A GENERALIZATION OF SUMMABILITY-(Z, p) OF SILVERMAN AND SZÁSZ

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1. Background

A sequence $x = \{s_i\}$ has been defined to be summable-(Z, p) to s if

$$\lim_n Z_n^p(x) = s,$$

where p is a positive integer and

$$Z_n^p(x) = (s_{n-p+1} + \ldots + s_n)/p \quad (n \geqq 0,\ s_i = 0 \text{ if } i < 0).$$

Ignoring the values of n for which $0 \leqq n < p-1$, the transformation (Z, p) coincides with the Nörlund transformation defined by the sequence

$$(1, 1, \ldots, 1, 0, 0, \ldots)$$

containing p initial 1's. This class of methods has been studied by Silverman and Szász (**5**) and by Hill and Sledd (**4**). We quote the following results for reference.

(1.1) Summability-(Z, p) is regular for $p = 2, 3, \ldots$ and $(Z, 1)$ is the identity transformation.

(1.2) ((**5**), Th. 11) Summability-(Z, p) implies summability-$(C, 1)$ to the same value for $p = 1, 2, 3, \ldots$.

(1.3) ((**5**), Th. 14) If p is a divisor of q, then the convergence field of (Z, p) is contained in that of (Z, q).

(1.4) ((**5**) Th. 15) If d is the greatest common divisor of p and q, then the convergence fields of (Z, p) and (Z, q) intersect in that of (Z, d).

(1.5) $Z_{n+p}^p(x) = Z_{p-1}^p(x) - \frac{1}{p}\sum_{i=0}^{n}(s_i - s_{i+p})$ for $p, n = 1, 2, 3, \ldots$, and therefore a sequence $x = \{s_i\}$ is summable-(Z, p) if and only if $\sum_{i=0}^{n}(s_i - s_{i+p})$ is convergent. Furthermore, $x = \{s_i\}$ is summable (Z, p) only if $s_i - s_{i+p} \to 0$ as $i \to \infty$.

2. (Z, p, k) Summability

We will be interested in generalizing the definition of (Z, p)-summability as well as the results (1.1)-(1.5). We write

$$Z_n^{p,k}(x) = (Z_{n-p+1}^{p,k-1}(x) + \ldots + Z_n^{p,k-1}(x))/p \quad (n \geqq 0),$$
$$Z_j^{p,k-1}(x) = 0 \text{ if } j < 0,$$

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where p and k are positive integers, $k>1$ and $Z_n^{p,1}(x) = Z_n^p(x)$. A sequence $x = \{s_i\}$ is summable (Z, p, k) to the value s if

$$\lim_n Z_n^{p,k}(x) = s.$$

For brevity, let us write $Z_n^{p,k}(x) = Z_n^{p,k}$. It is easy to express $Z_n^{p,k}$ in terms of s_n. We have

$$\sum_{n=0}^{\infty} Z_n^{p,k}x^n = p^{-1}\left(\frac{1-x^p}{1-x}\right)\sum_{n=0}^{\infty} Z_n^{p,k-1}x^n$$

$$= p^{-2}\left(\frac{1-x^p}{1-x}\right)^2\sum_{n=0}^{\infty} Z_n^{p,k-2}x^n,$$

and so on, giving

$$\sum_{n=0}^{\infty} Z_n^{p,k}x^n = p^{-k}\left(\frac{1-x^p}{1-x}\right)^k\sum_{n=0}^{\infty} s_n x^n. \tag{2.1}$$

Thus

$$Z_n^{p,k} = p^{-k}\sum_{i=0}^{n} b_i^{p,k}s_{n-i}, \quad n \geqq 0, \tag{2.2}$$

with k a positive integer, $b_i^{p,k} = 0$ for $i>k(p-1)$ and

$$\left(\frac{1-x^p}{1-x}\right)^k = \sum_{i=0}^{k(p-1)} b_i^{p,k}x^i.$$

Ignoring the values of n for which $0 \leqq n<k(p-1)$, the transformation (Z, p, k) coincides with the Nörlund transformation defined by the sequence

$$(b_0^{p,k}, b_1^{p,k}, \ldots, b_{k(p-1)}^{p,k}, 0, 0, \ldots).$$

We now wish to show that (2.1) and (2.2) remain significant for non-integral k and allow us to give a more general definition of (Z, p, k). In order to do this, we need information concerning the coefficients of the series expansion of $(1-x^p)^k(1-x)^{-k}$ when k is non-integral. We can find this by comparing the general case with the case $p = 2$. For $p = 2$, $(1-x^p)^k(1-x)^{-k}$ is the familiar $(1+x)^k$. If k is positive and non-integral then

