NATURAL PARTIAL ORDERS

R. A. DEAN AND GORDON KELLER

1. Introduction. Let n be an ordinal. A partial ordering P of the ordinals $T = T(n) = \{w: w < n\}$ is called *natural* if $x\,P\,y$ implies $x \leqslant y$.

A natural partial ordering, hereafter abbreviated NPO, of $T(n)$ is thus a coarsening of the natural total ordering of the ordinals. Every partial ordering of a finite set S is isomorphic to a natural partial ordering. This is a consequence of the theorem of Szpielrajn **(5)** which states that every partial ordering of a set may be refined to a total ordering. In this paper we consider only natural partial orderings. In the first section we obtain theorems about the lattice of all NPO's of $T(n)$. In the succeeding sections, for $n \leqslant \omega$ we associate a subgroup of the upper triangular group $TL(n)$ of $n \times n$ matrices. We obtain necessary and sufficient conditions for an NPO to be associated with a normal subgroup of $TL(n)$; we show that these "normal" NPO's form a distributive sublattice and for finite n we count the order of this lattice. We show that this number,

$$\binom{2n}{n}\Big/ (n+1),$$

is exactly the number of non-isomorphic partial orderings P of $\{0, \ldots, n-1\}$ which do *not* have a sub-partially ordered set like either of those shown in Figure 1.

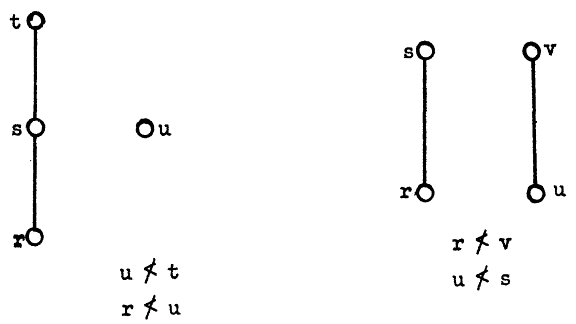

FIGURE 1.

Since a relation P on T is a subset of $T \times T$ we shall use whichever of the notations $(x, y) \in P$, $x\,P\,y$ or $(x, y) \notin P$, $x \sim Py$ seems most convenient. The collection of NPO's of T will be denoted by $\mathfrak{N}$ or by $\mathfrak{N}(n)$ if it is important

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to emphasize the ordinal n. It is natural to order $\mathfrak{N}$ itself by set inclusion: we write $P \subseteq Q$ if $x\,P\,y$ implies $x\,Q\,y$ for all (x, y) in $T \times T$. For any ordinal n we have the minimal ordering $\phi = \{(x, x): x < n\}$ and the maximal ordering $\Omega = \{(x, y): x \leqslant y < n\}$. Clearly $\phi \leqslant P \leqslant \Omega$ for every P. If $x < y$ we define the NPO $\Gamma(x, y) = \{(r, s): r = s \text{ or } (r, s) = (x, y)\}$, which is easily seen to be a minimal element over ϕ. In fact the following theorem is easily proved.

THEOREM 1.1. *For any ordinal n, the set $\mathfrak{N}(n)$ of all natural partial orderings of $T(n) = \{w: w < n\}$ is a complete, compactly generated,* point lattice in which the set of compact elements is the set of points $\{\Gamma(x, y): 0 \leqslant x < y < n\}$.*

We shall omit the proof of this theorem and the following standard useful result.

LEMMA 1.1. *If $\mathfrak{R}$ is a subset of $\mathfrak{N}$, then $(x, y) \in \bigcup\mathfrak{R}$ if and only if there exists an integer r and a sequence $x = u_1 < \ldots < u_r = y$ such that for each i, $1 \leqslant i < r$, there exists $R_i \in \mathfrak{R}$ such that $u_i\,R_i\,u_{i+1}$.*

Figure 2 shows $\mathfrak{N}(3)$.

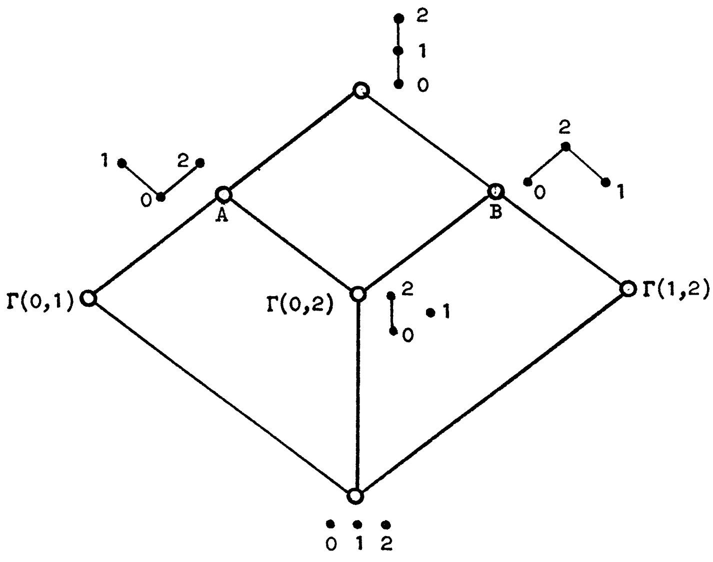

FIGURE 2.

The next sequence of lemmas characterizes coverings in $\mathfrak{N}(n)$ and shows that the lattice is lower semi-modular.

LEMMA 1.2. *Let $P \in \mathfrak{N}$. Let $x < y$ and $(x, y) \notin P$. If $Q = P \cup \Gamma(x, y)$ and $(u, v) \neq (x, y)$, then $u\,Q\,v$ if, and only if, $u\,P\,v$ or $u\,P\,x$ and $y\,P\,v$.*

*For a definition of "compactly generated" see **(2)**.

Proof. Clearly either of these two conditions is sufficient for $u\,Q\,v$. Let us suppose that $u\,Q\,v$ by means of the sequence $u = u_1 R_1 u_2 \ldots R_{r-1} u_r = v$ where $R_i = P$ or $\Gamma(x, y)$. First we may assume without loss of generality that $u_i \neq u_{i+1}$. Now it follows that $u_i < u_{i+1}$ and so for at most one index i we can have $R_i = \Gamma(x, y)$. Indeed if $u_i\,\Gamma(x, y)\,u_{i+1}$ and $u_i \neq u_{i+1}$, then $u_i = x$; the condition $u_i < u_{i+1}$ guarantees that at most one term is x. If no $R_i = \Gamma(x, y)$, then $u\,P\,v$. If $R_i = \Gamma(x, y)$, then

$$u = u_1\,P\,u_i, \qquad u_i = x\Gamma(x, y)y = u_{i+1}\,Pu_r = v$$

and the necessity is proved.

Clearly $P = \bigcup\{\Gamma(x, y)\colon x\,P\,y\}$ but more generally if $P \subseteq Q$, then $Q = P \cup \{\Gamma(x, y)\colon (x, y) \in P - Q\}$.

LEMMA 1.3. *If P and Q belong to $\mathfrak{N}$, then P is covered by Q if, and only if, $P - Q = \{(x, y)\}$.*

Proof. The sufficiency is clear since Q differs from P by exactly one ordered pair. Conversely, suppose that Q covers P and that (x, y) and $(u, v) \in Q-\mathrm{P}$. Clearly $P \subset P \cup \Gamma(x, y) \subseteq Q$. Thus $P \cup \Gamma(x, y) = Q$ and so

$$(u, v) \in P \cup \Gamma(x, y).$$

From Lemma 1.2, $u\,P\,v$ or $u\,P\,x$ and $y\,P\,v$. By assumption, the former cannot hold; hence the latter condition does and $u \leqslant x$ and $y \leqslant v$. Similarly we have $x \leqslant u$ and $v \leqslant x$; thus $(u, y) = (x, y)$.

To establish lower semi-modularity we need a better characterization of a covering.

