ON REPRESENTATIONS AS A SUM OF CONSECUTIVE INTEGERS

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1. Introduction. It is the object of this paper to investigate the function $\gamma(m)$, the number of representations of m in the form

$$(r+1)+(r+2)+\ldots+s, \tag{1}$$

where $s > r \geqslant 0$. It is shown that $\gamma(m)$ is always equal to the number of odd divisors of m, so that for example $\gamma(2^k) = 1$, this representation being the number 2^k itself. From this relationship the average order of $\gamma(m)$ is deduced; this result is given in Theorem 2. By a method due to Kac [2], it is shown in §3 that the number of positive integers $m \leqslant n$ for which $\gamma(m)$ does not exceed a rather complicated function of n and ω, a real parameter, is asymptotically $nD(\omega)$, where $D(\omega)$ is the probability integral

$$(2\pi)^{-\frac{1}{2}} \int_{-\infty}^{\omega} e^{-\frac{1}{2}x^2}\, dx.$$

In §4, these theorems are extended to $\gamma(m, s)$, the number of representations of m as the sum of positive consecutive terms in any of the s arithmetic progressions having constant difference s.

2. The average order of $\gamma(m)$. First we prove

THEOREM 1. *$\gamma(m) = \tau(\bar{m})$ where $\tau(u)$ is the number of divisors of u and* $m = 2^{a-1}\bar{m}$, $\bar{m}$ odd.

For by (1) we have

$$m = \frac{s^2+s}{2} - \frac{r^2+r}{2}, \quad 2m = (s-r)(s+r+1).$$

Putting $s - r = n$, this gives

$$2m = n(n+2r+1).$$

Since n and $n+2r+1$ have opposite parity, and since $n < (2m)^{\frac{1}{2}}$, $\gamma(m)$ is the number of ways of writing $2m$ as the product of an even and an odd number. That is,

$$\gamma(m) = \sum_{\substack{n \mid \bar{m} \\ n < (2m)^{\frac{1}{2}}}} 1 + \sum_{\substack{2m/n \mid \bar{m} \\ 2m/n > (2m)^{\frac{1}{2}}}} 1 = \sum_{d \mid \bar{m}} 1 = \tau(\bar{m}).$$

THEOREM 2. *The average order of $\gamma(m)$ is $\frac{1}{2}\log m$; more precisely,*

$$\frac{1}{n}\sum_{m=1}^{n} \gamma(m) = \frac{1}{2}\log n + \frac{2C + \log 2 - 1}{2} + O\,(n^{-\frac{1}{2}}),$$

where C is Euler's constant.

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For let l be the unique integer such that $2^l \leqslant n < 2^{l+1}$. Then by Theorem 1,

$$\sum_{m=1}^{n} \gamma(m) = \sum_{m=1}^{n} \tau(\overline{m})$$

$$= \sum_{\substack{1 \leqslant m \leqslant n \\ m \equiv 1 \pmod 2}} \tau(m) + \sum_{\substack{1 \leqslant m \leqslant n \\ m \equiv 2 \pmod 4}} \tau(m/2) + \sum_{\substack{1 \leqslant m \leqslant n \\ m \equiv 4 \pmod 8}} \tau(m/4) + \dots$$

$$+ \sum_{\substack{1 \leqslant m \leqslant n \\ m \equiv 2^l \pmod{2^{l+1}}}} \tau(m/2^l)$$

$$= \sum_{r=0}^{(n-1)/2} \tau(2r+1) + \sum_{r=0}^{(n-2)/4} \tau(2r+1) + \dots$$

$$+ \sum_{r=0}^{(n-2^l)/2^{l+1}} \tau(2r+1),$$

and since $l = \left[\dfrac{\log n}{\log 2}\right]$ this is

$$\sum_{m=1}^{n} \gamma(m) = \sum_{t=0}^{(\log n)/\log 2} \sum_{r=0}^{2^{-t}n-1} \tau(2r+1). \tag{2}$$

