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A NOTE ON SMALL BAIRE SPACES

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A Baire space is a topological space which satisfies the Baire Category Theorem, i.e., in which the intersection of countably many dense open sets is dense. In this note we shall be interested in the size of Baire spaces, so to avoid trivialities we shall consider only *non-atomic* spaces, that is, spaces X whose regular open algebras ro(X) are non-atomic. All natural examples of Baire spaces, such as complete metric spaces or compact spaces, seem to have sizes at least $2^{\aleph_0}$. So a natural question, asked first by W. Fleissner and K. Kunen [**5**], is whether there exists a Baire space of the minimal possible size $\aleph_1$. The purpose of this note is to show that such a space need not exist by proving the following result.

THEOREM. *If* ZF *is consistent, then so is* ZFC *plus the following two statements simultaneously*:

i) MA + $2^{\aleph_0} = \aleph_2$,

ii) *There is no non-atomic Baire space of size* $\aleph_1$.

This completes a list of a number of weaker or related results on the Fleissner-Kunen problem ([**5**], [**9**], [**3**], [**7**], [**1**]). The most general previous results in the direction of this problem are a result of P. Davies [**3**] who proved that $\mathrm{MA}_{\aleph_1}$ implies that there is no Baire ccc space of size $\aleph_1$, a result of K. Kunen [**7**] who proved the theorem assuming the consistency of the existence of an inaccessible cardinal, and a result of U. Abraham [**1**, p. 647] who proved the theorem for partially ordered sets with the forcing topology.

MA + $\neg$CH is not sufficient to solve the Fleissner-Kunen problem. This has been pointed out by K. Kunen ([**3**], [**7**]) who showed that an η_1-linear ordering remains Baire in ccc forcing extensions. In fact, this is an immediate consequence of the well-known result of W. Easton [**4**] that σ-closed posets remain σ-distributive in ccc forcing extensions.

The proof is given in a sequence of lemmas. We shall assume that our ground model V satisfies GCH and $\neg S(\omega_1, \omega_2)$, where $S(\kappa, \lambda)$ denotes the fact that there is a κ-complete λ-saturated nontrivial ideal on κ. Some information about $S(\kappa, \lambda)$ can be found in [**6**]. We shall assume the facts that $\neg S(\omega_1, \omega_2)$ holds, for example, in any set-forcing extension of L, and that $\neg S(\omega_1, \omega_2)$ is preserved under $\aleph_2$-cc forcing extensions ([**6**, p. 67]).

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The first lemma is due to K. Kunen [**7**] and it is included here with his kind permission. Its proof uses some ideas of P. Davies [**3**].

LEMMA 1 (Kunen). *Assume* $\mathrm{MA}_{\aleph_1}$ *and* $\neg S(\omega_1, \omega_2)$. *Let X be a space of size* $\aleph_1$. *Then X is Baire if and only if* $\mathrm{ro}(X)$ *is σ-distributive.*

Proof. Only the direct implication requires a proof. Let X be a Baire space such that $\mathrm{ro}(X)$ is not σ-distributive. We may assume that there are maximal antichains $A_n \subseteq \mathrm{ro}(X)$, $(n < \omega)$ with A_{n+1} refining A_n such that whenever $a_n \in A_n$ then

$$\mathrm{int}\left(\bigcap_n a_n\right) = \emptyset.$$

Let

$$Y = \bigcap_n \bigcup A_n,$$

and let $\mathscr{C}$ be the set of all nonempty intersections $\cap_n a_n$ where $a_n \in A_n$, $(n < \omega)$. Pick a continuous $f: Y \to \mathbf{R}$ such that $f^{-1}(x) \in \mathscr{C}$ for all $x \in \mathrm{rang}(f)$ ([**5**, p. 233]). By $\mathrm{MA}_{\aleph_1}$, every subset of $\mathrm{rang}(f)$ is relatively F_σ so every union of elements of $\mathscr{C}$ is F_σ in Y. Hence the union of every subset of $\mathscr{C}$ not in

$$\mathscr{I} = \{\mathscr{D} \subseteq \mathscr{C}: \cup \mathscr{D} \text{ is meager in } Y\}$$

has nonempty interior. Thus $\mathscr{I}$ is an $\aleph_1$-complete $\aleph_2$-saturated ideal on $\mathscr{C}$, a contradiction.