LEMMA 1.4. *If $Q = P \cup \Gamma(x, y)$ covers P in $\mathfrak{N}$, then y is a P-maximal element among $A(x) = \{z\colon (x, z) \notin P\}$; dually, x is a P-minimal element among $B(x) = \{z\colon (z, y) \notin P\}$.*

Proof. Suppose there were a z such that $x \sim Pz$ and yet $y\,P\,z$. Since we have $(x, z) \in P \cup \Gamma(x, y)$, then $P \subset P \cup \Gamma(x, z) \subseteq P \cup \Gamma(x, y)$. But since $P \cup \Gamma(x, y)$ covers P, it must be that $P \cup \Gamma(x, z) = P \cup \Gamma(x, y)$, or that $(x, y) \in P \cup \Gamma(x, z)$. But from Lemma 1.2 this entails $x\,P\,x$ and $z\,P\,y$; thus $z = y$.

Lemma 1.4 gives us a way to find all the covers, if any exists, for an arbitrary $P \in \mathfrak{N}$. We may proceed as follows. Let $(u, v) \notin P$ and $u \leqslant v$. Choose a P-maximal element y from $A(u, v) = \{z\colon u \sim P\,z \text{ and } v\,P\,z\}$. Having chosen y, choose a minimal element x from $B(u, y) = \{z\colon z\,P\,u \text{ and } z \sim P\,y\}$. We can now show that $P \cup \Gamma(x, y)$ covers P. We must show that y is a maximal element in $A(x)$. In the contrary case, as in Figure 3, suppose that $y\,P\,z$

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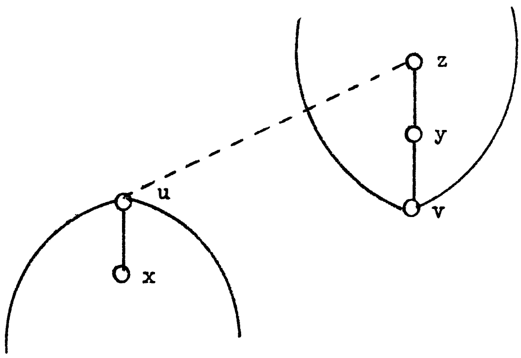

FIGURE 3.

and $x \sim P\ z$. It follows that $u \sim P\ z$ for otherwise $x\ P\ u$ implies $x\ P\ z$, a contradiction. But $u \sim P\ z$ and $v\ P\ z$ implies that $z \in A(u, v)$, while $y\ P\ z$ contradicts the choice of y. Similarly it follows that x is a P-minimal element in $B(y)$.

For infinite ordinals, P need have no covering element. Indeed, for $T(\omega + 1) = \{0, 1, 2, \ldots, \omega\}$, if P is as shown in Figure 4, then $Q \supseteq P$ implies that $P - Q$ has the form $\{(0, m): m > m_0\}$.

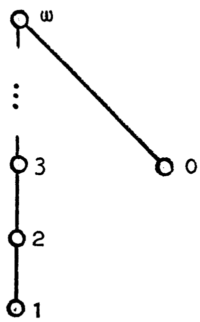

FIGURE 4.

THEOREM 1.2. *The lattice $\mathfrak{N}$ of natural partial orderings is lower semi-modular in the sense that if Q is covered by $P \cup Q$, then P covers $P \cap Q$.*

Proof. Let $P \cup Q = Q \cup \Gamma(u, v)$. We shall show that $P - P \cap Q = \{(u, v)\}$. We may suppose that $P \not\subseteq Q$ and so $P - Q \cap P$ is not empty. Now suppose that $(r, s) \in P - (P \cap Q)$. Thus $(r, s) \notin Q$; however, $(r, s) \in P \subset Q \cup \Gamma(u, v)$. Thus $r\ Q\ u$ and $v\ Q\ s$. Now $u \sim Q\ s$ (otherwise $r\ Q\ s$) and dually $r \sim Q\ v$. Thus $s \in A(u)$ and $r \in B(v)$, but by Lemma 1.4, v is a Q-maximal element in $A(u)$ and u is a Q-minimal element in $B(v)$. From $v\ Q\ s$ and $r\ Q\ u$ we conclude that $(r, s) = (u, v)$. Thus $P - P \cap Q = \{(u, v)\}$.

The lattice $\mathfrak{N}(n)$ is not complemented and the next result is a necessary condition for an element of $\mathfrak{N}(n)$ to possess a complement. If $n \leqslant \omega$, this condition is also sufficient. Necessary and sufficient conditions in the general case are not known.

LEMMA 1.5. *A necessary condition for an NPO, P, to have a complement in $\mathfrak{N}(n)$ is*

(*) *For all x, y such that $x\,P\,y$, either $y = x + 1$ or there exists $(u, v) \neq (x, y)$ with $x \leqslant u < v \leqslant y$ and $u\,P\,v$.*

Proof. Suppose that Q is a complement for P and that $x\,P\,y$ and $y \neq x + 1$. If there is a w such that $x < w < y$ and $x\,P\,w$ or $w\,P\,y$, then (*) holds. In the contrary case, suppose that for all w such that $x < w < y$, then $x \sim Pw$ and $w \sim Py$. However, since $x\,(P \cup Q)w$, it follows that either $x\,Q\,w$ or there exists u and v such that $x \leqslant u\,P\,v$ and (*) holds. Similarly from $w\,(P \cup Q)\,y$, either $w\,Q\,y$ or the condition (*) is satisfied. However, if $x\,Q\,w$ and $w\,Q\,y$, then $x\,Q\,y$ and hence $(x, y) \in P \cap Q \neq \phi$, a contradiction. Thus (*) is necessary.

THEOREM 1.3. *If $n \leqslant \omega$, there exists a complement for P in $\mathfrak{N}(n)$ if and only if (*) holds.*

Proof of the sufficiency of (*). Let $\hat{P} = \{(x, y): x = y$ or $(x < y$ and for all w, $x \leqslant w < y$ implies $w \sim P(w + 1))\}$. We claim that $\hat{P}$ is an NPO. Clearly it is reflexive and antisymmetric. To prove transitivity we suppose that $x\,\hat{P}\,y$ and $y\,\hat{P}\,z$. Suppose that $x < y$ and that there exists w such that $x \leqslant w < z$ and $w\,P\,(w + 1)$. By assumption $y\,\hat{P}\,z$, and thus $w < y$, a contradiction of $x\,\hat{P}\,y$. Thus $\hat{P}$ is a partial ordering.

To prove that $P \cap \hat{P} = \phi$ we note that since $n \leqslant \omega$, if $x\,P\,y$, then y is finite and a finite number of applications of (*) implies that there exists w such that $x \leqslant w < y$ and $w\,P\,(w + 1)$. Thus $x \sim \hat{P}\,y$. We prove that $P \cup \hat{P} = \Omega$ by showing by induction on $y - x$ that if $x \leqslant y$, then $x\,(P \cup Q)\,y$. If $y - x = 1$, then either $x\,P\,y$ or $x\,\hat{P}\,y$ by definition. In general, if $x \sim Py$, either $w \sim P(w + 1)$ for all w such that $x \leqslant w < y$, in which case $x\,\hat{P}\,y$, or there exist w such that $w\,P\,(w + 1)$. But then, by induction, $x\,(P \cup \hat{P})\,w$ and $(w + 1)\,(P \cup \hat{P})\,y$; hence $x\,(P \cup \hat{P})\,y$.

We remark that $\hat{P} = \bigcup\{\Gamma(u, u + 1): u \sim P(u + 1)\}$ and thus if Q is any complement for P, it follows that $Q \supseteq \hat{P}$. Condition (*) is not sufficient if $\omega < n$ as the example of Figure 5 shows.