We estimate the sum

$$\sum_{r=0}^{(w-1)/2} \tau(2r+1)$$

by counting the "odd" lattice points (x, y), i.e., those with both coordinates odd, for which $0 < xy \leqslant w$. (For a full account of this kind of reasoning, see Hardy and Wright, [1], p. 263). We put

$$u = 2[\tfrac{1}{2}w^{\frac{1}{2}}] + 1$$

and obtain

$$\sum_{r=0}^{(w-1)/2} \tau(2r+1) = 2\sum_{z=0}^{(u-1)/2}\left[\frac{1}{2}\left(\frac{w}{2z+1}\right)\right] - \frac{(u-1)^2}{4} + O(1)$$

$$= \frac{1}{4} w \log w + \frac{2C + 2\log 2 - 1}{4} w + O(w^{\frac{1}{2}}).$$

Putting this estimate in (2), we have

$$\sum_{m=1}^{n} \gamma(m) = \sum_{t=0}^{(\log n)/\log 2} \left\{\frac{1}{4}\frac{n \log (n/2^t)}{2^t} + \frac{2C + 2\log 2 - 1}{4}\frac{n}{2^t} + O(2^{-\frac{1}{2}t}n^{\frac{1}{2}})\right\}$$

$$= \frac{n \log n}{2} + \frac{2C + \log 2 - 1}{2} n + O(n^{\frac{1}{2}}),$$

and this completes the proof.

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3. A density theorem concerning $\gamma(m)$.

THEOREM 3. *Let ω be a real number, and let $s_n(\omega)$ be the number of positive integers $m \leqslant n$ for which*

$$\gamma(m) \leqslant 2^{\log\log n + \omega (\log \log n)^{\frac{1}{2}} - 1} = f(n, \omega).$$

Then

$$s_n(\omega) \sim nD(\omega).$$

The proof of this is quite similar to that given by Kac [2] in proving that the number of $m \leqslant n$ for which $\tau(m) \leqslant 2f(n, \omega)$ is asymptotic to $nD(\omega)$.

4. Representations in arithmetic progressions. We now turn our attention to $\gamma_1(m, s)$, the number of representations of m of the form

$$m = r + (r + s) + \ldots + \{r + (k - 1)\, s\}. \tag{3}$$

Although it was natural in the case $s = 1$ to restrict attention to positive representations (i.e., with $r > 0$), it turns out in the general case that this condition introduces complications. For this reason we shall consider separately the quantity $\gamma_1(m, s)$ and the quantity $\gamma(m, s)$, the number of positive representations of m in the form (3). In either case it is required that

$$2m = k\{2r + (k - 1)\, s\}. \tag{4}$$

THEOREM 4. $\gamma_1(m, s) = \tau(m)$ *if* $s \equiv 0 (\text{mod } 2)$, *and* $\gamma_1(m, s) = 2\, \tau(\overline{m})$ *if* $s \equiv 1$ (mod 2).

For if s is even, say $s = 2s_1$, then $\gamma_1(m, s)$ is the number of solutions k, r $(k > 0)$ of

$$m = k(r + (k - 1)s_1),$$

and k can clearly be any divisor of m. If s is odd, then k and $2r + (k - 1)s$ are of opposite parity, so that

$$\gamma_1(m, s) = \sum_{k|\overline{m}} 1 + \sum_{2m/k|\overline{m}} 1 = 2\tau(\overline{m}).$$

For example,

$$\gamma_1(6,1) = 4: \quad 6 = 1 + 2 + 3 = (-5) + (-4) + \ldots + 4 + 5 + 6 = 0 + 1 + 2 + 3;$$

and

$$\gamma_1(6,2) = 4: \quad 6 = 2 + 4 = 0 + 2 + 4 = (-4) + (-2) + 0 + 2 + 4 + 6.$$

As an immediate consequence of Theorems 2 and 4, and the fact that the average order of $\tau(m)$ is $\log m + 2C - 1 + O(m^{-\frac{1}{2}})$ ([1], *loc. cit.*), we have

THEOREM 5.

$$\frac{1}{n} \sum_{m=1}^{n} \gamma_1(m, s) = \begin{cases} \log n + (2C - 1) + O(n^{-\frac{1}{2}}) & \text{if } s \equiv 0 \text{ (mod 2)} \\ \log n + \frac{1}{2}(2C - 1 + \log 2) + O(n^{-\frac{1}{2}}) & \text{if } s \equiv 1 \text{ (mod 2)}. \end{cases}$$