The above proof, as Kunen [**7**] remarks, also explains a result of P. Davies [**3**] that $\mathrm{MA}_{\aleph_1}$ implies that there are no ccc Baire spaces of size $\aleph_1$. Namely, in this case in showing that $\mathrm{ro}(X)$ is σ-distributive one needs only $\neg S(\omega_1, \omega_1)$ which is a theorem of ZFC [**10**].

We say that a tree T is *special* [**2**] if and only if there is a decomposition

$$T = \bigcup_{n<\omega} D_n$$

such that for each n no element of D_n has two incomparable successors in D_n. We say that a non-atomic Boolean algebra $\mathscr{A}$ *embeds* a tree T of height ω_1 if and only if

1) each level T_α of T is a maximal antichain of $\mathscr{A}$;

2) if $\alpha < \beta < \omega_1$ then T_β everywhere properly refines T_α, and so $\leqq_T$ is equal to $\geqq_{\mathscr{A}} \upharpoonright T$;

3) $\wedge b = 0$ for every ω_1-branch b of T.

LEMMA 2. *If* $\mathscr{A}$ *embeds a special tree T, then* $\mathscr{A}$ *is not σ-distributive.*

Proof. Let $T = \cup_n D_n$ be a witness of T being special. For $n < \omega$, let

$\mathscr{A}_n$ be the set of all $a \in \mathscr{A}^+$ such that $t \notin D_n$ for any $t \in T$ with properties $a \wedge t \neq 0 \neq a \backslash t$. Then each $\mathscr{A}_n$ is a dense open subset of $\mathscr{A}^+$ and $\cap_n \mathscr{A}_n = \emptyset$. So $\mathscr{A}$ is not σ-distributive.

Our strategy in proving the theorem can shortly be described as follows. Given a space X of size $\aleph_1$ with ro(X) σ-distributive we shall try to embed a special tree into ro(X). This should be done via a poset which does not collapse cardinals. Clearly, we can easily embed an ω_1-tree T into ro(X), but in general T can have $> \aleph_1$ ω_1-branches so it cannot be made special in any cardinal preserving forcing extension. Our strategy is to project T onto a tree T' with $\leqq \aleph_1$ ω_1-branches which still embeds into ro(X) and which can be made special by a ccc poset of size $\aleph_1$. Moreover, ro(X) will embed T' in any forcing extension which decides $x \in$ int cl($\cup C$) for $x \in X$ and $C \in [T]^{\aleph_0}$, $C \in V$ in the same way as V does.

Let T be a fixed tree of size $\aleph_1$ and height ω_1 with the property that any point of T is contained in an ω_1-branch of T. Let B_T be the set of all ω_1-branches of T, and let $\mathscr{P}_T$ be the set of all pairs $p = \langle \pi_p, f_p \rangle$ where:

4) π_p is an order and level preserving map from a countable initial part of T onto a countable initial part of $(\omega_1)^{<\omega_1}$;

5) f_p is a countable partial map from B_T into ω_1 such that every member of dom(f_p) intersect each level of dom(π_p);

6) if $f_p(b) = f_p(c)$ then $\pi_p(s) = \pi_p(t)$ for all

$$s \in b \cap \text{dom}(\pi_p) \quad \text{and} \quad t \in c \cap \text{dom}(\pi_p)$$

of the same height.

For $p, q \in \mathscr{P}_T$ we let $q \leqq p$ if and only if $\pi_q \supseteq \pi_p, f_q \supseteq f_p$ and

7) $\pi_p(s) \neq \pi_q(t)$ for all $s \in \text{dom}(\pi_p)$ and $t \in \text{dom}(\pi_q \backslash \pi_p)$.

LEMMA 3. *Let $\mathscr{C} = \mathscr{C}_{\omega_1}$ be the standard poset for adding $\aleph_1$ Cohen reals and let $\dot{\mathscr{P}}_T$ be the poset $\mathscr{P}_T$ as defined in $V^{\mathscr{C}}$. Let $G = G_{\mathscr{C}*\dot{\mathscr{P}}_T}$ be a generic subset of $\mathscr{C} * \dot{\mathscr{P}}_T$ and let, in $V[G]$,*

$$T^* = \cup\{\text{rang}(\pi_p): p \in G_{\dot{\mathscr{P}}_T}\}$$

considered as a subtree of $(\omega_1)^{<\omega_1}$. Then every ω_1-branch of T^* has the form $\pi''b$ for some $b \in B_T$.