2. The upper triangular group. For this and subsequent sections we require that $n \leqslant \omega$. It will be convenient to let $T \subseteq \{1, 2, \ldots\}$ and $T(n) = \{1, 2, \ldots, n\}$. With each NPO, P, of T we associate a group G_P of non-singular $n \times n$, upper triangular matrices with entries in GF(2). Actually the field plays no role in our considerations; we could have used an arbitrary field, but

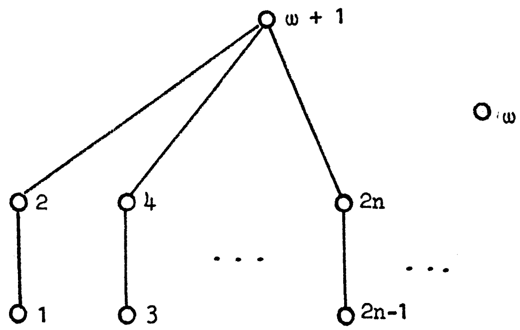

FIGURE 5.

by choosing GF(2), all our entries are either 0 or 1 and certain of our calculations are easier. The group of *all* non-singular upper triangular matrices we shall denote by $TL(n)$. In this section we show that the mapping $P \to G_P$ is a lattice isomorphism of $\mathfrak{N}(n)$ into $TL(n)$.

Definition. Let $P \in \mathfrak{N}(n)$. Let $G_P = \{M = (m_{ij}): m_{ii} = 1$, and if $i \sim P\, j$ then $m_{ij} = 0\}$. In particular, note that $G_P \subseteq TL(n)$.

Clearly every matrix in G_P has 1 on the main diagonal and the identity matrix $I \in G_P$ for all P. Only the entries which must be 0 have been specified by P.

THEOREM 2.1. *For all natural partial orderings, P, G_P, is a group.*

Proof. Let $A = (a_{ij})$ and $B = (b_{jk})$ belong to G_P. Let $AB = C = (c_{ik})$. Thus

$$c_{ik} = \sum_{1 \leqslant j} a_{ij}\, b_{jk} = \sum_{i \leqslant j \leqslant k} a_{ij}\, b_{jk}.$$

If $i \sim P\, k$, then clearly for each j either $i \sim P\, j$ or $j \sim P\, k$; thus $c_{ik} = 0$ and hence $C \in G_P$. If n is finite, closure of course suffices to establish that G_P is a group. If $n = \omega$, then we must also show that if $A \in G_P$, then $A^{-1} \in G_P$. Let $A = (a_{ij})$ and $A^{-1} = (b_{jk})$; we do know that $A^{-1} \in TL(n)$. We shall show, by induction on $(k - i)$, that if $i \sim Pk$, then $b_{ik} = 0$. For $(k - i) = 0$ the result is vacuously true. Suppose then that $i \sim P\, k$ while if $s - r < (k - i)$ and $r \sim P\, s$, then $b_{rs} = 0$. From $I = AA^{-1}$ we have

$$0 = \delta_{ik} = \sum_{i \leqslant j \leqslant k} a_{ij}\, b_{jk}.$$

When $i < j$ the term $a_{ij}\, b_{jk}$ is zero for if $a_{ij} = b_{jk} = 1$, then $i\, P\, j$ and by induction $j\, P\, k$; hence $i\, P\, k$, a contradiction. Thus $0 = a_{ii}\, b_{ik} = b_{ik}$.

It is easy to see that G_ϕ is the identity group and that if $P = \Gamma(x, y)$, then G_P, which we denote by $G(x, y)$, is a group of order 2 whose non-identity

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element is $I + E(x, y)$, where $E(x, y)$ is the $n \times n$ matrix whose only non-zero entry is in row x and column y. For our next result we need the familiar fact that $E(x, y)E(u, v) = \delta_{yu} E(x, v)$.

THEOREM 2.2. *Let n be a finite ordinal. Let P be a natural partial ordering of $T(n)$. The group G_P is generated by $\{I + E(x, y): x\,P\,y, x \neq y\}$;* indeed *$G_P$ is the subgroup union of the subgroups $\{G(x, y): x\,P\,y, x \neq y\}$.*

Proof. Since it is evident that if $x \neq y$ and $x\,P\,y$, then $I + E(x, y) \in G_P$, we have only to prove that $G_P \subseteq \bigcup\{G(x, y): x\,P\,y, x \neq y\}$. Now it is clear that if $M = (m_{ij}) \in G_P$, then $M = I + \sum E(u, v)$, the sum ranging over precisely those pairs (u, v) such that $u \neq v$ and $m_{uv} \neq 0$. First of all, note that if $m_{ij} \neq 0$, then $i\,P\,j$, and so $I + E(i, j)$ belongs to the set the theorem alleges to be a generating set for G_P. We shall show that if $M \in G_P$, then $M \in \bigcup\{G(x, y): x\,P\,y, x \neq y\}$ by an induction on the number λ of non-zero entries in M. Note that λ is the sum of n and the number of terms in $\sum E(u, v)$. If $\lambda = n$, then $M = I$ and the result holds. In general we may write

$$M = I + \sum (u, v) = I + \sum_{(u,v)\neq(r,s)} E(u, v) + E(r, s),$$

where (r, s) is chosen so that r is a minimal value occurring among the first coordinates of the pairs occurring in the sum. (Equivalently, r is a minimal row index occurring among all non-zero entries in M lying off the main diagonal.) Now we compute

$$\Bigl(I + \sum_{(u,v)\neq(r,s)} E(u, v)\Bigr)(I + E(r, s)) =$$
$$I + \sum_{(u,v)\neq(r,s)} E(u, v) + E(r, s) + \sum_{(u,v)\neq(r,s)} \delta_{vr} E(u, s).$$

However, in the last sum, $\delta_{vr} = 0$ for all v since $r \leqslant u < v$ by the choice of r. Thus

$$M = \Bigl(I + \sum_{(u,v)\neq(r,s)} E(u, v)\Bigr)(I + E(r, s)).$$

By the induction hypothesis the first factor belongs to $\bigcup\{G(x, y): x\,P\,y, x \neq y\}$ as does the second; hence M does as well. Thus the theorem holds.

THEOREM 2.3. *The mapping $P \to G_P$ is a lattice isomorphism from $\mathfrak{N}(n)$ into a sublattice of the lattice of all subgroups of $TL(n)$. Indeed $G_{P\cap Q} = G_P \cap G_Q$ and $G_{P\cup Q} = G_P \cup G_Q$.*

Proof. First note that $P \subseteq Q$ if and only if $G_P \subseteq G_Q$. This is an immediate corollary of the preceding theorem and from this remark we learn that the mapping $P \to G_P$ is one-to-one. The equality $G_{P\cap Q} = G_P \cap G_Q$ is easily seen to hold, primarily because both $P \cap Q$ and $G_P \cap G_Q$ are set intersections.

Because the mapping is order-preserving, to complete the proof we need only show that $G_{P \cup G} \subseteq G_P \cup G_Q$. From the preceding theorem it will suffice to show that $a\ (P \cup Q)\ b$ and $a \neq b$ implies that $I + E(a, b) \in G_P \cup G_Q$. We begin by computing the commutator:

$$\begin{aligned} C(x, y; u, v) &= [(I + E(x, y)), (I + E(u, r))] = [(I + E(x, y))(I + E(u, r))]^2 \\ &= (I + E(x, y) + E(u, v) + \delta_{yu} E(x, v))^2 \\ &= I + E(x, y) + E(u, v) + \delta_{yu} E(x, v) + E(x, y) \\ &\qquad + 0 + \delta_{vx} E(u, y) + 0 + E(u, v) + \delta_{yu} E(x, v) + 0 + 0 \\ &\qquad\qquad + \delta_{yu} E(x, v) + 0 + 0 + 0 \\ &= \begin{cases} I & \text{if } y \neq u \text{ and } v \neq x, \\ I + E(u, y) & \text{if } v = x \text{ and } u \neq y, \\ I + E(x, v) & \text{if } v \neq x \text{ and } u = y. \end{cases} \end{aligned}$$

Now we shall show that if $a = u_1 < u_2 < \ldots < u_r = b$ is a sequence such that $u_i\, R_i\, u_{i+1}$ where $R_i = P$ or Q, then $I + E(a, b) \in G_P \cup G_Q$. We proceed by induction on r. If $r = 1$, then $I + E(a, b) \in G_P$ or G_Q. If $r = 2$, then $I + E(a, b) = C(u_1, u_2; u_2, u_3) \in G_P \cup G_Q$. Thus, in any event we have that $I + E(u_1, u_3) \in G_P \cup G_Q$. By induction $I + E(u_3, u_r) \in G_P \cup G_Q$ and thus $G_P \cup G_Q$ contains $C(u_1, u_3; u_3, u_r) = I + E(a, b)$.