We now put on the restriction $r > 0$. Then by (4), k must be chosen so that

$$k(k-1)\, s < 2m,$$

or

$$k < \frac{1 + (1 + 8m/s)^{\frac{1}{2}}}{2}.$$

But

$$\left(\frac{2m}{s}\right)^{\frac{1}{2}} < \frac{1 + (1 + 8m/s)^{\frac{1}{2}}}{2} < \left(\frac{2m}{s}\right)^{\frac{1}{2}} + 1,$$

so that we will make an error of not more than 1 if, in computing $\gamma(m, s)$, we count the number of suitable k's which do not exceed $(2m/s)^{\frac{1}{2}}$. Thus by the argument used in proving Theorem 4, we find that if $s = 2s_1$ is even,

$$\gamma(m, s) = \sum_{\substack{k \mid m \\ k \leqslant (2m/s)^{\frac{1}{2}}}} 1 + \epsilon(m, s) = \tau(m, (2m/s)^{\frac{1}{2}}) + \epsilon(m, s),$$

where $\tau(m, x)$ is the number of divisors of m which do not exceed x, and $\epsilon(m, s)$ is either 0 or 1. We put

$$A(n, x) = \sum_{m=1}^{n} \gamma(m, s).$$

Then all those lattice points on the hyperbola $xy = m$ for which $x \leqslant (2m/s)^{\frac{1}{2}}$ are counted in the sum $\sum_1^n \tau(m, (2m/s)^{\frac{1}{2}})$, and by considering all positive m not exceeding n, we see that this sum is exactly the number of lattice points in the region $0 < xy \leqslant n$, $y \geqslant \frac{1}{2} sx$. Counting along vertical lines, we have

$$\begin{aligned}
\sum_{m=1}^{n} &\tau(m, (2m/s)^{\frac{1}{2}}) \\
&= \sum_{x=1}^{(2n/s)^{\frac{1}{2}}} \left\{\left[\frac{n}{x}\right] - \frac{sx}{2} + 1\right\} = n \sum_{x=1}^{(2n/s)^{\frac{1}{2}}} \frac{1}{x} + O(n^{\frac{1}{2}}) - \frac{s}{2} \sum_{x=1}^{(2n/s)^{\frac{1}{2}}} x + \left[\left(\frac{2n}{s}\right)^{\frac{1}{2}}\right] \\
&= n\left\{\log \left(\frac{2n}{s}\right)^{\frac{1}{2}} + C + O(n^{-\frac{1}{2}})\right\} - \frac{s}{4}\left\{\left[\left(\frac{2n}{s}\right)^{\frac{1}{2}}\right]^2 + \left[\left(\frac{2n}{s}\right)^{\frac{1}{2}}\right]\right\} + O(n^{\frac{1}{2}}) \\
&= \frac{n}{2} \log n + n\left(C - \frac{1}{2}\log\frac{s}{2} - \frac{1}{2}\right) + O(n^{\frac{1}{2}}).
\end{aligned}$$

As for the sum $\sum_1^n \epsilon(m, s)$, it does not exceed the number of lattice points on the curves $xy = m \leqslant n$ for which

$$(2m/s)^{\frac{1}{2}} < x \leqslant (2m/s)^{\frac{1}{2}} + 1,$$

i.e., the number of lattice points in the bounded region enclosed by the hyperbolas $xy = n$, $(x-1)^2 s = 2xy$ and the line $l_1 : y = \frac{1}{2}sx$. But the second of these hyperbolas is asymptotic to the line $l_2 : y = \frac{1}{2}s(x - 1)$. Let the intersections of l_1 and l_2 with $xy = n$ be (x_1, y_1) and (x_2, y_2) respectively, and let the chord joining these points be l_3. Then the sum in question is less than the number of lattice points in the triangle with vertices at $(0, 0)$, (x_1, y_1) and (x_2, y_2), plus the number of lattice points in the triangle with vertices at $(0, 0)$, $(x_2\, y_2)$ and the intersection of l_2 with the x-axis. This follows since l_3 is always above the curve $xy = n$. But it is easy to see that the number of lattice points in a triangle does not exceed one more than the sum of its area and perimeter. Hence

$$\sum_{m=1}^{n} \epsilon(m, s) < \tfrac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ 0 & 0 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} + \tfrac{1}{2}\begin{vmatrix} 1 & 0 & 0 \\ 1 & 0 & 2c_0/s \\ 1 & x_2 & y_2 \end{vmatrix} + (x_1^2 + y_1^2)^{\frac{1}{2}}$$
$$+ 2(x_2^2 + y_2^2)^{\frac{1}{2}} + \left\{(x_2 - x_1)^2 + (y_2 - y_1)^2\right\}^{\frac{1}{2}} + 2c_0/s$$
$$+ \left\{(x_2 - 2c_0/s)^2 + y_2^2\right\}^{\frac{1}{2}}.$$

