Bull. Aust. Math. Soc. **99** (2019), 284–292
doi:10.1017/S0004972718001077

AMENABLE SEMIGROUPS OF NONLINEAR OPERATORS IN UNIFORMLY CONVEX BANACH SPACES

KHADIME SALAME

(Received 29 June 2018; accepted 2 September 2018; first published online 28 November 2018)

Abstract

In 1965, Browder proved the existence of a common fixed point for commuting families of nonexpansive mappings acting on nonempty bounded closed convex subsets of uniformly convex Banach spaces. The purpose of this paper is to extend this result to left amenable semigroups of nonexpansive mappings.

2010 *Mathematics subject classification*: primary 47H20; secondary 47H09, 47H10.

Keywords and phrases: amenability, asymptotic centre, common fixed point, nonexpansive mapping, uniformly convex space.

1. Preliminaries and notation

Let K be a nonempty bounded closed convex subset of a uniformly convex Banach space E. Given a sequence $(x_j)_j$ of elements of K, we assign the nonnegative numbers

$$r((x_j)_j; x) := \limsup_j \|x_j - x\| \quad \text{for all } x \in K,$$

and

$$r((x_j)_j; K) := \inf_{x \in K} r((x_j)_j; x).$$

The real number $r((x_j)_j; K)$ is called the asymptotic radius of $(x_j)_j$ with respect to K. The asymptotic centre, $\mathcal{A}((x_j)_j; K)$, of $(x_j)_j$ in K is defined by

$$\mathcal{A}((x_j)_j; K) := \{x \in K : r((x_j)_j; x) = r((x_j)_j; K)\}.$$

It possesses the following properties: $\mathcal{A}((x_j)_j; K)$ is nonvoid, weakly compact and convex. These properties follow from the weak compactness of K (as a bounded closed convex subset of a reflexive Banach space), together with the fact that

$$\mathcal{A}((x_j)_j; K) = \bigcap_{p=1}^{\infty} \left\{x \in K : r((x_j)_j; x) \leq r((x_j)_j; K) + \frac{1}{p}\right\}$$

is an intersection of a decreasing sequence of nonempty weakly closed convex subsets of K. Convexity follows from that of the mapping $x \mapsto r((x_j)_j; x)$ from $K \to [0, \infty)$. For more details, the interested reader may see [1, 8, 12] or [13].

Let S be a semigroup. Let $\ell^\infty(S)$ denote the Banach algebra of all bounded real-valued functions on S equipped with the sup norm topology which is induced by $\|f\| = \sup_{s \in S} |f(s)|$ for $f \in \ell^\infty(S)$. To each member $s \in S$, let $\ell_s : \ell^\infty(S) \to \ell^\infty(S)$ be the left translation operator associated to s and defined by $\ell_s f(t) = f(st)$ for all $t \in S$ and $f \in \ell^\infty(S)$. The image $\ell_s f$ is called the left translate of f by s. An element m of the topological dual $\ell^\infty(S)^*$ of $\ell^\infty(S)$ is called a mean if it has the properties

$$m(f) \geq 0 \quad \text{whenever } f \geq 0, \text{ and } m(e) = 1.$$

The symbol e stands for the constant 1 function on S. One can show that this definition is equivalent to saying that

$$\|m\| = 1 = m(e).$$

A mean m is said to be left invariant if

$$m(\ell_s f) = m(f) \quad \text{for all } s \in S \text{ and for all } f \in \ell^\infty(S).$$

The semigroup S is called left amenable if there is a left invariant mean on $\ell^\infty(S)$. Examples of such semigroups include finite groups, commutative semigroups and compact groups. However, a semigroup need not be left amenable; for example, the free group on two generators is not left amenable. For amenability of semigroups, the interested reader can consult [3] and [5]. Given $s \in S$, let $\delta_s : \ell^\infty(S) \to \mathbb{R}$ denote the point measure at s (that is, the evaluation mapping at s), and let $M(S)$ denote the collection of all means on $\ell^\infty(S)$. It is known that $M(S)$ is a nonempty weak* compact convex subset of $\ell^\infty(S)^*$, and it has the property (see [3])

$$M(S) = \overline{co}^{\tau_{w^*}}(\{\delta_s : s \in S\}),$$

that is, $M(S)$ is exactly given by the weak* closed convex hull of all evaluation mappings of elements of S. We recall that members of the convex hull of point measures are usually called finite means.

