GENERALIZATION OF SCHWARZ-PICK LEMMA TO INVARIANT VOLUME

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1. Introduction. In this paper we give an extension of (**6**, Theorem 1), using a similar method of proof, to every homogeneous Siegel domain of second kind which can be mapped biholomorphically into a Kähler manifold of a certain class $\mathscr{K}$ (Theorem 1). Then by a well-known result of Vinberg, Gindikin, and Pjateckiĭ-Šapiro (**10**) that every bounded homogeneous domain D, contained in a complex euclidean space C^N, can be mapped biholomorphically onto an affinely homogeneous Siegel domain of second kind, the theorem follows for D (Theorem 2). (**6**, Theorem 1) is a generalization of the Ahlfors version of the Schwarz-Pick lemma in C^1 (**1**) to invariant volume for a star-like homogeneous bounded domain in C^N; see also (**4**). In § 3 we give the inequality for a special non-symmetric Siegel domain of second kind using an explicit form of $T_D(z, \bar{z})$ due to Lu (**7**). This domain is of interest because its Poisson kernel is not harmonic (with respect to the Laplacian corresponding to the Bergman metric of the domain).

The Siegel domain S of second kind was introduced by Pjateckiĭ-Šapiro in (**8**). It is given by

$$S = [\zeta = (z, u): z \in C^n, u \in C^m \text{ and } \operatorname{Im} z - F(u, u) \in V],$$

where V is a convex cone in real euclidean space R^n, containing no entire straight lines, and $F(u, v)$ is a V-hermitian form from $C^m \times C^m$ into C^n such that (i) $F(u, v) = \overline{F(v, u)}$, (ii) $F(\lambda u_1 + \mu u_2, v) = \lambda F(u_1, v) + \mu F(u_2, v)$, $\lambda, \mu \in C^1$, (iii) $F(u, u) \in \bar{V}$, and (iv) $F(u, u) = 0$ if and only if $u = 0$. Note that F is a bilinear form for $n = 1$. S is affinely homogeneous if it is homogeneous under the group of affine transformations of S onto itself, that is, given $z_1, z_2 \in S$ there is an affine automorphism α of S such that $\alpha z_1 = z_2$. Any affinely homogeneous Siegel domain of second kind is a homogeneous Siegel domain of second kind.

2. Generalization of Schwarz-Pick lemma. We extend (**6**, Theorem 1) to the class of homogeneous Siegel domains S of second kind. First we construct an increasing sequence of homogeneous subdomains whose union is S. Let t be a fixed point of V. By convexity, the points t/ν ($\nu = 1, 2, \ldots$) are elements of V. Set

$$(1)\quad S_\nu = [\zeta = (z, u) \in C^{n+m}: y - t/\nu - F(u, u) \in V] \qquad (z = x + iy).$$

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Then S_ν is a domain and it is easy to see that $S_\nu \subset S$, $S_\nu \subset S_{\nu+1}$ $(\nu = 1, 2, \ldots)$. Furthermore, we have the following result.

LEMMA 1. $\bar{S}_\nu \subset S_{\nu+1}$ $(\nu = 1, 2, \ldots)$.

Proof. We note that $\bar{S}_\nu$ is the closure of $S_\nu \subset C^{n+m}$ relative to the topology in C^{n+m}, and hence $\bar{S}_\nu = S_\nu \cup \partial S_\nu$, where ∂S_ν is the boundary of S_ν which lies in (finite) C^{n+m}. Let $\zeta \in \bar{S}_\nu$. Then there exists a sequence $\{\zeta^\alpha\}$ $(\alpha = 1, 2, \ldots)$ in S_ν such that $\zeta^\alpha \to \zeta$, $\zeta^\alpha = (z^\alpha, u^\alpha)$, $z^\alpha = x^\alpha + iy^\alpha$. Since $\zeta^\alpha \in S_\nu$ by definition of S_ν, $y^\alpha - t/\nu - F(u^\alpha, u^\alpha) \in V$ for all α. Letting $\alpha \to \infty$, we have $y - t/\nu - F(u, u) \in \bar{V}$, provided the V-hermitian form $F(u, v)$ on $C^m \times C^m$ into C^n is continuous. However, for fixed $v = v_0$ and $u = u_0$, $F(u, v_0)$ and the complex conjugate $\overline{F(u_0, v)}$ are linear maps from C^m into C^n over the complete field C^1 of complex numbers. Since C^m is finite-dimensional and Hausdorff (see **9**, p. 22, Theorem 3.4), $F(u, v_0)$ and $\overline{F(u_0, v)}$, and hence $F(u_0, v)$, are continuous. Thus, $F(u, v)$ is a separately continuous bilinear mapping of $C^m \times C^m$ into C^n and by (**9**, p. 88, Corollary 1 to Theorem 5.1), it is continuous.