Definition. Let P and Q be natural partial orderings of $T(n)$. P and Q are called *isomorphic* ($P \cong Q$) if there is a permutation π of $T(n)$ such that $i\,P\,j$ if and only if $\pi(i)\ Q\ \pi(j)$. We say that P is *unique* if $P \cong Q$ implies that $P = Q$. P and Q are called *dual* provided $i\,P\,j$ if and only if

$$(n + 1 - j)\ Q\ (n + 1 - i).$$

We note that $\Gamma(x, y)$ and $\Gamma(u, r)$ are always isomorphic while ϕ and Ω are unique. In Figure 2, NPO's A and B are unique and mutually dual.

THEOREM 2.4. *If P and Q are isomorphic, then G_P and G_Q are isomorphic, indeed $G_Q = \Pi^{-1} G_P \Pi$ if Π is the permutation matrix corresponding to the permutation π which takes P into Q.*

Proof. We need only verify that in general if Π is the permutation matrix $\Pi = (\pi_{ij})$, where $\pi_{ij} = 1$ if and only if $\pi(i) = j$, then

$$\Pi^{-1} E(x, y) \Pi = E(\pi(x), \pi(y)).$$

THEOREM 2.5. *If P and Q are dual, then the groups G_P and G_Q are isomorphic.*

Proof. It is easy to verify that the map

$$I + \sum E(x, y) \rightarrow I + E(n + 1 - y, n + 1 - x)$$

is an anti-isomorphism of G_P onto G_Q and so the mapping

$$I + \sum E(x, y) \rightarrow [I + \sum E(n + 1 - y, n + 1 - x)]^{-1}$$

is the desired group isomorphism.

3. Normal natural partial orders. In this section we determine necessary and sufficient conditions on P such that G_P is normal in $TL(n)$; moreover, Theorem 3.7 shows that if P has no restriction to a subset yielding either of the four-element abstract partially ordered sets of Figure 1, then there exists $Q \cong P$ such that G_Q is normal in $TL(n)$. The sublattice of normal NPO's is shown to be distributive.

Definition. A natural partial ordering P of $T(n)$ is called *normal* if G_P is a normal subgroup of $TL(n)$.

THEOREM 3.1. *A natural partial ordering P is normal if and only if the following two conditions hold*:
(1) *If $i\,P\,k$, $i \neq k$, and $r \leqslant i$, then $r\,P\,k$.*
(2) *If $k\,P\,i$, $i \neq k$, and $s \geqslant i$, then $k\,P\,s$.*

(This theorem asserts that P is normal if and only if it has no configurations like those in Figure 6.)

FIGURE 6.

Proof of Theorem 3.1. G_P is normal in $TL(n)$ if and only if for all pairs (i, k) such that $i \neq k$ and $i\,P\,k$, and for all pairs (r, s) such that $r < s$, it is the case that $(I + E(r, s))(I + E(i, k))(I + E(r, s)) \in G_P$.

A direct calculation shows this product to be equal to

$$\begin{array}{ll} I + E(i, k) & \text{if } r \neq k \text{ and } s \neq i, \\ I + E(i, k) + E(r, k) & \text{if } s = i, \\ I + E(i, k) + E(i, s) & \text{if } k = r. \end{array}$$

(The case $r = k$ and $s = i$ cannot arise.) Thus G_P is normal if and only if $i\,P\,k$ and $r < i$ implies $r\,P\,k$ and $i\,P\,k$ and $k < s$ implies $i\,P\,s$. In Figure 2, the normal partial orders are ϕ, $\Gamma(1, 3)$, A, B, and Ω.

Our next theorem characterizes normal NPO's as those that commute with all other natural partial orderings in a product of relations. If P and Q are relations on a set J, their product PQ is the set of pairs $\{(x, y)$: there exists a z such that $x\,P\,z$ and $z\,Q\,y\}$. It is well known **(1)** that if an algebra has permuting congruence relations, then the lattice of permuting congruence relations is modular.

THEOREM 3.2. *A natural partial ordering P is normal if and only if $PQ = QP$ for all natural partial orderings Q.*

Proof. Let us suppose that $PQ = QP$ for all $Q \in \mathfrak{N}(n)$. Suppose $i\,P\,k$ and $r < i$. Let $Q = \Gamma(r, i)$. Thus $r\,Q\,i$ and $i\,P\,k$. Since $PQ = QP$, there is an x such that $r\,P\,x$ and $x\,Q\,k$. But $x\,Q\,k$ implies $x = k$; hence $r\,P\,k$. Dually we verify that $i\,P\,k$ and $k < s$ implies $i\,P\,s$. Conversely, suppose that P is normal. If $i\,P\,k$ and $k\,Q\,s$, then $i < k < s$ and so $i\,P\,s$ from Theorem 3.1. Hence $i\,Q\,i$ and $i\,P\,s$, and thus $PQ \leqslant QP$. Similarly, if $r\,Q\,i$ and $i\,P\,k$, then $r\,P\,k$ so that $r\,P\,k$ and $k\,Q\,k$; thus $QP \leqslant PQ$. It is a well-known result (and easily proved) that if relations P and Q on a set T permute, then $P \cup Q = PQ = QP$.

THEOREM 3.3. *If either P or Q is a normal natural partial ordering then $x\,(P \cup Q)\,y$ if and only if $x\,P\,y$ or $x\,Q\,y$.*

Proof. If P is normal, Theorem 3.2 yields $P \cup Q = PQ$. Hence, if $x\,(P \cup Q)\,y$, we may suppose that $x\,P\,z$ and $z\,Q\,y$ for some z. If $z = x$, then $x\,Q\,y$ while if $z \neq x$, then, since $z \leqslant y$, Theorem 3.1 implies that $x\,P\,y$.

COROLLARY. *The normal natural partial orders form a distributive sublattice $\mathfrak{D}(n)$ of $\mathfrak{N}(n)$.*

Proof. From the lattice isomorphism $P \to G_P$ and the fact that the join and meet of normal subgroups is normal follows the fact that the normal partial orders form a sublattice. The lattice is distributive because we have shown that, as relations, the meet and join of *normal* partial orders are set meet and set join.

Our next theorem establishes the fact that if P is unique, then P is a normal NPO.

THEOREM 3.4. *Let $P \in \mathfrak{N}(n)$. If there is a triple $(i, k; j)$ in P such that $j < i \neq k$, $i\,P\,k$, and $j \sim P\,k$, then there is $Q \in \mathfrak{N}(n)$ such that $P \cong Q$ and $Q \neq P$.*

Proof. For a fixed k let $C(k) = \{(s, r) : r\,P\,k, s < r \text{ and } s \sim P\,k\}$. By hypothesis, $C(k)$ is non-empty. Let (u, v) be chosen in $C(k)$ such that $(v - u)$ is minimal. Let Q be the partial ordering obtained by interchanging u and v as indicated in Figure 7. Specifically

$$x\,Q\,y \text{ if } \begin{cases} \{x, y\} \cap \{u, v\} = \phi \text{ and } x\,P\,y. \\ x = v \text{ and } u\,P\,y \text{ or } y = v \text{ and } x\,P\,u. \\ x = u \text{ and } v\,P\,y \text{ or } y = u \text{ and } x\,P\,v. \end{cases}$$

(Note that $u \sim P\,v$ so that $u \sim Q\,v$ and $v \sim P\,u$.) We claim that Q is a natural

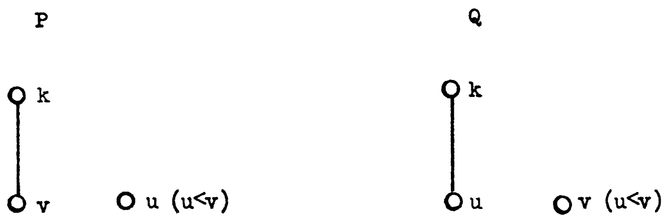

FIGURE 7.