An action of a semigroup S on K is a mapping $\sigma : S \times K \to K$ subject to the condition

$$\sigma(s, \sigma(t, x)) = \sigma(st, x) \quad \text{for all } s, t \in S \text{ and } x \in K.$$

In order to avoid cumbersome notation, for each $s \in S$ the mapping $\sigma(s, .) : K \to K$ will be abbreviated throughout this paper by the symbol σ_s. A subset X of K is called S-invariant if $\sigma_s(X) \subset X$ for all $s \in S$. An element $x \in K$ is said to be a common fixed point for S if $\sigma_s(x) = x$ for all $s \in S$ or, equivalently, if $\{x\}$ is S-invariant. The action σ is termed nonexpansive if for all $s \in S$ the mapping σ_s has the property

$$\|\sigma_s(x) - \sigma_s(y)\| \leq \|x - y\| \quad \text{for all } x, y \in K.$$

2. Main results

In this section, we shall prove our main results. From now on, we let $\tau_{\|.\|}$ and τ_w denote, respectively, the norm topology and the weak topology of a given Banach space.

THEOREM 2.1. *Let S be a countable semigroup. If S is left amenable, then it possesses the following fixed point property.*

(∗) *Whenever $\sigma : S \times K \to K$ is a nonexpansive action of S on a nonempty bounded closed convex subset K of a uniformly convex Banach space E, then there is in K a common fixed point for S.*

PROOF. By Zorn's lemma, choose K_o as a minimal (for inclusions of sets) bounded closed convex and S-invariant subset of K. A second application of Zorn's lemma shows the existence of a minimal nonempty weakly closed S-invariant set $X_o \subset K_o$. Since S is a left amenable countable semigroup, from [10], there is a left invariant mean m on $\ell^\infty(S)$ together with a sequence $(m_j)_j$ of finite means converging to m with respect to the weak* topology of $\ell^\infty(S)^*$. Put

$$m_j := \sum_{i=1}^{\alpha_j} t_i^j \delta_{s_i^j} \quad \text{with } t_i^j \geq 0 \text{ and } \sum_i t_i^j = 1.$$

Fix $x_o \in X_o$ and define a nonnegative nonzero linear functional

$$\begin{aligned} Ł : C(X_o) &\to \mathbb{R} \\ f &\mapsto m(f_{x_o}), \end{aligned}$$

where f_{x_o} is given by $f_{x_o}(s) = f(\sigma_s(x_o))$ for all $s \in S$. An application of the Riesz representation theorem to Ł yields the existence of a probability measure μ on the Borel sets of X_o representing Ł in the sense that

$$Ł(f) = \int_{X_o} f \, d\mu \quad \text{for all } f \in C(X_o).$$

It is easy to see that $\mu(X_o) = m(e) = 1$. Therefore

$$\{X \subset X_o : X \text{ is weakly closed and } \mu(X) = 1\} \ni X_o$$

is a nonempty family. Consider

$$\omega := \bigcap \{X \subset X_o : X \text{ is weakly closed and } \mu(X) = 1\}.$$

From [10, Lemma 2.12] and its proof, the following facts are known.

$$\mu(\omega) = 1 \quad \text{and} \quad \omega \subset \overline{\sigma_s(\omega)}^{\tau_w} \text{ for all } s \in S \tag{2.1}$$

(the closure being taken with respect to the weak topology of E). Thus

$$\emptyset \neq \omega \subset W := \bigcap_{s \in S} \overline{\sigma_s(\omega)}^{\tau_w}.$$

https://doi.org/10.1017/S0004972718001077 Published online by Cambridge University Press

We claim that W is separable in the norm topology. To prove this, it is enough to show that $\overline{\sigma_s(\omega)}^{\tau_w}$ is norm separable for all $s \in S$.

Step 1. We first show that $\omega \subset \Sigma$, where the latter set is defined by

$$\Sigma := \overline{\bigcup_{j\in\mathbb{N}} \{\sigma_{s_i^j}(x_o) : i = 1, \ldots, \alpha_j\}}^{\tau_w}.$$