Substituting the values $x_1 = (2n/s)^{\frac{1}{2}}$, $y_1 = (sn/2)^{\frac{1}{2}}$, $x_2 = \frac{1}{2}\{(8n/s + 1)^{\frac{1}{2}} + 1\}$ $y_2 = n/x_2$, it is easily verified that this upper bound is $O(n^{\frac{1}{2}})$.

We have thus shown that in case s is even,

$$A(n, s) = \frac{n}{2}\log n + \frac{n}{2}\left(2C - \log\frac{s}{2} - 1\right) + O(n^{\frac{1}{2}}). \tag{5}$$

On the other hand, if $s \equiv 1 \pmod 2$, then in (4) either k is even, in which case it contains the highest power 2^{α} of 2 which divides $2m$ and is such that r is positive, or k is odd, with r again positive. Hence

$$\gamma(m, s) = \sum_{\substack{k|\overline{m} \\ k \leqslant (2m/s)^{\frac{1}{2}}}} 1 + \sum_{\substack{k_1|\overline{m} \\ 2^{\alpha}k_1 \leqslant (2m/s)^{\frac{1}{2}}}} 1 + \epsilon(m, s)$$
$$= \tau(\overline{m}, (2^{\alpha}\overline{m}/s)^{\frac{1}{2}}) + \tau(\overline{m}, (2^{-\alpha}\overline{m}/s)^{\frac{1}{2}}) + \epsilon(m, s),$$

where $\epsilon(m, s)$, as before, is the error made in assuming that for r to be positive k must not exceed $(2m/s)^{\frac{1}{2}}$, rather than the actual upper bound. Since the bound for $\sum_1^n \epsilon(m, s)$ which we just computed did not depend on the parity of s, it holds also for odd s:

$$\sum_{m=1}^{n} \epsilon(m, s) = O(n^{\frac{1}{2}}). \tag{6}$$

We have

$$A(n, s) = \sum_{m=1}^{n} \tau(\overline{m}, (2^{\alpha}\overline{m}/s)^{\frac{1}{2}}) + \sum_{m=1}^{n} \tau(\overline{m}, (2^{-\alpha}\overline{m}/s)^{\frac{1}{2}}) + \sum_{m=1}^{n} \epsilon(m, s)$$
$$= A_1 + A_2 + A_3,$$

say. Summing over m's containing the same power of 2, we get

$$A_1 = \sum_{\lambda=1}^{(\log n)/\log 2} \sum_{r=1}^{2^{-\lambda}n - \frac{1}{2}} \tau(2r + 1, \{2^{\lambda}(2r + 1)/s\}^{\frac{1}{2}}).$$

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The sum

$$\sum_{r=0}^{(z-1)/2} \tau(2r+1, c^{\frac{1}{2}}(2r+1)^{\frac{1}{2}})$$

is the number of lattice points on the hyperbolas

$$xy = 2r+1, \qquad r = 0, 1, \ldots, \tfrac{1}{2}(z-1)$$

for which $x \leqslant c^{\frac{1}{2}}(2r+1)^{\frac{1}{2}}$, i.e., for which $x \leqslant cy$. This is the number of odd lattice points in this region, which is

$$\sum_{x=0}^{t} \left\{ \left[\frac{1}{2}\left(\frac{z}{2x+1}+1\right)\right] - \left[\frac{1}{2}\left(\frac{2x+1}{c}+1\right)\right] + \delta(x) \right\},$$

where $\delta(x)$ is 0 or 1 and

$$t = \left[\frac{1}{2}\left\{c^{\frac{1}{2}}\left(2\left[\frac{z-1}{2}\right]+1\right)^{\frac{1}{2}} - 1\right\}\right] \sim \frac{(cz)^{\frac{1}{2}}}{2}.$$