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partial ordering and so $Q \cong P$ since the permutation $\pi = (u, v)$ provides the isomorphism. First we show that $x\,Q\,y$ implies $x \leqslant y$. Since $u < v$, the questionable cases are $x = v$ or $y = u$. Consider first the case $v\,Q\,y$. Thus $u\,P\,y$ and so $u < y$. If $y\,P\,k$, then $u\,P\,k$, contrary to the choice of y. Thus $y \sim Pk$. If $y < v$, then $(y, v) \in C(k)$, but then since $u < y$, we should have $v - y < v - u$, contrary to the choice of (u, v). Similarly, suppose that $x\,Q\,u$, that is, $x\,P\,v$. Thus $x < v$ and if $u < x$, then $(u, x) \in C(k)$ while

$$x - u < v - u,$$

contrary to the choice of (u, v).

Now it is easy to see that Q is a partial ordering and since $u \sim Pk$ while $u\,Q\,k$, it is clear that $P \neq Q$.

COROLLARY. *If P is unique, then P is normal.*

Proof. This, together with Theorem 3.1, is essentially the contrapositive statement of the theorem or its dual.

Hereafter we shall suppose that n is always finite. Our notation is

$$T = T(n) = \{1, \ldots, n\}.$$

Definition. Let P be a partial ordering of a set S. The *depth* of $x \in S$, denoted $D_P(x)$ or more briefly $D(x)$, is defined by

$$D(x) = \max\{r\colon \text{there is a chain } x = x_1\,P\,x_2\,P \ldots P\,x_r \text{ such that } x_i \neq x_{i+1}\}.$$

LEMMA 3.1. *If $y\,P\,x$ and $y \neq x$, then $D(y) > D(x)$. If $D(y) = r$ and $y = y_1\,P\,y_2\,P \ldots P\,y_r$ is a chain such that $y_i \neq y_{i+1}$, then $D(y_i) = r - i + 1$.*

Definition. Let P be a natural partial ordering of $T(n)$. An element x is said to be of *class* 1 if $y\,P\,x$ whenever $D(y) > D(x)$. The elements of class 1 will be denoted $C1$, or C_P1 if it is important to denote the partial ordering.

An interesting result which we shall not prove since we make no use of it is that P is unique if and only if every element is of class 1. The following result is immediate.

LEMMA 3.2. *Let P be a natural partial ordering of $T(n)$. $x \in C1$ if and only if $y\,P\,x$ for all y such that $D(y) = D(x) + 1$.*

Now we can easily prove a lemma which turns out to be extremely useful.

LEMMA 3.3. *Let P be a natural partial ordering of $T(n)$. Let*

$$H(r) = \{x\colon D(x) = r\}.$$

If for some integer r for which $H(r) \neq \emptyset$ but $H(r) \cap C1 = \emptyset$, then P contains four elements, r, s, u, v, such that $r\,P\,s$, $u\,P\,v$, $r \sim Pv$, and *$u \sim Ps$.*

Proof. If m denotes the maximum depth of an element in $T(n)$, then every element of depth m is of class 1, since the condition of the definition is vacuous.

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Let $r < m$, and for each element x in $H(r+1)$, let

$$U(x) = \{y: x\,P\,y \text{ and } D(y) = r\}.$$

Of course $U(x)$ is non-empty. Now select $x_0 \in H(r+1)$ such that

$$|U(x_0)| \leqslant |U(x)| \text{ for all } x \in H(r+1).$$

Let $y_0 \in U(x_0)$. We shall show that $y_0 \in C1$ or the alleged configuration appears in P. Suppose that $x \in H(r+1)$ and $x \sim Py_0$. Consider $U(x)$. If there is a $y \in U(x)$ such that $y \notin U(x_0)$, then for x_0, y_0, x, y we have $x_0\,P\,y_0$, $x\,P\,y$, $x_0 \sim P\,y$ and $x \sim P\,y_0$. If this configuration is assumed not to occur, then it must be that $U(x) \subseteq U(x_0)$; however since $|U(x_0)| \leqslant |U(x)|$, it follows that $U(x) = U(x_0)$ and so $x \in H(r+1)$ must imply $x\,P\,y_0$. Thus $D(x) = D(y_0) + 1$ implies $x\,P\,y_0$ and so $y_0 \in C1$ by the previous lemma.

LEMMA 3.4. *If P is a normal NPO, then for all r, if $H(r) \neq \emptyset$, then*

$$H(r) \cap C1 \neq \emptyset.$$

Proof. Let $H(r) \cap C1 = \emptyset$, while $H(r) \neq \emptyset$. From the lemma we have that P has a configuration $r\,P\,s$, $u\,P\,v$, $r\,P\,v$, and $u\,P\,s$. Since P is normal, we must have $r \geqslant u$, and conversely $u \geqslant r$; hence $r = u$, a contradiction.

LEMMA 3.5. *Let P be a normal NPO. If $D(x) = D(y)$, $x \in C1$, and $y \notin C1$, then $x > y$. Moreover, if $D(z) > D(w)$, then $z < w$.*

Proof. Since $y \notin C1$, then there must exist z such that $D(z) > D(y)$ and $z \sim P\,y$. Since $x \in C1$, we have $z\,P\,x$, and now since P is normal, it must be that $y < x$. To prove the second assertion note first that if $w \in C1$, then $z\,P\,w$ and so $z < w$. Thus we may assume that $w \notin C1$. From Lemma 3.4 there is an $x \in C1$ such that $D(x) = D(w)$. Thus $z\,P\,x$ and if $w < z$, then from normality it follows that $w\,P\,x$; hence $D(w) > D(x)$, a contradiction.

THEOREM 3.5. *Let P be a normal NPO. Let r be an integer such that $H(r) \neq \emptyset$. Then there exist integers, $a \leqslant b < c$, such that $x \in H(r) \cap C1$ if and only if $b \leqslant x \leqslant c$ while $x \in H(r) - C1$ if and only if $a \leqslant x < b$.*

Proof. From Lemma 3.4, $H(r) \cap C1 \neq \emptyset$ and so let $c = \max\{x: x \in H(r) \cap C1\}$, $b = \min\{x: x \in H(r) \cap C1\}$, and $a = \min\{x: x \in H(r)\}$. Clearly $D(x) = r$ implies $a \leqslant r \leqslant c$ and Lemma 3.5 shows that if $x \in H(r) - C1$, then $x < b$. Conversely, suppose first that $b \leqslant x < c$. If $D(x) > r$, then $x\,P\,b$ since $b \in C1$ and thus $x < b$; hence $r \geqslant D(x)$. If $D(x) < r = D(b)$, then $c < x$, by Lemma 3.5. Hence $D(x) = r$ and again from Lemma 3.5, since $b \in C1$, it follows that $x \in C1$ also. Secondly suppose that $a \leqslant x < b$. If $D(x) > r = D(a)$, then $x < a$; if $D(x) < r = D(b)$, then $b < r$; hence $D(x) = r$. However, Lemma 3.5 and the choice of b show that $x \notin C1$; thus $x \in H(r) - C1$.

COROLLARY. *Let P be a normal NPO. If π is a permutation such that*
(1) *π fixes every element not in $H(1) \cap C1$,*
(2) *π fixes $H(1) \cap C1$,*
then $i\,P\,j$ if and only if $\pi(i)\,P\,\pi(j)$.

Proof. Suppose that $i \neq j$ and $i\,P\,j$. Since $i \notin H(1) \cap C1$, $\pi(i) = i$. If $j \notin H(1) \cap C1$, there is nothing to prove. If $j \in H(1) \cap C1$, then $\pi(j) \in H(1) \cap C1$ and so $i\,P\,\pi(j)$. The converse follows in a similar manner.