From the definition of ω, it is easy to see that ω is characterised by

$$x \in \omega \quad \text{if and only if} \quad \mu(U) > 0 \text{ for all } U \in \mathcal{U}(x), \tag{2.2}$$

where $\mathcal{U}(x)$ denotes the collection of all neighbourhoods of x in the relative weak topology of X_o. Assume that $x \in X_o$ and that x does not belong to Σ. Let U be a closed neighbourhood of x such that $U \cap \Sigma = \emptyset$. Such a neighbourhood exists because X_o is a normal topological space (in the relative weak topology). Using Urysohn's lemma, fix $f \in C(X_o)$ such that $f \geq 0$, $f \equiv 1$ on U and $f(\Sigma) = \{0\}$. Then

$$\begin{aligned}
\mu(U) = \int_U f\, d\mu \leq \int_{X_o} f\, d\mu& \\
&= m(f_{x_o}) = \lim_j m_j(f_{x_o}) = \lim_j \sum_{i=1}^{\alpha_j} t_i^j \delta_{s_i^j}(f_{x_o}) \\
&= \lim_j \sum_{i=1}^{\alpha_j} t_i^j f(\sigma_{s_i^j}(x_o)) = 0,
\end{aligned}$$

which shows that $\mu(U) = 0$ and by (2.2) it follows that x cannot be in ω. Consequently, $\omega \subset \Sigma$, showing that our assertion is true.

Step 2. We show that, for each $s \in S$ fixed, $\overline{\sigma_s(\omega)}^{\tau_w}$ is separable with respect to the norm topology. Fix $s \in S$. Since Σ is the closure in the weak topology of a countable set, it is norm separable and so *a fortiori* $\omega \subset \Sigma$. Let $D \subset \omega$ be a countable norm dense subset. Consider the norm closed convex hull

$$M := \overline{co}^{\tau_{\|\cdot\|}}(\sigma_s(\omega)) \supset \overline{\sigma_s(\omega)}^{\tau_w}$$

of $\sigma_s(\omega)$ and the countable subset

$$\mathrm{Conv}_{\mathbb{Q}}(\sigma_s(D)) := \left\{ \sum_{i=1}^{n} \lambda_i \sigma_s(d_i) : \lambda_i \in \mathbb{Q}, d_i \in D, n \in \mathbb{N} \right\}$$

of all finite linear combinations of elements of $\sigma_s(D)$ with rational coefficients. Given $x \in \mathrm{M}$ and a positive real number ϵ, let

$$\left\| x - \sum_{j=1}^{n} t_j \sigma_s(w_j) \right\| \leq \epsilon$$

https://doi.org/10.1017/S0004972718001077 Published online by Cambridge University Press

for some $w_1, \dots, w_n \in \omega$ with $t_j \geq 0$ and $\sum_j t_j = 1$. Fix $d_1, \dots, d_n$ in D such that $\|w_j - d_j\| \leq \epsilon$ for $j = 1, \dots, n$ and pick $q_1, \dots, q_n$ in $\mathbb{Q}$ such that $\sum_{j=1}^{n} |q_j - t_j| \leq \epsilon$. Then

$$\mathrm{Conv}_{\mathbb{Q}}(\sigma_s(D)) \ni y := \sum_{j=1}^{n} q_j \sigma_s(d_j)$$

and, moreover,

$$\begin{aligned}
\|x - y\| &\leq \Big\| x - \sum_{j=1}^{n} t_j \sigma_s(w_j) \Big\| + \Big\| \sum_{j=1}^{n} t_j (\sigma_s(w_j) - \sigma_s(d_j)) \Big\| + \Big\| \sum_{j=1}^{n} (t_j - q_j) \sigma_s(d_j) \Big\| \\
&\leq \Big\| x - \sum_{j=1}^{n} t_j \sigma_s(w_j) \Big\| + \sum_{j=1}^{n} t_j \|d_j - w_j\| + \Big(\sum_{j=1}^{n} |t_j - q_j| \Big) \cdot \sup_{x \in X_o} \|x\| \\
&\leq \Big(2 + \sup_{x \in X_o} \|x\| \Big) \epsilon.
\end{aligned}$$

As ϵ and x are freely chosen, it follows that $M \subset \overline{co}^{\tau_{\|\cdot\|}}(\sigma_s(D))$ and this shows that $\overline{\sigma_s(\omega)}^{\tau_w}$ is norm separable. Therefore, so are W and its norm closed convex hull.