$$\left(\frac{1-x^2}{1-x}\right)^k = (1+x)^k = \sum_{n=0}^{\infty}\binom{k}{n}x^n = \sum_{n=0}^{\infty} p_n x^n,$$

where

$$p_n \sim n^{-k-1}$$

On the other hand,

$$(1-x^2)^k(1-x)^{-k} = \sum_{n=0}^{\infty}(-1)^n\binom{k}{n}x^{2n}.\sum_{n=0}^{\infty}(-1)^n\binom{-k}{n}x^n$$

$$= \sum_{n=0}^{\infty} a_{2n}x^{2n}.\sum_{n=0}^{\infty} b_n x^n = \sum_{n=0}^{\infty} p_n x^n$$

Therefore

(2.3) $p_{2n} = b_0 a_{2n} + b_2 a_{2(n-1)} + \dots + b_{2n} a_0 \sim (2n)^{-k-1} \sim n^{-k-1},$

$$p_{2n+1} = b_1 a_{2n} + b_3 a_{2(n-1)} + \dots + b_{2n+1} a_0 \sim (2n+1)^{-k-1} \sim n^{-k-1}.$$

Considering the case $p = 3$, we have

$$(1-x^3)^k (1-x)^{-k} = \sum_{n=0}^{\infty} (-1)^n \binom{k}{n} x^{3n} \cdot \sum_{n=0}^{\infty} (-1)^n \binom{-k}{n} x^n$$

$$= \sum_{n=0}^{\infty} c_{3n} x^{3n} \sum_{n=0}^{\infty} b_n x^n = \sum_{n=0}^{\infty} q_n x^n.$$

Therefore

$$q_{3n+i} = b_i c_{3n} + b_{i+3} c_{3(n-1)} + \dots + b_{3n+i} c_0, \quad i = 0, 1, 2.$$

Since $c_{3n} = a_{2n}$, $n = 0, 1, 2, \dots$, and $b_{3n+i} \sim (3n+i)^{k-1} \sim (2n+j)^{k-1} \sim b_{2n+j}$ for $i = 0, 1, 2$; $j = 0, 1$, (2.3) gives us

$$q_{3n+i} \sim n^{-k-1}, \quad i = 0, 1, 2.$$

In general, if

(2.4) $$\left(\frac{1-x^p}{1-x}\right)^k = \sum_{i=0}^{\infty} b_i^{p,k} x^i$$

then

(2.5) $$b_i^{p,k} = b_{j+np}^{p,k} \sim n^{-k-1}, \quad j = 0, 1, \dots, p-1.$$

If $k > 0$ then $\sum_{i=0}^{\infty} | b_i^{p,k} | < \infty$. The series in (2.4) has radius of convergence 1 and Abel's theorem gives the result that $B_n^{p,k} = \sum_{i=0}^{n} b_i^{p,k} \to p^k$ as $n \to \infty$. In view of these results (2.1) and (2.2) can be extended to include all real positive values of k. (Z, p, k), with p a positive integer and k positive and real, is a Nörlund method associated with the sequence $\{b_i^{p,k}\}$ and the sequence is defined by (2.3).

We now quote two theorems given by Borwein and Boyd ((**2**), Ths. 16, 17) concerning Nörlund methods.

(2.6) The Nörlund method (N, p_n) is regular if and only if there is a constant H independent of n such that

$$\sum_{r=0}^{n} | p_r | < H | P_n | \quad \text{for } n \geqq M$$

and $p_n / P_n \to 0$ as $n \to \infty$.

(2.7) If (N, p_n) and (N, q_n) are regular and $p(x) = \sum_{n=0}^{\infty} p_n x^n$, $q(x) = \sum_{n=0}^{\infty} q_n x^n$, $q(x)/p(x) = \sum_{n=0}^{\infty} k_n x^n$, then summability-$(N, p_n)$ implies summability-(N, q_n) if and only if there is a constant H independent of n such that

$$\sum_{r=0}^{n} |k_{n-r}P_r| < H |Q_n| \quad \text{for } n \geqq M$$

and $k_n/Q_n \to 0$ as $n \to \infty$.