THEOREM 3.6. *If P and Q are normal NPO's, then $P \cong Q$ implies that $P = Q$.*

Proof. Let π be the permutation of $T(n)$ such that $i\,P\,j$ if and only if $\pi(i)\,Q\,\pi(j)$. Clearly we must have $D(i) = D(\pi(i))$ and $i \in C_P 1$ if and only if $\pi(i) \in C_Q 1$. By Theorem 3.5 the maximal elements in $C_P 1$ and in $C_Q 1$ satisfy $b \leqslant x \leqslant n$ for some integer b which must be the same for P and Q. Thus π can only permute these elements. Now if $b = 1$, the proof is complete. If $1 < b$, then consider the restriction of P and Q to $T(b-1)$. Let us denote these restrictions by $\hat{P}$ and $\hat{Q}$, respectively. It is easy to verify that $\hat{P}$ and $\hat{Q}$ are normal NPO's of $T(b-1)$; moreover the restriction $\tilde{\pi}$ of π to $\{1, \ldots, b-1\}$ is a permutation which shows that $\hat{P} \cong \hat{Q}$. By induction, $\hat{P} = \hat{Q}$ and hence $P = Q$ since the addition of $\{b, \ldots\}$ to $T(b-1)$ is the addition of class one elements with respect to both P and Q.

COROLLARY. *No two normal natural partial orderings are isomorphic.*

Our next theorem gives necessary and sufficient conditions for P to be normal based on the sub-partially ordered sets that can be obtained from P by restriction. We begin by noting that if P is normal, then P cannot have configurations like those of Figure 1. We shall now show that the absence of these configurations in P implies that there is an isomorphic NPO, Q, such that Q is normal.

The next lemma is an obvious "renumbering" principle which we list for easy reference.

LEMMA 3.6. *Let P be an NPO. Let i, j be such that*

$$\{x\colon x\,P\,i, x \neq i\} = \{x\colon x\,P\,j, x \neq j\} \text{ and } \{y\colon i\,P\,y, i \neq y\} = \{y\colon j\,P\,y, j \neq y\};$$

then P is left invariant under the transposition $\pi = (i, j)$.

Note that if i and j are maximal elements, then the second of these conditions is vacuous.

THEOREM 3.7. *Let P be an NPO of $T(n)$. There is an NPO, Q, of $T(n)$ such that $P \cong Q$ and Q is normal if and only if, for all sets of four distinct elements r, s, t, u*
(1) *$r\,P\,s\,P\,t$ implies $u\,P\,t$ or $r\,P\,u$*
and
(2) *$r\,P\,s$ and $t\,P\,u$ implies $r\,P\,u$ or $t\,P\,s$.*

Proof. If $P \cong Q$ and P satisfies conditions (1) and (2), then so does Q. However, if Q is normal, then, as we have remarked, the conditions are necessary.

Our proof is by induction on n and for fixed n on the number of triples $(i, k; j)$ such that $i\,P\,k$, $i \neq k$ and either $j < i$ and $j \sim P\,k$ or $j > k$ and $i \sim P\,j$. For brevity we call such triples "bad" P-triples. Of course, if there are no bad P-triples, then P is normal.

Our first reduction is to show that we may assume $n \in C1$. In any event it follows from condition (2) and Lemma 3.3 that there are depth one elements of class 1; let x be one such. If $n \notin C1$, let π be the transposition (x, n) and let R be the partial order defined by $u\,R\,v$ if and only if $\pi^{-1}(u)\,P\,\pi^{-1}(v)$. It is easily verified that R is a partial ordering. There are two interesting cases in the argument to show that R is natural: $u = n$ or $v = x$. If $n\,R\,v$, then $x\,P\,\pi^{-1}(v)$ and as $D(x) = 1$, then $\pi^{-1}(v) = x$, and hence $v = n$. If $u\,R\,x$, then $\pi^{-1}(u)\,P\,n$ and as $x \in C1$ it follows that either $D(\pi^{-1}(u)) = 1$ or $\pi^{-1}(u)\,P\,x$. In the former case we must have $\pi^{-1}(u) = n$, or $u = x$, while in the latter $\pi^{-1}(u) < x$, and hence $\pi^{-1}(u) = u < x$. It now follows that $n \in C_R\,1$, for if $D(y) > 1$, then $y\,P\,x$ and $y = \pi(y)\,R\,n$. Hereafter we shall assume that $n \in C_P\,1$.

Our second reduction is to assume that $\hat{P}$, the restriction of P to $T(n-1)$, is normal. Since conditions (1) and (2) hold for P, they hold a fortiori for $\hat{P}$, and hence by induction there is a normal NPO, $\hat{Q}$, isomorphic to $\hat{P}$. We now wish to adjoin n to $T(n-1)$ and extend $\hat{Q}$ to Q so that $Q \cong P$; the restriction of this Q to $T(n-1)$ is clearly $\hat{Q}$. To do this, let π be the permutation mapping $\hat{P}$ onto $\hat{Q}$. Now define

$$r\,Q\,s \text{ if and only if } \begin{cases} r\,\hat{Q}\,s \text{ if } s \neq n \text{ or} \\ \pi^{-1}(r)\,P\,n \text{ if } r \neq n \text{ and } s = n \\ r = s = n. \end{cases}$$

We omit the verification that Q is an NPO and that if π is extended to a permutation of $T(n)$ by defining $\pi(n) = n$, then $Q \cong P$ via π. Finally, $n \in C_Q\,1$ since in general $x \in C_P\,1$ if and only if $\pi(x) \in C_Q\,1$ and in this case $\pi(n) = n$. Thus we assume hereafter that $n \in C_P\,1$ and that the restriction $\hat{P}$ of P to $T(n-1)$ is a normal NPO.

Now suppose that P has a bad triple. Since $\hat{P}$ is normal, the bad triple must involve n and since $n \in C1$ it must be of the form $i\,P\,n$, $i \neq n$, $j < i$, and $j \sim Pn$. We shall show that we can permute i and j to obtain an NPO, Q, with fewer bad triples. Our result will then follow by induction.

First, it must be that $D(j) = 1$; otherwise $j\,P\,n$ since $n \in C1$. Second, it must be that $D(i) = 2$, and even more, that i is maximal in $\hat{P}$, for otherwise $\hat{P}$ would have a bad triple. Third, if $x\,P\,i$, then $x\,P\,j$ by condition (1). Conversely, if $x\,P\,j$, then $x\,P\,i$, for otherwise $(x, j; i)$ is a bad triple $(i > j)$ in $\hat{P}$. Thus $\{x\colon x\,P\,i, x \neq i\} = \{x\colon x\,P\,j, x \neq j\}$. Now in $\hat{P}$, since

$$D_{\hat{P}}(i) = D_{\hat{P}}(j) = 1,$$

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the conditions of Lemma 3.6 are satisfied so that the permutation $\pi = (i, j)$ of $T(n-1)$ leaves $\hat{P}$ invariant. Now regard π as a permutation on $T(n)$ and let Q be defined by $u\ Q\ v$ if and only if $\pi^{-1}(u)\ P\ \pi^{-1}(v)$. Q is an NPO, $Q \cong P$, $\hat{Q} = \hat{P}$, and Q is normal, and we claim that Q has fewer bad triples than P. We show that there is a 1-1 mapping of the bad triples of Q into and not onto the bad triples of P. Since $\hat{Q} = \hat{P}$, any bad triple of Q must include n. Suppose that $(x, n; y)$ is a bad Q-triple. If neither x nor y is i or j, $(x, n; y)$ is a bad P-triple. Suppose $(j, n; y)$ is a bad Q-triple because $y < j$ and $j \sim Qn$; then $y \neq i$. Now $(i, n; y)$ is a bad P-triple since $y < j < i$. Suppose that $(x, n; j)$ is a bad Q-triple because $j < x$ and $j \sim Qn$. Now $x \neq i$ as $j \sim Pn$; hence $(x, n; i)$ is a bad P-triple, if $i < x$. If $i > x$, we argue that $(x, n; j)$ is a bad P-triple. Suppose that $(x, n; i)$ with $i < x$ is a bad Q-triple; then $(x, n; j)$ is a bad P-triple. Note that here $i < x$ so that the triple $(x, n; j)$ with $i > x$ of the previous case is a different triple from this one. Thus to each bad Q-triple there corresponds a bad P-triple, different bad Q-triples corresponding to different bad P-triples. In addition the Q-triple $(j, n; i)$ corresponding to the old bad P-triple $(i, n; j)$ is not bad! Thus Q has at least one less bad triple than P and the proof is complete.