Case 1. ω is finite. Assume, by contradiction, that $|\omega| \geq 2$. Then the convex hull of ω possesses a normal structure as a norm compact set (see [6]). Pick $u \in co(\omega)$ such that

$$\sup_{x \in \omega} \|x - u\| < \sup_{x,y \in \omega} \|x - y\|.$$

Given $y \in \omega$, let $B[y, r]$ denote the closed ball with centre y and radius r, where $r := \sup_{x \in \omega} \|x - u\|$. Then

$$K_o^* := \bigcap_{y \in \omega} B[y, r]$$

is a nonempty, closed convex and S-invariant subset of K_o. Indeed, it is nonempty because it contains u, and closedness and convexity are clear. To see the S-invariance property, fix $x \in K_o^*$ and $s \in S$. For all $y \in \omega$, from (2.1), there is a net $(y_\alpha)_\alpha$ in ω such that $\sigma_s(y_\alpha) \to y$ weakly. Therefore, by lower semi-continuity of the norm together with the nonexpansiveness of the action,

$$\|y - \sigma_s(x)\| \leq \liminf_\alpha \|\sigma_s(x) - \sigma_s(y_\alpha)\| \leq \liminf_\alpha \|x - y_\alpha\| \leq r.$$

Thus $\sigma_s(x) \in B[y, r]$ for all $y \in \omega$ and, by minimality of K_o, we necessarily have $K_o = K_o^*$. But, since $r < \sup_{x,y \in \omega} \|x - y\|$, there are $x, y \in \omega$ such that $r < \|x - y\|$. It follows that $K_o \ni x \notin B[y, r] \supset K_o$, which is a contradiction. Therefore we must have $|\omega| = 1$. So, by minimality of X_o and (1), we have $X_o = \omega = \{\bar{x}\}$ for some $\bar{x} \in X_o$: that is, there is a common fixed point for S.

Case 2. $|\omega| = \infty$. By separability, let $\{a_j\ ;\ j \in \mathbb{N}\}$ be a countable norm dense subset of $\overline{co}^{\tau_{\|\cdot\|}}(W)$. For $i, j \in \mathbb{N}$ with $i \neq j$, put

$$\Gamma_{i,j} := \{t a_i + (1 - t) a_j : t \in \mathbb{Q} \cap [0, 1]\}.$$

https://doi.org/10.1017/S0004972718001077 Published online by Cambridge University Press

Then $\Gamma := \bigcup_{i \neq j} \Gamma_{i,j}$ is a countable dense family of $\overline{co}^{\tau_{\|\cdot\|}}(W)$ because $\{a_j : j \in \mathbb{N}\} \subset \Gamma$. We can write $\Gamma := \{\gamma_j : j \in \mathbb{N}\}$ because Γ is countable and infinite (since $\omega \subset W$ is infinite). Consider the asymptotic centre $\mathcal{A}(\Gamma; K_o) := \mathcal{A}((\gamma_j)_j; K_o)$. We claim that this asymptotic centre is S-invariant. In fact, let $x \in \mathcal{A}(\Gamma; K_o)$ and $s \in S$. Given $\epsilon > 0$, let $j_\epsilon \in \mathbb{N}$ be sufficiently large so that

$$\limsup_j \|\gamma_j - \sigma_s(x)\| \leq \|\gamma_{j_\epsilon} - \sigma_s(x)\| + \epsilon$$

and

$$\sup_{j \geq j_\epsilon} \|\gamma_j - x\| \leq r(\Gamma; K_o) + \epsilon$$

with $r(\Gamma; K_o) := r((\gamma_j)_j; K_o)$. Since $W \subset \overline{\sigma_s(\omega)}^{\tau_w}$, we can write

$$\gamma_{j_\epsilon} = \tau_w\text{-}\lim_\alpha \sigma_s(x_\alpha)$$

for some net $(x_\alpha)_{\alpha \in J}$ in ω. By nonexpansiveness of the action and the lower semi-continuity of the norm,

$$\|\gamma_{j_\epsilon} - \sigma_s(x)\| \leq \liminf_\alpha \|\sigma_s(x_\alpha) - \sigma_s(x)\| \leq \liminf_\alpha \|x_\alpha - x\|.$$

Fix $\alpha_\epsilon \in J$ such that $\liminf_\alpha \|x_\alpha - x\| \leq \|x_{\alpha_\epsilon} - x\| + \epsilon$ and choose an integer $j \geq j_\epsilon$ such that $\|\gamma_j - x_{\alpha_\epsilon}\| \leq \epsilon$. Such an integer j exists because, by density, there is a $\gamma_k \in \Gamma_{i,j}$ for some i, j such that $\|x_{\alpha_\epsilon} - \gamma_k\| \leq \frac{1}{2}\epsilon$. Let $\gamma_k = t_o a_i + (1 - t_o) a_j$ for some $t_o \in \mathbb{Q} \cap [0, 1]$. Then, for $t \in \mathbb{Q} \cap [0, 1]$,

$$\|ta_i + (1 - t)a_j - \gamma_k\| = |t - t_o|\,\|a_i - a_j\| \to 0 \quad \text{as} \quad t \to t_o.$$

