A self-reciprocal function

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1. The object of this note is to discover a new function which is its own reciprocal in the Hankel Transform of order zero.

I will make use of the following theorem of Hardy and Titchmarsh[1]:—

A necessary and sufficient condition that a function $f(x)$ should be its own J_ν transform is that it should be of the form

$$f(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} 2^{\frac{1}{2}s}\,\Gamma\left(\tfrac{1}{4} + \tfrac{1}{2}\nu + \tfrac{1}{2}s\right)\psi(s)\,x^{-s}\,ds, \qquad (1.1)$$

where $0 < c < 1$, *and*

$$\psi(s) = \psi(1 - s). \qquad (1.2)$$

2. I start with the pair of Mellin Transforms[2]

$$I_n(x)\,K_n(x),\quad \frac{\Gamma\left(\frac{1}{2}s\right)\Gamma\left(\frac{1}{2}s + n\right)\Gamma\left(\frac{1}{2} - \frac{1}{2}s\right)}{4\pi^{\frac{1}{2}}\,\Gamma\left(1 + n - \frac{1}{2}s\right)},\qquad \left(0 < \sigma < n + \tfrac{3}{2}\right),$$

where $I_n(x)$ and $K_n(x)$ are the usual Bessel Functions with imaginary argument.

This gives rise to the integral formula

$$I_n(x)\,K_n(x) = \frac{1}{2\pi i}\cdot\frac{1}{4\pi^{\frac{1}{2}}}\int_{k-i\infty}^{k+i\infty}\frac{\Gamma\left(\frac{1}{2}s\right)\Gamma\left(\frac{1}{2}s + n\right)\Gamma\left(\frac{1}{2} - \frac{1}{2}s\right)}{\Gamma\left(1 + n - \frac{1}{2}s\right)}\,x^{-s}\,ds$$

$$= \frac{1}{2\pi i}\int_{k-i\infty}^{k+i\infty}\frac{\Gamma(s)\,\Gamma\left(\frac{1}{2}s + n\right)\Gamma\left(\frac{1}{2} - \frac{1}{2}s\right)}{\Gamma\left(\frac{1}{2} + \frac{1}{2}s\right)\Gamma\left(1 + n - \frac{1}{2}s\right)}\cdot\frac{x^{-s}}{2^{s+1}}\,ds,$$

on using the Duplication Formula for Gamma Functions.

Hence

$$I_n(x)\,K_n(x) = \frac{1}{2\pi i}\int_{2k-i\infty}^{2k+i\infty}\frac{\Gamma\left(\frac{1}{2}u\right)\Gamma\left(\frac{1}{4}u + n\right)\Gamma\left(\frac{1}{2} - \frac{1}{4}u\right)}{\Gamma\left(\frac{1}{2} + \frac{1}{4}u\right)\Gamma\left(1 + n - \frac{1}{4}u\right)}\,\frac{x^{-\frac{1}{2}u}}{2^{\frac{1}{2}u+2}}\,du,$$

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[1] E. C. Titchmarsh: *The Theory of the Fourier Integral* (Oxford, 1937), §(9.1.9.).
[2] *Ibid.*, §(7.10.8.).

so that

$$x^{\frac{1}{2}} I_n(\tfrac{1}{4}x^2) K_n(\tfrac{1}{4}x^2)$$

$$= \frac{1}{2\pi i}\int_{2k-i\infty}^{2k+i\infty} \frac{\Gamma(\frac{1}{2}u)\,\Gamma(n+\frac{1}{4}u)\,\Gamma(\frac{1}{2}-\frac{1}{4}u)}{\Gamma(\frac{1}{2}+\frac{1}{4}u)\,\Gamma(1+n-\frac{1}{4}u)}\, 2^{\frac{1}{2}u-2}\, x^{\frac{1}{2}-u}\, du$$

$$= \frac{1}{2\pi i}\int_{2k-\frac{1}{2}-i\infty}^{2k-\frac{1}{2}+i\infty} 2^{\frac{1}{2}s-\frac{7}{4}} \frac{\Gamma(\frac{1}{2}s+\frac{1}{4})\,\Gamma(n+\frac{1}{8}+\frac{1}{4}s)\,\Gamma(\frac{3}{8}-\frac{1}{4}s)}{\Gamma(\frac{5}{8}+\frac{1}{4}s)\,\Gamma(\frac{7}{8}+n-\frac{1}{4}s)}\, x^{-s}\, ds.$$

Putting $n = 0$, we get

$$x^{\frac{1}{2}} I_0(\tfrac{1}{4}x^2) K_0(\tfrac{1}{4}x^2) = \frac{1}{2\pi i}\int_{2k-\frac{1}{2}-i\infty}^{2k-\frac{1}{2}+i\infty} 2^{\frac{1}{2}s}\, \Gamma(\tfrac{1}{4}+\tfrac{1}{2}s)\, \psi_1(s)\, x^{-s}\, ds, \quad (2.1)$$

where

$$\psi_1(s) = \frac{\Gamma(\frac{3}{8}-\frac{1}{4}s)\,\Gamma(\frac{1}{8}+\frac{1}{4}s)}{\Gamma(\frac{7}{8}-\frac{1}{4}s)\,\Gamma(\frac{5}{8}+\frac{1}{4}s)}\, 2^{-\frac{7}{4}}.$$

As $\psi_1(s)$ satisfies (1.2) it follows that the integral in (2.1) is of the same form as (1.1) with

$$\psi(s) = \psi_1(s)$$

and $\frac{1}{4} < k < \frac{3}{4}$.

It follows that the function on the left-hand side of (2.1) is its own J_0 transform. That is, it satisfies the integral equation

$$x^{\frac{1}{2}} I_0(\tfrac{1}{4}x^2) K_0(\tfrac{1}{4}x^2) = \int_0^\infty (xy)^{\frac{1}{2}} J_0(xy)\, y^{\frac{1}{2}} I_0(\tfrac{1}{4}y^2) K_0(\tfrac{1}{4}y^2)\, dy.$$

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