Applying these theorems to (Z, p, k), we have

Theorem 1. *Summability-(Z, p, k) is regular for $k>0$ and $p = 2, 3, \ldots$ and $(Z, 1, k)$ is the identity transformation.*

Theorem 2. *If p is a divisor of q, then the convergence field of (Z, p, k) is properly contained in that of (Z, q, k) for $k>0$.*

Theorem 3. *If $0<k<k'$, then the convergence field of (Z, p, k) is properly contained in that of (Z, p, k') for $p = 2, 3, \ldots$.*

Theorem 4. *Summability-(Z, p, k) implies summability-(C, k) to the same value for $k>0$ and $p = 2, 3, 4, \ldots$.*

In order to generalize (1.4) and (1.5) we restrict k to be a positive integer. With this restriction the following theorem follows directly from a theorem of Borwein ((**1**), Th. 3).

Theorem 5. *If d is the greatest common divisor of p and q, then the convergence fields of (Z, p, k) and (Z, q, k) intersect in that of (Z, d, k) for $k = 1, 2, 3, \ldots$.*

Using induction, it can be proved that

$$b_i^{p,k} = b_{k(p-1)-i}^{p,k}, \quad i = 0, 1, \ldots, k(p-1); \; k = 1, 2, 3, \ldots.$$

Therefore we can write

$$Z_{n+k(p-1)+1}^{p,k}(x) = p^{-k} \sum_{i=0}^{k(p-1)} b_i^{p,k} s_{n+1+i} \quad (n \geqq -k(p-1)-1; \; s_k = 0 \text{ if } k>0).$$

Furthermore we have the following identity:

$$\sum_{i=0}^{k(p-1)} b_i^{p,k} x^{n+1+i} = \sum_{i=0}^{k(p-1)} b_i^{p,k} x^i - \sum_{j=0}^{n} \left[\sum_{i=0}^{(k-1)(p-1)} b_i^{p,k-1} (x^{i+j} - x^{i+j+p}) \right].$$

In view of these results we are justified in writing

$$Z_{n+k(p-1)+1}^{p,k}(x) = Z_{k(p-1)}^{p,k}(x) - p^{-1} \sum_{j=0}^{n} \left[p^{1-k} \sum_{i=0}^{(k-1)(p-1)} b_i^{p,k-1} (s_{i+j} - s_{i+j+p}) \right].$$

Inspection of the expression in the brackets will reveal that if $y = \{s_i - s_{i+p}\}$ then the expression is the $(Z, p, k-1)$ transform of y and we have

Theorem 6. *If $x = \{s_i\}$ and $y = \{s_i - s_{i+p}\}$, then*

$$Z_{n+k(p-1)+1}^{p,k}(x) = Z_{k(p-1)}^{p,k}(x) - p^{-1} \sum_{j=0}^{n} Z_{j+(k-1)(p-1)}^{p,k-1}(y), \quad p, k, n = 1, 2, 3, \ldots$$

and therefore $x = \{s_i\}$ *is summable-*(Z, p, k) *if and only if*

$$\sum_{j=0}^{\infty} Z^{p,k-1}_{j+(k-1)(p-1)}(y)$$

is convergent. Further, a necessary condition for $x = \{s_i\}$ *to be summable-*(Z, p, k) *is that* $y = \{s_i - s_{i+p}\}$ *be summable-*$(Z, p, k-1)$ *to zero.*

If we define $(Z, p, 0)$ to be ordinary convergence, then Theorem 6 reduces to (1.5) when $k = 1$.

Finally, we note that when p and k are positive integers, the $b_i^{p,k}$ may be considered generalizations of the binomial coefficients since $b_i^{2,k} = \binom{k}{i}$. With p fixed, a generalized Pascal's triangle may be constructed using

$$b_i^{p,1} = 1, \quad i = 0, 1, \ldots, p-1,$$

$$b_i^{p,k} = b_i^{p,k-1} + \ldots + b_{i-p+1}^{p,k-1}, \quad k > 1, \quad i = 0, 1, \ldots, k(p-1),$$

where

$$b_i^{p,k-1} = 0 \text{ if } i < 0 \text{ or } i > (k-1)(p-1).$$

We might also mention that $(Z, 2, k)$, k a positive integer, was considered by Hutton in 1812. For a reference to this, see Hardy ((**3**), 21-22).

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