4. The order of $\mathfrak{D}(n)$. In this section we shall show that if n is finite, the number of elements in $\mathfrak{D}(n)$ is

$$\binom{2n}{n} \Big/ (n+1).$$

In view of Theorems 3.6 and 3.7, this number is the number of abstract non-isomorphic partially ordered sets which do not contain copies of either configurations in Figure 1 as sub-partially ordered sets.

Our first count is obtained by showing a 1-1 correspondence between the NPO's of $T(n)$ and the set of lattice paths counted by Feller **(3)**. Let $\mathfrak{M}(n)$ be the set of non-negative, integer-valued step functions, f, defined on $[0, n]$ such that

(1) $0 < f(x) \leqslant x$,
(2) $f(x) = f([x])$,
(3) $f(0) = 0$ and $f(n) = n$,
(4) f is monotone non-decreasing.

The graph of f can be thought of as a polygonal path from $(0, 0)$ to (n, n) with jumps at points with integer coordinates, always lying below the line $y = x$. Through the transformation $(i, j) \to (i + j, j - i)$ the path corresponds to what is called a lattice path $\{s_1, s_2, \ldots, s_{2n}\}$ from the origin through the points (a, s_a) to the point $(2n, 0)$ such that all $s_a \geqslant 0$ and $s_{a+1} - s_a = \pm 1$. The number of these paths has been determined **(3**, p. 72**)** to be

$$\binom{2n}{n} \Big/ (n+1).$$

We shall now show that the relation $\subseteq$ defined by $f \subseteq g$ if and only if $f(x) \leqslant g(x)$ makes $\mathfrak{M}(n)$ into a lattice and that there is a lattice isomorphism from $\mathfrak{M}(n)$ onto $\mathfrak{D}(n)$.

Definition. Let $f \in \mathfrak{M}(n)$. Let P_f be defined as follows:
(i) $x \, P_f \, x$ for all $x \in T(n)$.
(ii) If $x \neq y$, then $x \, P_f \, y$ if and only if $x - 1 < f(y - 1)$.

LEMMA 4.1. *For all f, P_f is a normal partial ordering of $T(n)$.*

Proof. First note that if $x \, P_f \, y$ and $x \neq y$, then $x - 1 < f(y - 1) \leqslant y - 1$ implies that $x < y$ and thus P_f is a natural relation. P_f is then easily seen to be reflexive and antisymmetric. Suppose $x \, P_f \, y$ and $y \, P_f \, x$. We exclude trivial cases by supposing that $x \neq y$ and $y \neq z$. Then

$$x - 1 < f(y - 1) \leqslant (y - 1) < f(z - 1);$$

hence $x \, P_f \, z$. Thus P_f is a natural partial ordering.

To show that P_f is normal we verify the two conditions of Theorem 3.1. First, suppose that $i \, P \, k$ and $r \leqslant i$. Thus $r - 1 \leqslant i - 1 < f(k - 1)$; hence $r \, P \, k$. Second, if $k \, P \, i$ and $s > i$, then $f(s - 1) \geqslant f(i - 1) > k - 1$; hence $k \, P \, s$.

LEMMA 4.2. *Let P be a normal natural partial ordering. Let f be the function defined by: $f(x) = 0$ if $0 \leqslant x < 1$, $f(n) = n$, and for $1 \leqslant x < n$;*

$$f(x) = f([x]) = \max\{y \colon y = 0 \text{ or } y \, P \, (x + 1) \text{ and } y \neq x + 1\}.$$

Then $f \in \mathfrak{M}(n)$ and $P_f = P$.

Proof. It is clear that f has integer values, that $0 \leqslant f(x) \leqslant x$, that $f(0) = 0$ and $f(n) = n$, and that $f([x]) = f(x)$. We shall prove that $x_1 < x_2$ implies that $f(x_1) \leqslant f(x_2)$. We may suppose that $f(x_1) = y_1 \neq 0$. Thus $y_1 \, P \, (x_1 + 1)$ and since $x_1 + 1 < x_2 + 1$ it follows from the normality of P that $y_1 \, P \, x_2 + 1$. Hence by definition $y_1 \leqslant f(x_2)$.

Now suppose $x \, P_f \, y$ and $x \neq y$. Then $x - 1 < f(y - 1)$. The case $f(y - 1) = 0$ is thus impossible since $x \geqslant 1$. Thus $f(y - 1) = \max\{z \colon z \, P \, y, z \neq y\}$. Thus $f(y - 1) \, P \, y$ and since $x \leqslant f(y - 1)$ and P is normal, it follows that $x \, P \, y$. Thus $P_f \subseteq P$. Conversely, suppose $x \, P \, y$ and $x \neq y$. Then $(x - 1) \, P \, y$ since P is normal and so $x - 1 \leqslant f(y - 1) = \max\{z \colon z \, P \, y, z \neq y\}$. Hence $x \, P_f \, y$ and so $P_f = P$.

LEMMA 4.3. *$f \subseteq g$ in $\mathfrak{M}(n)$ if and only if $P_f \subseteq P_g$ in $\mathfrak{N}(n)$.*

Proof. Suppose first that $f \subseteq g$. If $x \, P_f \, y$, then $x - 1 < f(y - 1) \leqslant g(y - 1)$ and so $x \, P_g \, y$. Conversely, suppose $P_f \subseteq P_g$ and consider x. If $g(x) < f(x)$, then $(g(x) + 1) \, P_f \, (x + 1)$ and so $(g(x) + 1) \, P_g \, (x + 1)$, but this entails $g(x) < g(x)$, a contradiction. Hence $f(x) \leqslant g(x)$.

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COROLLARY. *$f = g$ if and only if $P_f = P_g$. Moreover, given $f \in \mathfrak{M}(n)$, if f_1 is the function constructed in Lemma* 4.2 *corresponding to the partial order P_f, then $f = f_1$.*

Proof. The first part of the corollary is clear. Lemma 4.3 gives $P_{f_1} = P_f$ and thus $f_1 = f$.

THEOREM 4.1. *$\mathfrak{M}(n)$ is a lattice under the partial ordering $\subseteq$. The mapping $f \to P_f$ is a lattice isomorphism of $\mathfrak{M}(n)$ onto $\mathfrak{D}(n)$.*

Proof. It is a standard result that $\mathfrak{M}(n)$ is a complete lattice in which the greatest lower bound of a set $\mathfrak{R}$ of functions in $\mathfrak{M}(n)$ is the function g such that $g(x) = \min\{r(x): r \in \mathfrak{R}\}$; dually the least upper bound is the function h such that $h(x) = \max\{r(x): r \in \mathfrak{R}\}$. From Lemma 4.2 the mapping is onto and from Lemma 4.3 it is a lattice isomorphism.

The integer

$$\binom{2n}{n} \Big/ (n+1)$$

is the number of ways of inserting parentheses in a string of $n + 1$ symbols so as to interpret the string as an element in a binary (non-associative) system. This number is also the number of lattice paths and the connection between these two problems is through the Lukasiewicz parenthesis-free notation.* Thus, for example, $((a + b) + (c + d))$ becomes $+ +ab + cd$. By replacing $+$ by 1 and a, b, c by -1 and deleting the superfluous last symbol d, we obtain the sequence $1, 1, -1, -1, 1, -1$. The ith partial sums of this series is the integer s_i of the associated path $(s_1, s_2, s_3, s_4, s_5, s_6)$. The formula

$$\binom{2n}{n} \Big/ (n+1)$$

for the number of ways of parenthesizing $a_1 + a_2 + \ldots + a_{n+1}$ is developed in **(4)**.

It is interesting to note that Feller's count of the number of paths $(s_1, \ldots, s_{2n})$ and Hall's count of the number of ways of inserting parentheses in $a_1, \ldots, a_{n+1}$ coincide since each shows that this number, f_{n+1}, satisfies the recursion $f_{n+1} = f_0 f_n + f_1 f_{n-1} + \ldots + f_n f_0$. Our second development of the count of $\mathfrak{D}(n)$ proceeds along somewhat different lines and is of interest in itself.