From $y - t/\nu - F(u, u) \in \bar{V}$ follows $y - t/(\nu + 1) - F(u, u) \in \bar{V} + t/\nu(\nu + 1)$. However, it is easy to see that $\bar{V} + v_0 \subset V$ if $v_0 \in V$. In fact, for any $v \in \bar{V}$ the open segment $\lambda v + (1 - \lambda)v_0 \subset V$ $(0 < \lambda < 1)$ (**9**, p. 38). For $\lambda = \frac{1}{2}$, $\frac{1}{2}(v + v_0) \in V$, and hence $v + v_0 \in V$ or $\bar{V} + v_0 \subset V$. Since $v_0 = t/\nu(\nu + 1) \in V$, $y - t/(\nu + 1) - F(u, u) \in V$ which implies that $\zeta \in S_{\nu+1}$ and $\bar{S}_\nu \subset S_{\nu+1}$.

By (1), $\zeta \in S$ implies that there exists ν such that $\zeta \in S_\nu$. Hence,

$$\bigcup_1^\infty S_\nu = \lim_{\nu\to\infty} S_\nu = S.$$

Also the map $\zeta' = \zeta'(\zeta)$ taking S onto S_ν is the translation

$$z' = z + it/\nu, \qquad u' = u, \tag{2}$$

and hence is biholomorphic so that S_ν is homogeneous.

By a well-known theorem of Pjateckiĭ-Šapiro (**8**), the Siegel domain of second kind is biholomorphically equivalent to a bounded domain D. Let $\eta = \eta(\zeta)$ be the mapping of S onto D. Since under any biholomorphic mapping the boundaries of domains correspond, S_ν maps onto a domain D_ν such that $\bar{D}_\nu \subset D_{\nu+1}$, where $\bar{D}_\nu = D_\nu \cup \partial D_\nu^0$, ∂D_ν^0 being that part of the boundary of D_ν which corresponds to ∂S_ν. Since η maps S onto D, it follows that $\bigcup_1^\infty D_\nu = D$. Also, D and D_ν are homogeneous, and

$$\eta' = \eta\zeta'\eta^{-1} \tag{3}$$

is a biholomorphic interior mapping of D onto D_ν which takes η into $\eta' = \eta'(\eta)$. Since from (2) and (3) the Jacobian $J_\eta(\eta') = 1$, the Bergman kernel function of D transforms by

$$K_{D_\nu}(\eta', \bar{\eta}') = K_D(\eta, \bar{\eta})$$

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(**3**) and the invariant $I_D = K_D/T_D$ by

$$I_D(\eta', \bar{\eta}') = I_D(\eta, \bar{\eta}). \tag{4}$$

Here, $T_D = T_D(\eta, \bar{\eta})$ is the determinant of the Bergman metric tensor of D. Since D and D_ν are homogeneous, I_D is constant on D and K_{D_ν} becomes infinite on ∂D_ν (**6**).

Let $w = w(\zeta)$ be a biholomorphic mapping of S into a Kähler manifold Δ with metric given by

$$d\sigma^2 = g_{\alpha\bar{\beta}} dw^\alpha d\bar{w}^\beta, \qquad g_\Delta = g_\Delta(w, \bar{w}) = \det(g_{\alpha\bar{\beta}}), \tag{5}$$

where w is a local coordinate of a point on Δ. Under the inverse mapping of S into Δ, to (5) corresponds the hermitian form,

$$G_{\alpha\bar{\beta}} d\zeta^\alpha d\bar{\zeta}^\beta,$$

on S, where

$$G_S(\zeta, \bar{\zeta}) = \det(G_{\alpha\bar{\beta}}) = g_\Delta(w, \bar{w}) |J_w(\zeta)|^2,$$

$J_w(\zeta)$ being the Jacobian of the mapping $w = w(\zeta)$.

Let $\mathscr{K}$ be the class of Kähler manifolds Δ with metric given by (5) which satisfy the following conditions. The components of the Ricci curvature tensor, $r_{\alpha\bar{\beta}} = -\partial^2 \log g_\Delta / \partial w^\alpha \partial \bar{w}^\beta$, satisfy the boundedness conditions

$$-r_{\alpha\bar{\beta}} u^\alpha \bar{u}^\beta \geqq 0, \qquad \det(-r_{\alpha\bar{\beta}}) \geqq g_\Delta. \tag{6}$$

Let ζ_∞ be a boundary point of S which is a "point at infinity" of C^{n+m} (**2**, Chapter I) and let η_∞ be a boundary point of D which is an image of ζ_∞. Then for every such point η_∞ we assume that

$$\overline{\lim_{\eta \to \eta_\infty}}\, G_D(\eta, \bar{\eta}) / T_D(\eta, \bar{\eta}) = L(\eta_\infty) \leqq 1, \tag{7}$$

where $G_D(\eta, \bar{\eta})$ is the corresponding function on D to $G_S(\zeta, \bar{\zeta})$. The class $\mathscr{K}$ is not empty.