So there are infinitely many rationals t in [0,1] (and, therefore, infinitely many γ_j) such that the distance from γ_k to each such γ_j is within $\frac{1}{2}\epsilon$. Therefore, there is a $j \geq j_\epsilon$ such that $\|\gamma_k - \gamma_j\| \leq \frac{1}{2}\epsilon$. For such an integer, $\|x_{\alpha_\epsilon} - \gamma_j\| \leq \epsilon$. Then using (4),

$$\begin{aligned}\limsup_j \|\gamma_j - \sigma_s(x)\| &\leq \liminf_\alpha \|x_\alpha - x\| + \epsilon \leq \|x_{\alpha_\epsilon} - x\| + 2\epsilon \\ &\leq \|\gamma_j - x\| + 3\epsilon \leq \sup_{j \geq j_\epsilon} \|\gamma_j - x\| + 3\epsilon \leq r(\Gamma; K_o) + 4\epsilon.\end{aligned}$$

Since $\epsilon > 0$ is arbitrary, it follows that $\limsup_j \|\gamma_j - \sigma_s(x)\| = r(\Gamma; K_o)$, which shows that $\sigma_s(x) \in \mathcal{A}(\Gamma; K_o)$. Consequently, the asymptotic centre of Γ with respect to K_o is S-invariant, which proves our claim. Thus, the asymptotic centre of Γ in K_o is a nonempty weakly compact convex and S-invariant subset of K_o. The minimality of K_o forces the equality $\mathcal{A}(\Gamma; K_o) = K_o$. On the other hand, since the underlying space E is uniformly convex, it is known that $\mathcal{A}(\Gamma; K_o)$ is a singleton (see [1]). Put $\mathcal{A}(\Gamma; K_o) = \{\bar{x}\}$ for some $\bar{x} \in K_o$. Then $\bar{x}$ is a common fixed point for S. □

Corollary 2.2 (Browder, [4, Theorem 1]). *Let X be a uniformly convex Banach space and let U be a nonexpansive mapping of the bounded closed convex subset C of X into C. Then U has a fixed point in C.*

https://doi.org/10.1017/S0004972718001077 Published online by Cambridge University Press

Proof. If $U : C \to C$ is nonexpansive, then the set $S = \{U^j : j \in \mathbb{N}\}$ of all iterates of U is an abelian (therefore amenable; see [2]) countable semigroup. By Theorem 2.1, S has a common fixed point in C that is *a fortiori* a fixed point for U. □

Remark 2.3. Usually, for noncommuting families, one has to assume weak continuity of the action to ensure the existence of a fixed point. The interest of Theorem 2.1 is that it completely avoids the use of such a strong condition.

Theorem 2.4. *Let S be a semigroup. If S is left amenable, then S possesses the following fixed point property.*

(∗) *Whenever $\sigma : S \times K \to K$ is a nonexpansive action on a nonempty bounded closed convex subset K of a uniformly convex Banach space E, there exists a common fixed point for S in K.*

Proof. Let $\Gamma := \{\varsigma \subset S : \varsigma \neq \emptyset \text{ and } |\varsigma| < \infty\}$. Order Γ by $\varsigma_1 \leq \varsigma_2 \iff \varsigma_1 \subset \varsigma_2$. Given $\varsigma \in \Gamma$, let $\langle\varsigma\rangle$ denote the semigroup generated by ς and $F(s) := \{x \in K : \sigma_s(x) = x\}$. From [7], for each $\varsigma \in \Gamma$ there is a subsemigroup $S_\varsigma \subset S$ such that:

- S_ς is countable and left amenable; and
- $\langle\varsigma\rangle \subset S_\varsigma$.

