Can. J. Math., Vol. XLII, No. 4, 1990, pp. 646–658

RESTRICTED DETERMINANTAL HOMOMORPHISMS AND LOCALLY FREE CLASS GROUPS

VICTOR SNAITH

1. Introduction. Let K be a number field and let O_K denote the integers of K. The locally free class groups, $\mathrm{Cl}(O_K[G])$, furnish a fundamental collection of invariants of a finite group, G. In this paper I will construct some new, non-trivial homomorphisms, called *restricted determinants*, which map the N_GH-invariant idèlic units of $O_K[H^{ab}]$ to $\mathrm{Cl}(O_K[G])$. These homomorphisms are constructed by means of the Hom-description of $\mathrm{Cl}(O_K[G])$, which describes the locally free class group in terms of the representation theory of G, and the technique of Explicit Brauer Induction, which was introduced in [**5**].

Let $J^*(O_K[H^{ab}])$ denote the idèles of $O_K[H^{ab}]$ and let $U^*(O_K[H^{ab}])$ denote the subgroup of unit idèles (see (2.3)/(2.4)). Let N_GH denote the normaliser of the subgroup, H, in G and let $W_GH = (N_GH)/H$ act (by conjugation) on $J^*(O_K[H^{ab}])$. For each subgroup, H of G there is a restricted determinant map (see 4 for details)

$$(1.1) \qquad \mathrm{Det}_H : J^*(O_K[H^{ab}])^{W_GH} \longrightarrow \mathrm{Cl}(O_K[G]).$$

When $H = G$ (1.1) may be described simply without the use of Explicit Brauer Induction. As explained in 2, $\mathrm{Cl}(O_K[G])$ is describeable as a quotient of $\mathrm{Hom}_{\Omega_K}(R(G), J^*(E))$. If $u \in J^*(O_K[G^{ab}])$ we may define a map

$$(1.2) \qquad \mathrm{Det}_G(u) : R(G) \longrightarrow J^*(E)$$

by sending a representation, ν, to

$$(1.3) \qquad \prod_{i=1}^{t} \phi_i(u) \in J^*(E)$$

where ν decomposes into irreducibles as $\nu = \phi_1 \oplus \ldots \oplus \phi_t \oplus \rho_1 \oplus \ldots (\dim(\phi_1) = 1, \dim(\rho_j) \geqq 2)$. In (1.3) $\phi_i(u)$ means the element of $J^*(E)$ (for some splitting field, E) obtained by evaluating ϕ_i on the group elements in $u = \sum_{g\in G} \lambda_g g$. In (1.1) Det_G is given by sending u to the class represented by (1.2) (see 5.4 (proof)).

This paper is arranged in the following manner. In 2 we recall the Hom-description of the class group and the classical determinantal homomorphisms. In 3 we summarise the properties of Explicit Brauer induction from [**5**] and

Received April 24, 1989.
Research partially supported by NSERC grant #OGP0042580

the improvements due to R. Boltje [**1**; **2**]. In 4 the restricted determinants are constructed and their main properties are collected in Theorem 4.6. In 5 we study a simple example and deduce some further properties. In particular, in 5 we study the quotient group

$$(1.4)\qquad B(G) = \mathrm{Cl}(O_K[G])\Big/\left\{\sum_{H \lneqq G} \mathrm{im}\,(\mathrm{Cl}(O_K[H]) \longrightarrow \mathrm{Cl}(O_K[G])\right\}.$$

We show that there is a partially ordered filtration, $F_{(H)}B(G)$, on $B(G)$, indexed by the poset of conjugacy classes of subgroups of G and we show that Det_H induces a surjection

$$(1.5)\qquad \mathrm{Det}_H : \hat{H}^0(W_GH;\, j^*(O_K[H^{ab}]) \longrightarrow \mathrm{Gr}\,_{(H)}B(G) \text{ where } \mathrm{Gr}_{(H)} = F_{(H)}\Big/\left(\sum_{(Y)>(H)} F_{(Y)}\right).$$

2. The locally free class group. Let G be a finite group and let O_K denote the algebraic integers in G number field, K. Let $\mathrm{Cl}(O_K[G])$ denote the class group of finitely generated $O_K(G)$ – modules which are locally free [**3**, p. 219; 4].

Let us recall from [**3**, p. 334; 4] Fröhlich's Hom-description of $\mathrm{Cl}(O_K[G])$.

Suppose that E/K is a Galois extension of number fields, where E is chosen large enough so as to be a splitting field for G. Let $J(E)$ denote the idèle group of E. Suppose that E lies within a fixed algebraic closure, K^c, of K and let $\Omega_K = \mathrm{Gal}(K^c/K)$. Also let $R(G)$ denote the Grothendieck ring of finite dimensional K^c–representations of G. Hence Ω_K acts (on the left, say) on $R(G)$ by means of its action entry by entry on GL_nK^c and Ω_K acts also on E and on $J(E)$.