But this sum is equal to

$$\frac{z}{2}\sum_{x=0}^{t} \frac{1}{2x+1} + O(t) - \frac{1}{2c}\sum_{x=0}^{t}(2x+1) + O(t)$$
$$= \frac{z}{4}\log\left\{c\left(2\left[\frac{z-1}{2}\right]+1\right)\right\}^{\frac{1}{2}} + \frac{z}{4}(C+\log 2) + O(z^{\frac{1}{2}}) - \frac{t^2}{2c} + O(t),$$

so that

$$\sum_{r=0}^{(z-1)/2} \tau(2r+1, c^{\frac{1}{2}}(2r+1)^{\frac{1}{2}}) = \frac{z\log z}{8} + \frac{z}{4}(C + \log 2 + \log c^{\frac{1}{2}}) - \frac{z}{8} + O(z^{\frac{1}{2}}). \tag{7}$$

Hence

$$A_1 = \sum_{\lambda=1}^{(\log n)/\log 2} \left\{\frac{n}{2^{\lambda-1}}\frac{1}{8}\log\frac{n}{2^{\lambda-1}} + \frac{n}{2^\lambda}\frac{1}{4}\left(C + \log 2 + \log\frac{2^{\frac{1}{2}\lambda}}{s^{\frac{1}{2}}}\right) - \frac{n}{8.2^{\lambda-1}} + O\left(\frac{n^{\frac{1}{2}}}{2^{\frac{1}{2}(\lambda-1)}}\right)\right\}$$

$$= \frac{n\log n}{8}\sum_{\lambda=1}^{\log n}\frac{1}{2^{\lambda-1}} - \frac{n}{8}\log 2\sum_{\lambda=1}^{\log n}\frac{\lambda-1}{2^{\lambda-1}} + \frac{n}{4}(C+\log 2)\sum_{\lambda=1}^{\log n}\frac{1}{2^{\lambda-1}}$$
$$+ \frac{n\log 2}{8}\sum_{\lambda=1}^{\log n}\frac{\lambda}{2^{\lambda-1}} - \frac{n\log s}{8}\sum_{\lambda=1}^{\log n}\frac{1}{2^{\lambda-1}} - \frac{n}{8}\sum_{\lambda=1}^{\log n}\frac{1}{2^{\lambda-1}} + O(n^{\frac{1}{2}})$$
$$= \frac{n\log n}{4} + n\left(\frac{C}{2} + \frac{\log 2}{2} - \frac{\log s}{4}\right) - \frac{n}{4} - \frac{n\log 2}{4}$$
$$+ \frac{n\log 2}{4} + O(n^{\frac{1}{2}}),$$

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and finally

$$A_1 = \frac{n \log n}{4} + n\left(\frac{C + \log 2}{2} - \frac{1}{4} - \frac{\log s}{4}\right) + O(n^{\frac{1}{2}}). \tag{8}$$

Turning now to A_3, we have

$$A_3 = \sum_{\lambda=1}^{(\log n)/\log 2} \sum_{r=0}^{2^{-\lambda}n - \frac{1}{2}} \tau\left(2r + 1, \left(\frac{2r+1}{2^\lambda s}\right)^{\frac{1}{2}}\right),$$

and using (7) with $z = n/2^{\lambda-1}$, $c = s/2^\lambda$, we have

$$A_2 = \sum_{\lambda=1}^{(\log n)/\log 2} \left\{\frac{n}{8.2^{\lambda-1}} \log \frac{n}{2^{\lambda-1}} + \frac{n}{4.2^{\lambda-1}} (C + \log 2 + \log (2^\lambda s)^{-\frac{1}{2}}) - \frac{n}{4.2^{\lambda-1}} + O\left(\frac{n^{\frac{1}{2}}}{2^{\frac{1}{2}(\lambda-1)}}\right)\right\}$$

$$= \frac{n \log n}{4} + n\left(\frac{C}{2} - \frac{1}{4} - \frac{\log s}{4}\right) + O\,(n^{\frac{1}{2}}).$$

Combining this with (5), (6) and (8), we have

THEOREM 6. *For every s,*

$$\frac{1}{n} \sum_{m=1}^{n} \gamma(m, s) = \frac{1}{2} \log n + \left(C - \frac{1}{2} \log \frac{s}{2} - \frac{1}{2}\right) + O\,(n^{-\frac{1}{2}}).$$

Theorem 2 is, of course, the special case of Theorem 6 with $s = 1$.

REFERENCES

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