By Theorem 2.1, the restriction $S_\varsigma \times K \to K$ of the S-action on K possesses a common fixed point in K. Therefore

$$\bigcap_{s\in\varsigma} F(s) \neq \emptyset \quad \text{for all } \varsigma \in \Gamma.$$

On the other hand, since E is uniformly convex (*a fortiori* strictly convex), each fixed point set F(s) is convex (see [8]). So each set of type (2.1) is weakly closed as an intersection of norm closed convex sets. Thus, the collection

$$\mathcal{F} := \Big\{\bigcap_{s\in\varsigma} F(s) : \varsigma \in (\Gamma, \leq)\Big\},$$

being a decreasing family of nonempty weakly closed subsets of K, must have a nonempty intersection. Hence

$$\emptyset \neq \bigcap_{\varsigma\in\Gamma}\Big(\bigcap_{s\in\varsigma} F(s)\Big) \subset \{x \in K : \sigma_s(x) = x \text{ for all } s \in S\}.$$

□

Corollary 2.5 (Browder, [4, Theorem 2]). *Let X be a uniformly convex Banach space and let (U_λ) be a commuting family of nonexpansive mappings of a given bounded closed convex subset C of X into C. Then the family of mappings (U_λ) has a common fixed point in C.*

Proof. Given a commuting family $\mathcal{U}$ of self-nonexpansive mappings of C, the semigroup $S := \langle\mathcal{U}\rangle$ spanned by $\mathcal{U}$ is commutative and therefore amenable. Hence Theorem 2.4 ensures the existence of a common fixed point for S and so for $\mathcal{U}$. □

https://doi.org/10.1017/S0004972718001077 Published online by Cambridge University Press

REMARK 2.6. If we make use of the fact that left amenable semigroups are left reversible, then Theorem 2.4 follows from our result in [10] for left reversible semitopological semigroups. It is much easier to prove Theorem 2.1 using left reversibility techniques rather than amenability methods. But the interest in our approach here is that it shows once more how amenability techniques can lead to strong existence results in fixed point theory.

REMARK 2.7. Theorem 2.4 follows from [9] and [11] (note that a left amenable discrete semigroup is always left reversible) if K is assumed to be compact with respect to the norm topology.

REMARK 2.8. In general, the uniform convexity condition of the underlying space in Theorems 2.1 and 2.4 cannot be removed. In fact, Alspach [2] has constructed a fixed point free nonexpansive self-mapping of a nonempty weakly compact convex subset of $L^1([0, 1])$. Alspach's counterexample, together with Theorem 2.4, provide a fixed point proof of the fact that the Lebesgue space L^1 fails, in general, to be uniformly convex.

Acknowledgements

The author would like to thank the referee for careful reading of the manuscript, and also Professor Anthony Lau for his kind suggestions for a better presentation of the paper.

References

[1] R. P. Agarwal, D. O'Regan and D. R. Sahu, *Fixed Point Theory for Lipschitzian-type Mappings with Applications* (Springer, New York, 2009).

[2] D. Alspach, 'A fixed point free nonexpansive map', *Proc. Amer. Math. Soc.* **82** (1981), 423–424.

[3] J. F. Berglund, H. D. Junghenn and P. Milnes, *Analysis on Semigroups* (John Wiley & Sons, New York, 1989).

[4] F. E. Browder, 'Nonexpansive nonlinear operators in a Banach space', *Proc. Natl Acad. Sci. USA* **54** (1965), 1041–1044.

[5] M. M. Day, 'Amenable semigroups', *Illinois J. Math.* **1** (1957), 509–544.

[6] R. DeMarr, 'Common fixed points for commuting contraction mappings', *Pacific J. Math.* **13** (1963), 1139–1141.

[7] E. Granirer, 'On amenable semigroups with a finite dimensional set of invariant means II', *Illinois J. Math.* **7** (1963), 49–58.

[8] W. A. Kirk and K. Goebel, *Topics in Metric Fixed Point Theory* (Cambridge University Press, Cambridge, 1990).

[9] T. Mitchell, 'Fixed point of reversible semigroups of non-expansive mappings', *Kodai Math. Sem. Rep.* **22** (1970), 322–323.

[10] K. Salame, 'Nonlinear common fixed point properties of semitopological semigroups in uniformly convex spaces', *J. Fixed Point Theory Appl.* **19** (2017), 1041–1057.

[11] W. Takahashi, 'Fixed point theorem for amenable semigroups of nonexpansive mappings', *Kodai Math. Rep.* **21** (1969), 383–386.

[12] W. Takahashi, *Nonlinear Functional Analysis: Fixed Point Theory and its Applications* (Kindai-Kagakusha, Tokyo, 1988), (in Japanese).

[13] W. Takahashi, *Nonlinear Functional Analysis: Fixed Point Theory and its Applications* (Yokohama Publishers, Yokohama, 2000).

KHADIME SALAME, Department of Mathematical and Statistical Sciences,
University of Alberta, Edmonton, Alberta T6G 2G1, Canada
e-mail: khadime@ualberta.ca, khadime.salame1313@gmail.com