Let $J^*(E)$ denote the multiplicative group of the idéles and consider the group of Ω_K-equivariant homomorphisms.

$$(2.1)\qquad \mathrm{Hom}_{\Omega_K}(R(G),\ J^*(E)) = \{f : R(G) \longrightarrow J^*(E) | f(w(\chi)) = \mathrm{wf}(\chi)), w \in \Omega_K\}.$$

The diagonal embedding of $E^* = E - \{0\}$ into $J^*(E)$ induces an inclusion of abelian groups

$$(2.2)\qquad \mathrm{Hom}_{\Omega_K}(R(G),\ E^*) \longrightarrow \mathrm{Hom}_{\Omega_K}(R(G),\ J^*(E)).$$

For each prime, P (finite or infinite), of K let O_{K_P} and K_P denote the completions of O_K and K at P. When P is infinite we adopt the familiar convention that $O_{K_P} = K_p$.

Define the group ring unit idéles by

$$(2.3)\qquad U^*(O_K[G]) = \prod_P (O_{K_P}[G])^*.$$

The group, $U^*(O_K[G])$, is a subgroup of the group-ring idèles

$$(2.4)\qquad J^*(O_K[G]) = \{(u_p) \in \prod_P (K_p[G])^* | u_p \in (O_{K_P}[G])^* \text{a.e.}\}$$

If $\{u_P\} \in J^*(O_K[G])$ we may define a homomorphism

$$(2.5)\qquad \mathrm{Det}\{u_P\} \in \mathrm{Hom}_{\Omega_K}(R(G),\ J^*(E))$$

in the following manner. Let

$$(2.6)\qquad \chi : G \longrightarrow GL_n E$$

denote a representation of G. If $u_p = \sum_{g \in G} \lambda_g g$ then

$$(2.7)\qquad \det\left(\sum_{g \in G} \lambda_g \chi(g)\right) \in E_Q^*$$

for each prime, Q, of E lying over P.

The homomorphism of (2.5) is defined by setting the Q-component of $\mathrm{Det}\{u_P\}(\chi)$ equal to (2.7). Hence we obtain a homomorphism

$$(2.8)\qquad \mathrm{Det} : J^*(O_K[G]) \longrightarrow \mathrm{Hom}_{\Omega_K}(R(G),\ J^*(E)).$$

With the notation introduced above there is an isomorphism [3, p. 334]

$$(2.9)\qquad \mathrm{Cl}(O_K[G]) \cong \frac{\mathrm{Hom}_{\Omega_K}(R(G),\ J^*(E))}{\mathrm{Hom}_{\Omega_K}(R(G),\ E^*)\ \mathrm{Det}(U^*(O_K[G]))}$$

Remark 2.10.

When G is abelian the map $\mathrm{Det} : U^*(O_K[G]) \longrightarrow \mathrm{Det}(U^*(O_K[G]))$ is an isomorphism. Later we will require the following consequence of this observation. Suppose that H is a subgroup of G and that $N_G H$ is its normaliser. Set $W_G H = N_G H/H$. If H^{ab} denotes the abelianisation of H then the conjugation action of $N_G H$ on H induces a $W_G H$-action on H^{ab}. Det induces an isomorphism on the $W_G H$-invariant elements

$$(2.11)\qquad \mathrm{Det} : U^*(O_K[H^{ab}])^{W_G H} \xrightarrow{\cong} \mathrm{Det}(U^*(O_K[H^{ab}]))^{W_G H}.$$

3. Explicit Brauer Induction. Explicit Brauer Induction is a canonical form of Brauer's induction theorem. The first such canonical form appeared in [**5**] (see also [**2**; **4**; **6**; **7**]). However, in this paper it will be convenient to use a related construction due to R. Boltje. Since the latter construction has not yet appeared in print I will describe it in terms of [**5**; **6**; **8**].

Denote by $R_+(G, (K^c)^*)$ the free abelian group on the G-conjugacy classes of subhomomorphisms.

(3.1) $$G \supset H \xrightarrow{\phi} (K^c)^*$$

where K^c is as in 2. $R_+(G, (K^c)^*)$ is a ring-valued functor when endowed with the following structure.

The product is defined by

(3.2) $$(G \supset H \xrightarrow{\phi} (K^c)^*)(G \supset J \xrightarrow{\Psi} (K^c)^*) = \sum_{z \in_H \backslash G/J} (G \supset H \cap zJz^{-1} \xrightarrow{\phi((z^{-1})^*(\Psi))} (K^c(^*)$$

where $(z^{-1})^*(\Psi)(\alpha) = \Psi(z^{-1}\alpha z)$.