Proof. Working in $V[G_{\mathscr{C}}]$ and going to a dense subset of $\mathscr{P} = \mathscr{P}_T$, we may assume that dom(f_p) $\in V$ for every $p \in \mathscr{P}$. Assume there is a $\mathscr{P}$-name $\dot{b}^*$ for an ω_1-branch of $\dot{T}^*$ which is not of the form $\dot{\pi}''b$ for any $b \in B_T$. Let θ be a large enough regular cardinal, and let $N \prec H_\theta$ be countable such that $N \cap V \in V$ and $T, \mathscr{P}, \dot{b}^* \in N$. Let $\delta = N \cap \omega_1$, and let $s \in T_\delta$. Let $\mathscr{D}_s$ be the set of all $p \in \mathscr{P} \cap N$ for which there exist $\alpha < \delta$ and $t \in \text{dom}(\pi_p) \cap T_\alpha$ such that

$$p \Vdash \dot{\pi}(t) \in \dot{b}^*,$$

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but $u \not<_T s$ for any $u \in \mathrm{dom}(\pi_p) \cap T_\alpha$ with $\pi_p(u) = \pi_p(t)$.

Claim. $\mathscr{D}_s$ is dense open in $\mathscr{P} \cap N$.

Proof. Assume $p \in \mathscr{P} \cap N$ cannot be extended to a condition in $\mathscr{D}_s$. Let

$$S = \{t \in T: \nexists q \leqq p(t \in \mathrm{dom}(\pi_q) \text{ and } q \Vdash \dot{\pi}(t) \notin \dot{b}^*)\}.$$

Then $S \in N$ and $\{t: t <_T s\} \subseteq S$, so S is an uncountable initial part of T which contains no ω_1-branch. Since every point of T is contained in an ω_1-branch, we can find a $q \leqq p$ in N such that for some $\alpha < \delta$ and $t, u \in \mathrm{dom}(\pi_q) \cap T_\alpha \cap S$ we have $\pi_q(t) \neq \pi_q(u)$. Extending q to a condition r which forces that π_r of something from $\mathrm{dom}(\pi_r) \cap T_\alpha$ is in $\dot{b}^*$ gives a contradiction with one of the facts $t \in S$ or $u \in S$.

Define $F \subseteq (\mathscr{P} \times T)$ by

$$F(p, t) \equiv t \in \mathrm{dom}(\pi_p) \text{ and } p \Vdash \dot{\pi}(t) \in \dot{b}^*.$$

Pick a $\xi < \omega_1$ such that $\mathscr{P} \cap N$, $F \restriction (\mathscr{P} \times T) \cap N \in V[G_{\mathscr{C}_\xi}]$. Then

$$\mathscr{D}_s \in V[G_{\mathscr{C}_\xi}] \quad \text{for all } s \in T_\delta,$$

so we can use the generic subset of $\mathscr{C}_{\xi,\xi+\omega}$ to build a condition q of $\mathscr{P}$ which intersect each of $\mathscr{D}_s$ for $s \in T_\delta$. Then by our definition of $\mathscr{D}_s$, q forces

$$\dot{b}^* \cap \dot{T}^*_\delta = \emptyset,$$

a contradiction. This completes the proof of Lemma 3.

