lambda(b_i)$ for all $1 \leqslant i \leqslant n$. So, $\Lambda'(\mathbf{b}) = \bigotimes_{i=1}^n \Lambda(b_i)$. By theorem 1.1 it follows that for any $\mathbf{b} \in (0, 1/2)^n$ the set $\Lambda'(\mathbf{b})$ is a Cantor set in $\mathbb{R}^n$, i.e., it is a non-empty compact, totally disconnected and perfect set in $\mathbb{R}^n$. Furthermore, by [**7**, product formula 7.2] and theorem 1.2 we obtain that $\Lambda'(\mathbf{b})$ has Lebesgue measure zero and $\dim_H \Lambda'(\mathbf{b}) = n$ for any $\mathbf{b} \in (0, 1/2)^n$.

Note that in examples 5.1 and 5.2 the higher dimensional self-similar sets all have the product form $\bigotimes_{i=1}^n K_{\lambda_i}$. However, if the higher dimensional self-similar sets do not have the product form, our results theorems 1.1–1.4 can not be applied directly. Some recent progress on the extension of Newhouse thickness theorems to higher dimensions may be useful in this direction (cf. [**2**, **8**, **9**, **29**]).

Acknowledgements

The authors thank the anonymous referee for many useful comments which greatly improve the paper. K. Jiang was supported by NSFC No. 11701302, Zhejiang Provincial NSF No. LY20A010009, and the K.C. Wong Magna Fund in Ningbo University. D. Kong was supported by NSFC No. 11971079. W. Li was supported by NSFC No. 12071148 and Science and Technology Commission of Shanghai Municipality (STCSM) No. 22DZ2229014. Z. Wang was supported by Fundamental Research Funds for the Central Universities No. YBNLTS2023-016.

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