If H is a subgroup of G the restriction homomorphism

(3.3) $$\mathrm{Res}^G_H : R_+(G, (K^c)^*) \longrightarrow R_+(H, (K^c)^*)$$

is given by the formula

$$\mathrm{Res}^G_H(G \supset J \xrightarrow{\phi} (K^c)^*) = \sum_{z \in_H \backslash G/J} (H \supset H \cap zJz^{-1} \xrightarrow{(z^{-1})^*(\phi)} (K^c)^*).$$

Induction

(3.4) $$\mathrm{Ind}^G_H : R_+(H, (K^c(^*) \longrightarrow R_+(G, (K^c)^*)$$

is given by

$$\mathrm{Ind}^G_H(H \supset J \xrightarrow{\phi} (K^c)^*) = (G \supset J \xrightarrow{\phi} (K^c)^*).$$

If $\Pi : P \longrightarrow G$ is a surjection then we have an inflation map

(3.5) $$\mathrm{Inf}^P_G : R_+(G, (K^c)^*) \longrightarrow R_+(P, (K^c)^*)$$

given by

$$\mathrm{Inf}^P_G(G \supset J \xrightarrow{\phi} (K^c)^*) = P \supset \Pi^{-1}(J) \xrightarrow{\phi\pi} (K^c)^*).$$

With the structure of (3.2) – (3.5) $R_+(-, (K^c)^*)$ is a Mackey functor in the sense of [**1**].

There is a canonical homomorphism, which is surjective, to the representation ring,

(3.6) $b_G : R_+(G,\ (K^c)^*) \longrightarrow R(G)$

$$b_G(G \supset J \xrightarrow{\phi} (K^c)^*) = \mathrm{Ind}_J^G(\phi).$$

The homomorphism, b_G, commutes with the usual restriction, induction and inflation maps of $R(G)$ so that

$$b_H \mathrm{Res}_H^G(z) = \mathrm{Res}_H^G(b_G(z))(z \in R_+(G,\ (K^c)^*)),$$

$$b_G \mathrm{Ind}_H^G(y) = \mathrm{Ind}_H^G b_H(y)(y \in R_+(H,\ (K^c)^*)),$$

and

$$b_p \mathrm{Inf}_G^P(z) = \mathrm{Inf}_G^P b_G(z).$$

The explicit Brauer induction map of R. Boltje is a homomorphism

(3.7) $a_G : R(G) \longrightarrow R_+(G,\ (K^c)^*)$

which is characterised by the following properties.
(3.8) (i) If H is a subgroup of G then

$$\mathrm{Res}_H^G a_G = a_H\ \mathrm{Res}_H^G,$$

(ii) Let $\nu : G \longrightarrow GL_n(K^c)$ be a representation and write $a_G(\nu) = \sum_i \alpha_i(G \supset J_i \xrightarrow{\phi_i} (K^c)^*)$. For each i such that $G = J_i$ then $\alpha_i = \langle \nu,\ \phi_i \rangle = \{$multiplicity of ϕ_i in $\nu\}$,

(iii) If ν is one-dimensional then $a_G(\nu) = (G \supset G \xrightarrow{\nu} (K^c)^*)$.

(iv) $b_G a_G = 1 : R(G) \longrightarrow R(G)$.

(3.9) The relation between a_G and the Explicit Brauer induction formulae of [**5**; **7**; **8**] is as follows. Each n-dimensional representation of G, ν, over K^c determines a unique complex, unitary representation

$$\nu : G \longrightarrow U(n).$$

Let $R_+(G,\ NT^n)$ denote the free abelian group of G- and NT^n-conjugacy classes of subhomomorphisms, $(G \supset J \xrightarrow{\Psi} NT^n)$, where NT^n is the normaliser of the torus, T^n, of diagonal matrices in the unitary group, $U(n)$. G acts, via ν, on $U(n)T^n$ and from this action an element

(3.10) $\tau_G(\nu) \in R_+(G,\ NT^n)$

is defined in [**5**]. The map from NT^n to the trivial group induces a map

$$R_+(G,\ NT^n) \longrightarrow R_+(G,\ \{1\})$$

which sends $\tau_G(\nu)$ to $\epsilon_G(\nu)$. Also $R_+(G,\ \{1\})$ is naturally a subring of $R_+(G,\ S^1)$, where S^1 is the 1-torus. In [**5**] I defined a homomorphism

(3.11) $\quad \rho_G : R_+(G,\ NT^n) \longrightarrow R_+(G,\ S^1) \cong R_+(G,\ (K^c)^*).$

The following properties summarise the results of [**2**; **5**; **9**]:–

(3.12) (i) Define $T_G(\nu) = \rho_G(\tau_G(\nu)) \in R_+(G,\ (K^c)^*)$ then $T_G(\nu)$ and $\epsilon_G(\nu)$ are well-defined and natural in G. If dim $\nu = 1$, $T_G(\nu) = (G > G \xrightarrow{\nu} (K^c)^*)$.