Note that the above proof also shows that $\mathscr{C}_{\omega_1} * \dot{\mathscr{P}}_T$ does not add new ω_1-branches to T, so the tree T^* has $\leqq \aleph_1$ ω_1-branches. Thus we may specialize T^* by a ccc poset $\mathscr{S}_{T^*}$ of size $\aleph_1$ [2]. It is clear that the compatibility relation of $\mathscr{P}_T$ is essentially determined by the π-part of the conditions from $\mathscr{P}_T$, that is, $p, q \in \mathscr{P}_T$ are compatible in $\mathscr{P}_T$ if $\pi_p = \pi_q$. Using this observation one easily shows that $\mathscr{P}_T$ satisfies the $\aleph_2$-proper isomorphism condition of [**8**, Chapter VIII]. Hence we can iterate $\mathscr{P}_T$ with other ccc posets of size $\aleph_1$ ω_2 times preserving properness and the $\aleph_2$-chain condition ([**8**, Chapters III and VIII]). So, let $\langle \mathscr{P}_\alpha : \alpha \leqq \omega_2 \rangle$ be a countable support iteration of ccc posets of size $\aleph_1$ and the posets of the form $\mathscr{P}_T$ such that $\mathscr{P}_{\omega_2}$ forces MA plus $2^{\aleph_0} = \aleph_2$, and such that for any $\mathscr{P}_{\omega_2}$-name $\dot{T}$ of a tree of size $\aleph_1$ and height ω_1 there exist stationarily many $\delta < \omega_2$ with cf $\delta = \omega_1$ such that

$$\mathscr{P}_{\delta+1} = \mathscr{P}_\delta * \dot{\mathscr{C}}_{\omega_1} * \dot{\mathscr{P}}_{\dot{T}} * \dot{\mathscr{S}}_{\dot{\mathscr{T}}^*}.$$

We shall show that

$$\Vdash_{\mathscr{P}_{\omega_2}} \text{ there is no non-atomic Baire space of size } \aleph_1,$$

but first we need to prove a lemma.

Let X be a space and let T be a tree of height ω_1. We say that ro(X) *strongly embeds* T if and only if ro(X) embeds T and:

8) for any $x \in X$ there is a $\xi < \omega_1$ such that $x \notin \cup T_\xi$,

9) for any $x \in X$ and $B \in [B_T]^{\aleph_0}$ there is a $\xi < \omega_1$ such that

$$x \notin \text{int cl}(\cup((\cup B) \cap T_\xi)).$$

LEMMA 4. *If X has size $\aleph_1$ and if* ro(X) *is σ-distributive, then* ro(X) *strongly embeds a tree T of height ω_1.*

Proof. Fix an enumeration $\{x_\xi : \xi < \omega_1\}$ of X. The tree T is constructed by induction on the levels T_ξ $(\xi < \omega_1)$ which are maximal antichains in ro(X) with T_η everywhere properly refining T_ξ, $(\xi < \eta < \omega_1)$ and such that for every $\xi < \omega_1$, $x_\xi \notin \cup T_{\xi+1}$. Moreover, for any $\xi < \omega_1$ one of the following two conditions holds:

10) there is an $\eta < \omega_1$ such that $x_\xi \notin \text{int cl}(\cup C)$ for all $C \in [T_\eta]^{\aleph_0}$, or

11) for every maximal antichain $A \subseteq$ ro(X) there is a $C \in [A]^{\aleph_0}$ such that $x_\xi \in \text{int cl}(\cup C)$.

We claim that ro(X) strongly embeds T. So let $B \subseteq B_T$ be countable and let $\xi < \omega_1$. If (10) holds we are done, so assume (11) holds. Let A be the set of all minimal points of $T \setminus \cup B$. A simple argument shows that A is a maximal antichain of ro(X). So there is a $C \in [A]^{\aleph_0}$ such that

$$x_\xi \in \text{int cl}(\cup C).$$

Let $\eta > \xi$ be above the height of any member of C. Then we must have

$$x_\xi \notin \text{int cl}(\cup((\cup B) \cap T_\eta)).$$

This completes the proof of Lemma 4.

Now we are ready to finish the proof that $\mathscr{P}_{\omega_2}$ forces that there is no non-atomic Baire space of size $\aleph_1$. To simplify the notation, let V^α denote $V^{\mathscr{P}_\alpha}$ for $\alpha \leqq \omega_2$. Assume V^{ω_2} contains a non-atomic Baire space $X = \langle \omega_1, \tau \rangle$. By Lemma 1 ro($X$) is σ-distributive so by Lemma 4 ro(X) strongly embeds an $\aleph_1$-splitting tree T of height ω_1. Since subtrees of T without ω_1-branches are special, it follows from Lemma 2 that every point of T is contained in an ω_1-branch of T. For $x \in X$, $\xi < \omega_1$ and $C \in [T_\xi]^{\aleph_0}$ we define, in V^{ω_2}, $I(x, \xi, C)$ to hold if and only if