LEMMA 2. *The class $\mathscr{K}$ contains a homogeneous Kähler manifold Δ, admitting the Bergman metric ds_Δ, with metric given by*

$$d\sigma_\Delta{}^2 = k\, ds_\Delta{}^2, \tag{8}$$

where k is a constant such that

$$0 < k \leqq \min[1, (I_\Delta I_D{}^{-1})^{-N}], \tag{9}$$

I_D and I_Δ invariants of D and Δ, respectively, and N the complex dimension of Δ.

Proof. Let Δ be a homogeneous manifold with metric given by (8). Then Δ is a Kähler manifold. Let $w = w(\eta)$ be a biholomorphic mapping of D onto $B \subset \Delta$. Then $ds_B{}^2(w, \bar{w}) = ds_D{}^2(\eta, \bar{\eta})$ for $\eta \in D$. From (8), $g_\Delta(w, \bar{w}) = k^N T_\Delta(w, \bar{w})$. By a well-known property of the kernel function, $K_B(w, \bar{w}) \geqq K_\Delta(w, \bar{w})$, $w \in B$, or $I_B T_B(w, \bar{w}) \geqq I_\Delta T_\Delta(w, \bar{w}) = I_\Delta k^{-N} g_\Delta(w, \bar{w})$, where I_B

is an invariant of B and $I_B = I_D$. Since $G_D(\eta, \bar{\eta}) = g_\Delta(w, \bar{w})|J_w(\eta)|^2$ and $T_D(\eta, \bar{\eta}) = T_B(w, \bar{w})|J_w(\eta)|^2$, we have

$$\frac{G_D(\eta, \bar{\eta})}{T_D(\eta, \bar{\eta})} = \frac{g_\Delta(w, \bar{w})}{T_B(w, \bar{w})} \leqq k^N I_D I_\Delta^{-1}.$$

If k is chosen to be any positive number such that $k \leqq I_\Delta I_D^{-1}$, then (7) holds. In order that the metric (8) satisfies the boundedness conditions (6) for the Ricci curvature tensor, it is necessary and sufficient to take $k \leqq 1$. This completes the proof.

An extension of (**6**, Theorem 1) to domains S is given by the following theorem.

THEOREM 1. *Let S be a homogeneous Siegel domain of second kind. If $w = w(\zeta)$ maps S biholomorphically into a Kähler manifold $\Delta \in \mathscr{K}$, then*

$$g_\Delta(w, \bar{w})|J_w(\zeta)|^2 \leqq T_S(\zeta, \bar{\zeta}). \tag{10}$$

Proof. Let $\eta \in D$. There exists ν so that $\eta \in D_\nu$. Set

$$U = \log G_D(\eta, \bar{\eta}), \qquad V_\nu = \log T_{D_\nu}(\eta, \bar{\eta}), \qquad \Psi = U - V_\nu.$$

Define $\Psi(\eta_\infty) = \log L(\eta_\infty)$. Set $E = [\eta \in \bar{D}_\nu\colon \ \Psi(\eta) > 0]$. Since T_{D_ν} is a relative invariant of D_ν, by (**6**, Lemma 1) it is infinite on ∂D_ν whereas G_D is continuous on D so that $\Psi(\eta) = -\infty$ on ∂D_ν^0. Let $\partial D_\nu^\infty = \partial D_\nu - \partial D_\nu^0$ be the set of boundary points of D_ν which correspond to the set of boundary points $\{\zeta_\infty\}$ of S. By (7), $\Psi(\eta_\infty) \leqq 0$ at each $\eta_\infty \in \partial D_\nu^\infty$. This implies that $E \subset D_\nu$ and $\bar{E} \subset D_\nu \cup \partial D_\nu^\infty$. Also, E is open since Ψ is continuous on D_ν. Let O be any non-empty component of E. Since $\bar{O}$ is bounded, it is compact.