(ii) In $R(G)$, $b_G T_G(\nu) = \nu$ and $b_G\epsilon_G(\nu) = 1$.

(iii) $\epsilon_G(\nu \oplus \mu) = \epsilon_G(\nu)\epsilon_G(\mu) \in R_+(G,\ \{1\})$.

(iv) $T_G(\nu \oplus \mu) = T_G(\nu)\epsilon_G(\mu) + \epsilon_G(\nu)T_G(\mu)$ in $R_+(G,\ (K^c)^*)$, and

(v) $a_G(\nu)\epsilon_G(\nu) = T_G(\nu)$ in $R_+(G,\ (K^c)^*)$.

Adams operations 3.13. (see [**2**; **8**]).

Let $\Psi^k : R(G) \longrightarrow R(G)$ be the k-th Adams operation. If ν is a representation $\Psi^k(\nu)$ is the k-th Newton polynomial in the exterior powers of ν, $\{\lambda^i(\nu)\}$. In terms of characters, if χ_ν is the character function of ν then $\chi_{\Psi^k(\nu)}(g) = \chi(g^k) \quad (g \in G)$.

If $a_G(\nu)$ or $T_G(\nu)$ is equal to $\sum_i \alpha_i(G > J_i \xrightarrow{\phi_i} (K^c)^*) \in R_+(G,\ (K^c)^*)$ then, for all $k \geqq 0$,

(3.14) $\quad \Psi^k(\nu) = \sum_i \alpha_i\ \mathrm{Ind}_{H_i}^G(\phi_i^k) \in R(G)$

(where ϕ_i^k is the k-th power, $\phi_i^k = \Psi^k(\phi_i)$).

The formula of (3.14), first proved in [**8**], is used in [**10**] to prove a conjecture of M. J. Taylor on determinantal congruences [**9**, p. 469, Remark 2] (see also [**3**, p. 364 (54.12); **4**, p. 79, l. 6]).

PROPOSITION 3.15. *Let* $\Omega_K = \mathrm{Gal}(K^c/K)$ *act on* $R_+(G,\ (K^c)^*)$ *via its action on* $(K^c)^*$. *Then* $a_G : R(G) \longrightarrow R_+(G,\ (K^c)^*)$ *is* Ω_K*-equivariant.*

Proof. It is shown in [1] that a_G is uniquely characterised by the properties of 3.8(i)–(iv). However, if $w \in \Omega_K$, then the homomorphism $(\nu \longrightarrow w(a_G(w^{-1}(\nu))))$ also satisfies §3.8(i)–(iv) so that $a_G(\nu) = wa_G(w^{-1}(\nu))$ for all representations, ν, in $R(G)$.

4. Restricted determinants. Let H be a subgroup of G. In the notation of 2 we will define a *restricted determinant homomorphism*

(4.1) $\quad \mathrm{Det}_H : J^*(O_K[H^{ab}]))^{W_G H} \longrightarrow \mathrm{Cl}(O_K[G]).$

Here, as in (2.10), W_GH is the Weyl group of H in G, $W_GH = (N_GH)/H$. Give an idèle

$$\nu \in J^*(O_K[H^{ab}])^{W_GH}$$

we may assign to it the homomorphism which assigns to $\nu \in R(G)$

$$(4.2) \qquad \mathrm{Det}(\nu)\left(\sum \alpha_i(H^{ab} \xrightarrow{\phi_i} (K^c)^*)\right) \in J^*(E)$$

where the sum in (4.2) is over all the terms of $a_G(\nu) = \sum_j \alpha_j(G > H_j \xrightarrow{\phi_j} (K^c)^*)$ for which H_j is conjugate to H. This is well-defined because ν is W_GH-invariant and the homomorphisms

$$(4.3) \qquad (H \longrightarrow H^{ab} \xrightarrow{\phi_i} (K^c)^*)$$

which appear in $a_G(\nu)$ are well-defined up to conjugation by elements of G so that, once we have chosen H to represent H_i, then (4.3) is defined up to the action by N_GH. By 3.15 the resulting homomorphism, which will be denoted by $\mathrm{Det}_H(\nu) : R(G) \longrightarrow J^*(E))$ actually lies in $\mathrm{Hom}_{\Omega_K}(R(G), J^*(E))$. Passing to class groups via (2.8) we obtain

$$\mathrm{Det}_H(\nu) \in \mathrm{Cl}(O_K[G])$$

and obtain the required homomorphism of (4.1).