$$x \notin \text{int cl}(\cup C).$$

So the fact that ro(X) strongly embeds T means that for all $x \in X$ and $B \in [B_T]^{\aleph_0}$ there is a $\xi < \omega_1$ such that

$$I(x, \eta, (\cup B) \cap T_\eta) \text{ holds for all } \eta \geqq \xi.$$

Let E be the set of all $\delta < \omega_2$ with cf $\delta = \omega_1$ such that

$$T \in V^{\delta} \quad \text{and} \quad I_{\delta} = I \cap (\omega_1 \times \omega_1 \times [T]^{\aleph_0})^{V^{\delta}} \in V^{\delta}.$$

It follows directly that for any $\delta \in E$, V^{δ} satisfies the following sentence

12) for all $x \in X$ and $B \in [B_T]^{\aleph_0}$ there is a $\xi < \omega_1$ such that

$I_{\delta}(x, \eta, (\cup B) \cap T_{\eta})$ holds for all $\eta \geqq \xi$.

Since $\langle \mathscr{P}_{\alpha}:\alpha \leqq \omega_2\rangle$ is a proper $\aleph_2$-cc iteration in which no real can first appear at a limit stage of cofinality ω_1, it follows easily that E is closed and unbounded relative to

$$\{\delta < \omega_2 : \text{cf } \delta = \omega_1\}.$$

So by our definition of $\langle \mathscr{P}_{\alpha}:\alpha \leqq \omega_2\rangle$ there must be a $\delta \in E$ such that

$$\mathscr{P}_{\delta+1} = \mathscr{P}_{\delta} * \dot{\mathscr{C}}_{\omega_1} * \dot{\mathscr{P}}_{\dot{T}} * \dot{\mathscr{S}}_{\dot{T}^*}$$

where $\dot{T}$ is a $\mathscr{P}_{\delta}$-name for our tree T. Define in V^{ω_2}

$$T' = \{\text{int cl}(\cup\{s \in T:\pi(s) = t\}) : t \in T^*\}.$$

Then T' is a special tree isomorphic to T^*. Moreover, the levels of T' are maximal antichains of ro(X) with T'_{η} everywhere properly refining T'_{ξ} for $\xi < \eta < \omega_1$. So by Lemmas 1 and 2 this will give us a contradiction if we can show, in V^{ω_2}, that $\wedge c = 0$ for every ω_1-branch c of T'.

So let c be an ω_1-branch of T' and let $x \in X$. By the way T^* is defined there is an ordinal $\gamma < \omega_1$ such that if $\xi < \omega_1$ and if $\{c_{\xi}\} = c \cap T'_{\xi}$, then there is a $p \in G_{\dot{\mathscr{P}}_T}$ such that if

$$B = \{b \in \text{dom}(f_p): f_p(b) = \gamma\}$$

and if $C = (\cup B) \cap T_{\xi}$, then

$$C \subseteq \text{dom}(\pi_p) \quad \text{and} \quad c_{\xi} = \text{int } cl(\cup C).$$

Since every countable set of ω_1-branches of T in $(V^{\delta})^{\dot{\mathscr{C}}_{\omega_1}}$ is covered by a countable set of branches from V^{δ}, and since (12) holds in V^{δ}, a simple density argument in $(V^{\delta})^{\mathscr{C}_{\omega_1}}$ over $\dot{\mathscr{P}}_T$ shows that there exist $\xi < \omega_1$ and $p \in G_{\dot{\mathscr{P}}_T}$ such that

$$B = \{b \in \text{dom}(f_p): f_p(b) = \gamma\} \quad \text{and}$$

$$C = (\cup B) \cap T_{\xi} \subseteq \text{dom}(\pi_p)$$

are members of V^{δ} and $I_{\delta}(x, \xi, C)$ holds. Since $\delta \in E$, we have that $I(x, \xi, C)$ holds in V^{ω_2}, and so

$$V^{\omega_2} \models x \notin \text{int cl}(\cup C) = c_{\xi} \geqq \wedge c.$$

Since $x \in X$ was arbitrary it follows that $\wedge c = 0$. This completes the proof.

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