By definition, Ψ is an upper semi-continuous real function on $\bar{D}_\nu$, since at each $\eta_\infty \in \partial D_\nu^\infty$ we have

$$\overline{\lim_{\eta\to\eta_\infty}}\, \Psi(\eta) = \overline{\lim_{\eta\to\eta_\infty}} \log \frac{G_D(\eta, \bar{\eta})}{T_{D_\nu}(\eta, \bar{\eta})} \leqq \overline{\lim_{\eta\to\eta_\infty}} \log \frac{G_D(\eta, \bar{\eta})}{T_D(\eta, \bar{\eta})} = \Psi(\eta_\infty)$$

and

$$\overline{\lim_{\eta\to\eta_\infty}}\, \Psi(\eta) = \lim_{\eta\to\eta_\infty} \Psi(\eta)$$

exists for all $\eta_0 \in D_\nu \cup \partial D_\nu^0$. Thus, Ψ has a maximum at, say, $\eta^* \in \bar{O}$ and $\Psi(\eta^*) > 0$. Since $\Psi(\eta) \leqq 0$ on ∂D_ν, $\eta^* \in D_\nu \cap E$, and hence to O. The same procedure as in the proof of (**6**, Theorem 1) now yields the inequality

$$G_D(\eta, \bar{\eta}) \leqq T_{D_\nu}(\eta, \bar{\eta})$$

on D_ν. By (4),

$$T_{D_\nu}(\eta, \bar{\eta}) = K_{D_\nu}(\eta, \bar{\eta})I_{D_\nu}^{-1} = K_{D_\nu}(\eta, \bar{\eta})I_D^{-1} = K_D(\eta_\nu, \bar{\eta}_\nu)I_D^{-1},$$

where η_ν corresponds to η under the inverse mapping to (3). Hence,

$$G_D(\eta, \bar{\eta}) \leqq K_D(\eta_\nu, \bar{\eta}_\nu)T_D(\eta, \bar{\eta})/K_D(\eta, \bar{\eta}). \tag{11}$$

Let $\nu \to \infty$ in (11). From the continuity of K_D on D and the biholomorphy of the map (3), $\lim_{\nu\to\infty} K_D(\eta_\nu, \bar{\eta}_\nu) = K_D(\eta, \bar{\eta})$ so that from (11),

$$G_D(\eta, \bar{\eta}) \leqq T_D(\eta, \bar{\eta})$$

on D. Inequality (10) follows by mapping from D onto S and using the formula $G_S(\zeta, \bar{\zeta}) = g_\Delta(w, \bar{w})|J_w(\zeta)|^2$.

Since every bounded homogeneous domain $D \subset C^N$ can be mapped biholomorphically onto an affinely homogeneous Siegel domain of second kind (**10**), we have the following result.

THEOREM 2. *If a bounded homogeneous domain D can be mapped into a Kähler manifold Δ in $\mathscr{K}$ under a biholomorphic mapping $w = w(z)$, then for any $z \in D$,*

$$g_\Delta(w, \bar{w})|J_w(z)|^2 \leqq T_D(z, \bar{z}).$$

3. Example. Let

$$S = S_{p,m,n} = [\mathfrak{z} = (Z, U, V): i(Z^* - Z) - \tfrac{1}{2}(UU^* + V^*V) > 0] \quad (Z^* = \bar{Z}')$$

in the space C^N of $N = p(m + n + p)$ complex variables; here, Z is a $p \times p$ matrix of complex numbers, U is a $p \times m$ matrix of complex numbers, and V is an $n \times p$ matrix of complex numbers.

This domain is an affinely homogeneous Siegel domain of the second kind which is non-symmetric for $n \neq 0$, and has the Bergman kernel function

$$K_S(\mathfrak{z}, \bar{\mathfrak{z}}) = c \det[\tfrac{1}{2}i(Z^* - Z) - \tfrac{1}{4}(UU^* + V^*V)]^{-(m+n+2p)}$$

and

$$T_S(\mathfrak{z}, \bar{\mathfrak{z}}) = [\tfrac{1}{4}(m + n + 2p)]^{p(m+n+2p)} \det[\tfrac{1}{2}i(Z^* - Z) - \tfrac{1}{4}(UU^* + V^*V)]^{-(m+n+2p)},$$

where $c = K_S(\mathfrak{z}, \bar{\mathfrak{z}})$ at the point $\mathfrak{z} = (iI, 0, 0)$ in S (**7**). Therefore, the inequality in Theorem 1 becomes

$$g_\Delta(w, \bar{w})|J_w(\mathfrak{z})|^2 \leqq [\tfrac{1}{4}(m + n + 2p)]^{p(m+n+2p)} \det[\tfrac{1}{2}i(Z^* - Z) - \tfrac{1}{4}(UU^* + V^*V)]^{-(m+n+2p)}$$

for a biholomorphic mapping $w = w(\mathfrak{z})$ of S into a Kähler manifold $\Delta \in \mathscr{K}$ of dimension $N = p(m + n + 2p)$.

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