Suppose the H is a subgroup of G then we have canonical maps of representation groups

$$(4.4) \qquad R(H^{ab}) \longrightarrow R(H) \xrightarrow{\mathrm{Ind}_H^G} R(G)$$

which may be assembled to induce, via (2.9), a map

$$(4.5) \qquad \beta_G : \mathrm{Cl}(O_K[G]) \longrightarrow \bigoplus_{(H)} \mathrm{Cl}(O_K[H^{ab}])^{W_GH}$$

where (H) denotes the G-conjugacy class of H and the sum in (4.5) runs over all conjugacy classes of subgroups of G.

We are now ready to state and prove our main result on restricted determinants.

THEOREM 4.6. *With the notation introduced in* 2 *there is for each subgroup, H, of G a homomorphism*

$$\mathrm{Det}_H : J^*(O_K[H^{ab}])^{W_GH} \longrightarrow \mathrm{Cl}(O_K[G]).$$

(i) *The images of the* $\{\mathrm{Det}_H : H \leqq G\}$ *generate* $\mathrm{Cl}(O_K[G])$.

(ii) *If β is not injective in* (4.5) *then at least one of the homomorphisms (from the unit ideles)* $\mathrm{Det}_H : U^*(O_K[H^{ab}])^{W_G H} \longrightarrow \mathrm{Cl}(O_K[G])$ *is non-trivial.*

(iii) *Suppose that J is a subgroup of G then for each subgroup, H, of J the following diagram commutes:*

$$\begin{array}{ccc} J^*(\mathcal{O}_K[H^{ab}])^{W_J H} & \xrightarrow{\mathrm{Det}_H} & \mathrm{Cl}(\mathcal{O}_K[J]) \\ \Big\downarrow \bigoplus_Y N_Y^H & & \Big\downarrow \mathrm{Ind}_J^G \\ \bigoplus_Y J^*(\mathcal{O}_K[Y^{ab}])^{W_G H} & \xrightarrow{\sum_Y \mathrm{Det}_Y} & \mathrm{Cl}(\mathcal{O}_K[G]) \end{array}$$

Here, if $u \in J^*(O_K[H^{ab}])^{W_J H}$, then

$$N_Y^H(u) = \prod_{\substack{z \in J\backslash G/Y \\ zYz^{-1}\cap J = H}} (z^{-1}uz) \in J^*(O_K[Y^{ab}])W_G Y.$$

Also Ind_J^G *is the map induced, via* (2.9), *by the map* Res_J^G *of representation rings.*

Proof. The isomorphism of (2.9), as described in [**4**, p. 20] for example, is given by constructing from locally free $O_K[G]$-module an element, $u \in J^*(O_K[G])$, and then taking its image under determinant homomorphism of (2.8). Hence $\{\mathrm{Det}(u) | u \in J^*(O_K[G])\}$ generates $\mathrm{Cl}(O_K[G])$.

Using the homomorphism, a_G, of (3.7) we may refine this fact to establish part (i). Let the subgroup, H, vary through a set, $\sum$, of conjugacy class representatives. Write, for $\chi \in R(g)$,

$$a_G(\chi) = \sum_{H\in\Sigma} a_{G,H}(\chi)$$

where $a_{G,H}(\chi)$ is the sum of all the terms in $a_G(\chi) = \sum_i \alpha_i(G \supset H_i \xrightarrow{\phi_i} (K^c)^*)$ for which $(H_i) = (H)$. We may express 3.8 (iv) as

$$\chi = \sum_{H\in\sigma} \mathrm{Ind}_H^G(a_{G,H}(\chi)) \in R(G), \tag{4.7}$$

where $\mathrm{Ind}_H^G(a_{G,H}(\chi)) = b_G(a_{G,H}(\chi))$ in terms of (3.6). Hence $\mathrm{Cl}(O_K[G])$ is generated by the images of the homomorphisms

$$\left\{\chi \longrightarrow \mathrm{Det}(u)(\mathrm{Ind}_H^G(\alpha_{G,H}(\chi))) | u \in J^*(O_K[G]),\ H \in \sum\right\}. \tag{4.8}$$

We must show that each of these maps is given by applying $a_{G,H}(\chi)$ to a $W_G H$-invariant unit of $J^*(O_K[H^{ab}])$. It suffices for this to examine each local component separately. However, the Q-component (Q lying over P) of the determinant map factors through the algebraic K-group, $K_1(S_P[G])$, where $S_P = K_P$ or O_{K_P}. Furthermore, the following diagram of canonical maps commutes, since it is the adjoint of [**3**, p. 340].