We begin with the observation that since $\mathfrak{D}(n)$ is a finite distributive lattice, every element of $\mathfrak{D}$ has a unique expression as an irredundant meet of meet irreducibles. Moreover, in a distributive lattice a meet of meet irreducibles is irredundant if and only if no two irreducibles in the set are comparable. We shall characterize the meet irreducibles of $\mathfrak{D}$ and count the number of subsets of this set in which no two elements are comparable.

*We are indebted to Professors Marshall Hall, Jr., and Donald Knuth for these remarks.

LEMMA 4.4. *A normal NPO, P, is a meet irreducible in $\mathfrak{D}(n)$ if and only if $P = \Omega$ or there exists a pair (r, s), $r < s$, such that $P = \bigcup\{\Gamma(u, v): u < r$ or $s < v\}$.*

Proof of sufficiency. Let (r, s) be given. Define P by the formula of the lemma. We omit the proof that P is a normal NPO. It is easy to see that $u\,P\,v$ if and only if $u = v$ or $u < r$ or $s < v$. A typical P is shown in Figure 8.

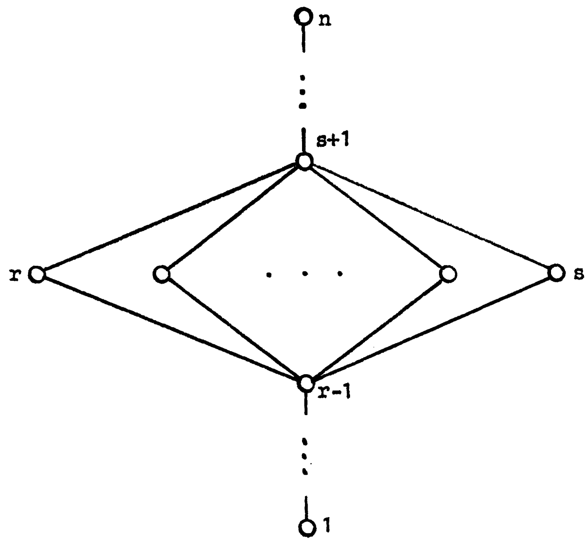

FIGURE 8.

Suppose that $P \subset R$ in $\mathfrak{D}$ and $(a, b) \in R - P$. Thus $r \leqslant a < b \leqslant s$. Since R is normal, it follows (Theorem 3.1) that $r\,R\,b$ and thus, for the same reason, that even $r\,R\,s$. Now if P is meet reducible, say, $P = R_1 \cap \ldots \cap R_k$ with $P \neq R_i$, then $r\,R_i\,s$ and hence $r\,P\,s$, a contradiction.

Proof of necessity. Let $P \neq \Omega$ be meet irreducible. Choose r and s such that $r \sim P\,s$ and $(s - r)$ is maximal. Let $\hat{P} = \bigcup\{\Gamma(u, v): u < r$ or $s < b\}$. We claim that $P = \hat{P}$. To show that $P \subseteq \hat{P}$ suppose that $a\,P\,b$. If $a < r$ or $s < b$, then $a\,\hat{P}\,b$. However, if $r \leqslant a < b \leqslant s$ it follows from normality that $r\,P\,b$ and this in turn implies that $r\,P\,s$; a contradiction. Hence $a\,P\,b$ implies either $a < r$ or $s < b$ and so $P \subseteq \hat{P}$.

Our proof that $P \supseteq \hat{P}$ is in three stages. First, from the choice of the pair (r, s) we have that $u < r$ and $x \geqslant s$ imply $u\,P\,x$; similarly $s < v$ and $y \leqslant r$ imply $y\,P\,s$. Second, we claim that $Q = P \cup \Gamma(r, s)$ $(= P \cup \{(r, s)\}$ by Theorem 3.3) is itself normal. We prove this by verifying the criteria of Theorem 3.1. Since P is normal, we need only check a triple $(r, s; t)$; however, our first remark shows that $x > s$ implies $r\,P\,t$, and dually. In particular, note that Q covers P in $\mathfrak{N}(n)$ from Lemma 1.3; a fortiori Q covers P in $\mathfrak{D}$. Thus since P is meet irreducible, Q is the unique element of $\mathfrak{D}$ covering P.

Third, we claim that for all u, if $u < r$, then $u\,P\,y$ for all $y \geqslant u$. This, and its dual which we omit, proves that $P \supseteq \hat{P}$. If our assertion is not true, choose u minimal such that $u < r$ and there exists a y such that $u \sim P\,y$. For this u choose $w = \max\{x \colon u \sim P\,x\}$. Now we claim that $R = P \cup \Gamma(u, w)$ is normal. Indeed, from Theorem 3.1, arguing as before, if R is not normal, it follows that either there exists s with $s < u$ and $s \sim P\,w$ (a contradiction of the choice of u) or there exists t with $w < t$ such that $u \sim P\,t$ (a contradiction of the choice of w). But manifestly, $Q \neq R$, a contradiction of the meet irreducibility of P.

In view of this lemma the following notation is sensible.

Notation. If P is a meet irreducible in $\mathfrak{D}$ and $P = \bigcup\{\Gamma(u, v) \colon u < r$ or $s < v\}$ we write $P = [r, s] = P_{r,s}$. The set of intervals

$$\mathfrak{R}(n) = \mathfrak{R} = \{[a, b] \colon 1 \leqslant a < b \leqslant n\}$$

is then isomorphic to the set of meet irreducibles of $\mathfrak{D}(n)$.

The ordering of $\mathfrak{D}$ induces a partial ordering of $\mathfrak{R}$: $[r, s] \leqslant [u, v]$ if and only if $P_{r,s} \subseteq P_{u,v}$. Clearly $[r, s] \leqslant [u, v]$ if and only if $r \leqslant u < v \leqslant s$; thus this ordering is dual to subset inclusion of the intervals on the real line.

From our previous remarks it now remains only to count the number of subsets of $\mathfrak{R}$ in which no two intervals are comparable. To this end we define a function $K(r, s)$ for all pairs (r, s), $r \leqslant s$. If $r < s$, let $K(r, s)$ be the number of subsets $E \subseteq \mathfrak{R}$ such that

(1) $E \subseteq \{[u, v] \colon u \leqslant r, u < v$, and $v \leqslant s\}$,

(2) no two intervals in E are comparable.

If $r = s$, let $K(s, s) = K(s - 1, s)$, and $K(1, 1) = 0$.

It is clear that $K(1, s) = s - 1$ if $1 < s$ and that the number of elements in $\mathfrak{D}$ is $1 + K(n, n)$ since we have not yet counted the meet irreducible Ω.

THEOREM 4.2. *If $r < s$, then $K(r, s)$ satisfies the recursion relation*

$$K(r, s) = K(r, s - 1) + K(r - 1, s) + 1$$

which is satisfied by

$$K(r, s) = -1 + \frac{1 + (s - r)}{1 + (s + r)} \binom{r + s + 1}{r}.$$

Proof. We observe that

$$K(r, s) = K(r, s - 1) + r + \sum_{i=1}^{r-1} K(i, s - 1).$$

The first term counts all sets E in which every index is at most $n - 1$. The second term counts all singleton sets E of the form $E = \{[i, s]\}$. The third sum counts all sets E which are not singletons and in which an interval $[i + 1, s]$ occurs. Note that since E cannot contain two comparable intervals, only one interval of the form $[i + 1, s]$ can occur. Also if $[i + 1, s] \in E$ and

$[h, j] \in E$, then $j \leqslant s - 1$ and $h \leqslant i$ and so $E - \{[i + 1, s]\}$ is counted in $K(i, s - 1)$. From this equation the recursion of the theorem follows by a direct calculation. Similarly, it is a simple matter to verify that

$$-1 + \frac{1 + (s - r)}{1 + (r + s)} \binom{1 + s + r}{r}$$

satisfies both the recursion and the initial conditions $K(1, 1) = 0$ and $K(1, s) = s - 1$.

COROLLARY.
$$1 + K(n, n) = 1 + K(n - 1, n) = \binom{2n}{n - 1} \Big/ n = \binom{2n}{n} \Big/ (n + 1).$$

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California Institute of Technology,
Pasadena, California