(4.9):
$$\begin{array}{ccccc}
K_1(S_P[G]) \otimes R(H) & \xrightarrow{1 \otimes \mathrm{Ind}} & K_1(S_P[G]) \otimes R(G) & & \\
\uparrow & & & \searrow^{\mathrm{Det}} & \\
K_1(S_P[G]) \otimes R(H^{ab}) & & & & \\
\downarrow{\scriptstyle \mathrm{Res} \otimes 1} & & & & \\
K_1(S_P[H])^{N_G H} \otimes R(H^{ab}) & \longrightarrow & K_1(S_P[H])^{N_G H} \otimes R(H) & \xrightarrow{\mathrm{Det}} & (K_Q^c)^* \\
\downarrow & & & \nearrow_{\mathrm{Det}} & \\
K_1(S_P[H^{ab}])^{W_G H} \otimes R(H^{ab}) & & & &
\end{array}$$

However, in (4.9), $K_1(S_P[H^{ab}]) \cong (S_P[H^{ab}])^*$ by [**3**, (46.24)]. Therefore, if we start with $u \otimes a_{G,H}(\chi) \in J^*(O_K[G]) \otimes R(H^{ab})$ then the image via the clockwise route in (4.9) is just the map of (4.8). Since the final map in the anticlockwise route is the Q-component of Det_H we have established part (i).

Now we turn to the proof of part (ii). Let A be a $\mathbf{Z}[G]$-module and let A_G denote the group of coinvariants

$$A_G = A/\{g(a) - a | g \in G,\ a \in A\}.$$

We have a homomorphism, which is split injective,

(4.10) $$\alpha_G : R(G) \longrightarrow \bigoplus_{H \in \Sigma} R(H^{ab})_{W_G H}$$

given by

$$\alpha_G(z)_H = \sum \alpha_i(\phi_i : H^{ab} \longrightarrow (K^c)^*)$$

where

$$a_{G,H}(z) = \sum \alpha_i(H = H_i \xrightarrow{\phi_i} (K^c)^*).$$

When $X = E^*$ or $J^*(E)$ there is an induced (surjective) map

(4.11) $$\alpha_G^* : \bigoplus_{H \in \Sigma} \mathrm{Hom}_{\Omega_K}(R(H^{ab}),\ X)^{W_G H} \longrightarrow \mathrm{Hom}_{\Omega_K}(R(G),\ X)$$

https://doi.org/10.4153/CJM-1990-034-7 Published online by Cambridge University Press

since

$$\mathrm{Hom}_{\Omega_K}(R(H^{ab})_{W_GH},\ X) \cong \mathrm{Hom}_{\Omega_K}(R(H^{ab}),\ X)^{W_GH}.$$

The map, α_G^*, of (4.11) is split by the maps induced by (4.4). Therefore, if we temporarily set

$$\Lambda(G) = \mathrm{Hom}_{\Omega_K}(R(G),\ J^*(E))/\mathrm{Hom}_{\Omega_K}(R(G),\ E^*)$$

we obtain a split surjection

$$(4.12)\qquad \alpha_G^* : \bigoplus_{H\in\Sigma} \Lambda(H^{ab})^{W_GH} \longrightarrow \Lambda(G).$$

To prove part (ii) we need to show that the groups $a_G^*(\mathrm{Det}(U^*(O_K[H^{ab}])^{W_GH})))$ do not all lie within the image of $\mathrm{Det}(U^*(O_K[G]))\ \mathrm{Hom}_{\Omega_K}(R(G),\ E^*)$ in $\Lambda(G)$. However, if this were not so we would receive an induced map

$$(4.13)\qquad \alpha_G^* : \bigoplus_{H\in\Sigma} \Lambda(H^{ab})^{W_\epsilon H}/\mathrm{Det}(U^*(O_K[H^{ab}])^{W_GH} \longrightarrow \mathrm{Cl}(O_K[G]).$$

The H-summand of the domain of (4.13) is a subgroup of $\mathrm{Cl}(O_K[H^{ab}])^{W_GH}$ and the map, a_G^*, of (4.13) would be a left inverse to β_G of (4.5). This is impossible, since β_G is not injective, by hypothesis.

Finally, consider part (iii). Let $u \in J^*(O_K[H^{ab}])^{W_JH}$ be an idèle. In $\mathrm{Hom}_{\Omega_K}(R(G),\ J^*(E))$ the image of this element by the clockwise route in the diagram sends $\chi \in R(G)$ to

$$\mathrm{Det}(u)(a_{G,H}(\mathrm{Res}_J^G(\chi))) \in J^*(E).$$

However, if $a_G(\chi) = \sum_{Y\in\Sigma} a_{G,Y}(\chi)$ then, by 3.8(i),

$$(4.14)\qquad a_J(\mathrm{Res}_J^G(\chi)) = \sum_{Y\in\Sigma}\ \sum_{z\in J\backslash G/Y} \alpha_i(J \supset J\cap zYz^{-1} \xrightarrow{(z^{-1^*})(\phi_i)} (K^c)^*)$$

where $a_{G,Y}(\chi) = \sum_i \alpha_i(G \supset Y \xrightarrow{\phi_i} (K^c)^*)$. Therefore

$$\begin{aligned}\mathrm{Det}(u)(\alpha_{G,H}(\mathrm{Res}_J^G(\chi))) &= \prod_{Y\in\Sigma}\ \prod_{\substack{z\in J\backslash G/Y\\ zYz^{-1}\cap J=H}} \mathrm{Det}(z^{-1}uz)(a_{G,Y}(\chi))\\ &= \prod_Y \mathrm{Det}_Y(N_Y^H(u)),\end{aligned}$$

as required. This completes the proof of Theorem 4.6.

5. The class group of G versus those of its subgroups In this section we will study the manner in which we may filter the group of (1.5)

$$(5.1) \qquad B(G) = \mathrm{Cl}(O_K[G]) \Big/ \left\{ \sum_{H \subsetneq G} \mathrm{im}(\mathrm{Cl}(O_K[H])) \longrightarrow \mathrm{Cl}(O_K[G]) \right\}$$

by means of restricted determinants.

Firstly let us pause for an elementary example.

Example 5.2. Let Q_8 denote the quaternion group of order eight

$$Q_8 = \{x,\ y | x^2 = y^2,\ x^4 = 1,\ xyx^{-1} = y^{-1}\}.$$

Hence $Q_8^{ab} \cong \mathbf{Z}/2 \times \mathbf{Z}/2$ and the subgroups of Q_8 are cyclic of order 1, 2 or 4. The class group of $\mathbf{Z}[H]$ is trivial for $\mathbf{Z}/2 \times \mathbf{Z}/2$, $\mathbf{Z}/4, \mathbf{Z}/2$ and $\{1\}$ so that

$$\mathrm{Det}_{Q_8} : U^*(\mathbf{Z}[\mathbf{Z}/2 \times \mathbf{Z}/2]) \longrightarrow \mathrm{Cl}(\mathbf{Z}[Q_8]) \cong \mathbf{Z}/2$$

must be non-trivial. From [**3**, p. 349 (53.17)] the nontrivial element of this class group is the Swan module $\langle 3,\ \sigma\rangle \subset \mathbf{Z}[Q_8]$ where $\sigma = (1 + x + x^2 + x^3)(1 + y)$. By [**3**, p. 335 (52.13)] $\langle 3,\ \sigma\rangle = \langle -3,\ \sigma\rangle$ is represented by the homomorphism which sends the trivial to the idèle which equals (-3) in the 2-adic coordinate and 1 elsewhere; all other irreducibles are sent to 1.

$$\text{In } \mathbf{Z}_2[\mathbf{Z}/2 \times \mathbf{Z}/2] \text{ let } u_2 = x + xy + y$$

where x, y are generators. Since u_2 has augmentation $3 \in \mathbf{Z}_2^*$ we see that $u_2 \in (\mathbf{Z}_2[\mathbf{Z}/2 \times \mathbf{Z}/2])^*$. Defining $u_p = 1$ for all places different from 2 we obtain $u = (u_p) \in U^*(\mathbf{Z}[\mathbf{Z}/2 \times \mathbf{Z}/2])$.

From [**6**, p. 186 and p. 207] one finds the following formula for $T_{Q_8}(\chi)(= a_{Q_8}(\chi))$ for χ irreducible.

$$(5.3) \qquad a_{Q_8}(\phi) = (Q_8 \xrightarrow{\phi} (K^c)^*) \text{ if } \dim \phi = 1,$$
$$a_{Q_8}(\nu) = \sum_{g=x,y,xy} (Q_8 \supset \langle g\rangle \cong \mathbf{Z}/4 \xrightarrow{\lambda} (K^c)^*) - (Q_8 \supset \langle x^2\rangle \xrightarrow{\mu} (K^c)^*)$$

where $\mu(x^2) = -1$ and $\lambda(g) = i(i^2 = -1)$.

Therefore, from (5.3), $\mathrm{Det}_{Q_8}(u)(\chi) = 1$ unless $\dim \chi = 1$ (χ irreducible) and when $\dim \chi = 1$, $\chi \neq 1$ it is also trivial but $\mathrm{Det}_{Q_8}(u)(1) = u$ which equals (-3) at the 2-adic place and 1 elsewhere.

Hence $\mathrm{Det}_{Q_8}(u) = \langle 3,\ \sigma\rangle$.

A similar calculation shows that the Swan modules for the generalized quaternion 2-group, Q_{2^n}, all lie in $\mathrm{Det}_{Q_{2^n}}(U^*(\mathbf{Z}[\mathbf{Z}/2 \times \mathbf{Z}/2]))$.

PROPOSITION 5.4. *If* $H \lneqq G$ *then the composition* $J^*(O_k[H^{ab}]) \longrightarrow \mathrm{Cl}(O_K[G]) \longrightarrow \mathrm{Cl}(O_K[G^{ab}])$ *is trivial. If* $H = G$ *this composition is surjective and annihilates* $U^*(O_K[G^{ab}])$.

Proof. Let ν be an irreducible representation of G. If $a_G(\nu) = \sum_i \alpha_i(G \supset H_i \xrightarrow{\phi_i} (K^c)^*)$ then no H_i equals G unless $\dim(\nu) = 1$ in which case $a_G(\nu) = (G \xrightarrow{\nu} (K^c)^*)$. From this the statement for the case $H = G$ follows at once. Furthermore, if χ is a representation inflated from G^{ab} then $a_G(\chi) = \sum \alpha_j(G = G \xrightarrow{\phi_j} (K^c)^x)$ so that $\mathrm{Det}_H(u)(\chi)$ is trivial for all such χ if $H \neq G$.

5.5 Let $B(G)$ denote the quotient of $\mathrm{Cl}(O_K(G))$ by the images of the class groups of the proper subgroups, as in (5.1). Consider the poset whose elements are conjugacy classes, (H), of subgroups of G. We set $(H) \leqq (J)$ if $zHz^{-1} \leqq J$ for some $z \in G$. Define a filtration on $B(G)$, indexed by this poset.

$$(5.6) \qquad F_{(H)}B(G) = \left\{ \sum_{(Y) \geqq (H)} \mathrm{im}(\mathrm{Det}_Y : J^*(O_K[Y^{ab}])^{W_G Y} \longrightarrow B(G) \right\}.$$

Define an associated graded object

$$(5.7) \qquad Gr_{(H)}B(G) = F_{(H)}B(G) \Big/ \left(\sum_{(Y) \gneqq (H)} F_{(Y)}B(G) \right).$$

THEOREM 5.8. *With the notation introduced above*
(i) $\bigcup_{(H)} F_{(H)}B(G) = B(G)$.
(ii) Det_H *induces a surjection*

$$\mathrm{Det}_H : \hat{H}^0(W_G H;\ J^*(O_K[H^{ab}])) \longrightarrow Gr_{(H)}B(G).$$

In (ii) $\hat{H}^0$ *denotes Tate cohomology.*

Proof. Part (i) follows from 4.6(i). Recall that $\hat{H}^0(Z; A) = A^Z / \{\prod_{z \in Z} z(a) | a \in A\}$. By definition Det_H will induce a surjection

$$J^*(O_K[H^{ab}])^{W_G H} \longrightarrow Gr_{(H)}B(G)$$

and, by 4.6(ii) (with $H = J \leqq G$), this surjection kills $\mathrm{im}(N_H^H)$ which is the image of map which averages over $W_G H$. This completes the proof of 5.8.

REFERENCES

1. R. Boltje, Thesis, Universität Augsburg (1989).
2. R. Boltje, V. Snaith and P. Symonds, *Algebraicisation of Explicit Brauer Induction, with applications*, submitted to J. of Algebra.
3. C. W. Curtis and I. Reiner, *Methods of representation theory*, Vol II; Wiley (1987).

4. A. Fröhlich, *Galois module structure of algebraic integers*, Ergeb. Math. (Folge 3, band 1), Springer-Verlag (1983).
5. V. Snaith, *Explicit Brauer Induction*, Inventiones Math. *94* (1988) 455–478.
6. ———, *Topological methods in Galois representation theory*, C.M. Soc. Monographs, Wiley (1989).
7. ———, *Applications of Explicit Brauer Induction*, A.M. Soc. Proc. Symp. Pure Math *47* (1987) 495–531.
8. ———, *Invariants of representations*, Proc. Lake Louise K-theory conference 445–508 (1987), NATO ASI series vol. 279, Kluwer (1989).
9. M. J. Taylor, *Locally free class groups of groups of prime power order*, J. Alg. *50* (2) (1978) 463–487.
10. V. P. Snaith: On the class group and Swan subgroup of an integral group-ring; McMaster preprint #3 (1989/90).

McMaster University,
Hamilton, Ontario,
Canada L8S 4K1