Canad. Math. Bull. Vol. **47** (2), 2004 pp. 257–263

A Geometric Characterization of Nonnegative Bands

Alka Marwaha

Abstract. A band is a semigroup of idempotent operators. A nonnegative band $\mathcal{S}$ in $\mathcal{B}(\mathcal{L}^2(\mathcal{X}))$ having at least one element of finite rank and with rank $(S) > 1$ for all S in $\mathcal{S}$ is known to have a special kind of common invariant subspace which is termed a standard subspace (defined below).

Such bands are called decomposable. Decomposability has helped to understand the structure of nonnegative bands with constant finite rank. In this paper, a geometric characterization of maximal, rank-one, indecomposable nonnegative bands is obtained which facilitates the understanding of their geometric structure.

1 Introduction

$\mathcal{X}$ will denote a separable, locally compact Hausdroff space and μ a Borel measure on $\mathcal{X}$. Let us write $\mathcal{L}^2(\mathcal{X})$ for the Hilbert space of (equivalence classes of) complex-valued measurable functions on $\mathcal{X}$ which are square-integrable relative to μ. Let us assume for simplicity that $\mu(\mathcal{X}) < \infty$. This is not a great restriction, and almost all our considerations will be valid for the case of a σ-finite measure with obvious modifications. We shall denote by $\mathcal{B}(\mathcal{L}^2(\mathcal{X}))$ the space of all bounded linear operators on $\mathcal{L}^2(\mathcal{X})$.

It has been the endeavour of several mathematicians in the past few years to find sufficient conditions under which a semigroup can be reduced, meaning thereby that the members of the semigroup have a common nontrivial invariant subspace. The next step is to see if these conditions are strong enough to give (simultaneous) triangularizability of the semigroup $\mathcal{S}$. This means the existence of a chain $\mathcal{C}$ of closed subspaces of $\mathcal{L}^2(\mathcal{X})$ such that

(a) $\mathcal{C}$ is maximal (as a chain of closed subspaces of $\mathcal{L}^2(\mathcal{X})$), and
(b) every member of $\mathcal{C}$ is invariant for $\mathcal{S}$.

Semigroups of $n \times n$ matrices with nonnegative entries were studied in [5] and conditions were obtained to give reducibility for them. Also it has been proved [5] that submultiplicativity of spectral radius on the members of a semigroup of compact operators represented by matrices with nonnegative entries results in the reducibility of the semigroup, although it may not yield decomposability (see definition below). In [1], it has been shown that under certain conditions, semigroups of nonnegative quasinilpotent operators are not only decomposable but simultaneously triangularizable with a maximal subspace chain consisting of standard subspaces. Some attempts

Received by the editors October 8, 2001.

This paper is a part of the Ph.D thesis submitted under the supervision of Professor Heydar Radjavi at Dalhousie University, Halifax, Nova Scotia

AMS subject classification: Primary: 47D03; Secondary: 47A15.

https://doi.org/10.4153/CMB-2004-025-0 Published online by Cambridge University Press

have been made to study the structure of bands, *e.g.*, in [2] and [3]. The extra condition of nonnegativity does throw a light on their structure. The author proved in [4] that a maximal nonnegative band of constant rank r under the special condition of fullness is the direct sum of r maximal rank-one nonnegative indecomposable bands. In this paper, a geometric characterization of a maximal, nonnegative, indecomposable rank-one band is obtained. This result completely determines the geometric structure of maximal, nonnegative, indecomposable, finite-rank bands.

But first and foremost, a brief review of the terminology and definitions.

1.1 Definition and Preliminary Results

A function $f \in \mathcal{L}^2(\mathcal{X})$ is said to be *nonnegative* (resp. *positive*), written $f \geq 0$ (resp. $f > 0$) if

$$\mu\{x \in \mathcal{X} : f(x) < 0\} = 0 \; \big(\text{resp. } \mu\{x \in \mathcal{X} : f(x) \leq 0\} = 0\big)$$

By a *standard subspace* of $\mathcal{L}^2(\mathcal{X})$, we mean a norm-closed linear manifold in $\mathcal{L}^2(\mathcal{X})$ of the form

$$\mathcal{L}^2(\mathcal{U}) = \{f \in \mathcal{L}^2(\mathcal{X}) : f = 0 \text{ a.e. on } \mathcal{U}^c\}$$

for some Borel subset $\mathcal{U}$ of $\mathcal{X}$. This space is nontrivial if $\mu(\mathcal{U}).\mu(\mathcal{U}^c) > 0$.

An operator $A \in \mathcal{B}(\mathcal{L}^2(\mathcal{X}))$ is said to be *decomposable* if there exists a nontrivial standard subspace of $\mathcal{L}^2(\mathcal{X})$ invariant under A.

Similarly, a semigroup $\mathcal{S}$ in $\mathcal{B}(\mathcal{L}^2(\mathcal{X}))$ is decomposable if every member of $\mathcal{S}$ has a common nontrivial standard invariant subspace; otherwise $\mathcal{S}$ is indecomposable.

A *band* in $\mathcal{B}(\mathcal{L}^2(\mathcal{X}))$ is a semigroup of idempotents, *i.e.*, operators E on $\mathcal{L}^2(\mathcal{X})$ such that $E^2 = E$.

An operator A in $\mathcal{B}(\mathcal{L}^2(\mathcal{X}))$ is nonnegative if $Af \geq 0$ whenever $f \geq 0$ in $\mathcal{L}^2(\mathcal{X})$.

Similarly, A is called positive if $Af > 0$ whenever $0 \neq f \geq 0$ in $\mathcal{L}^2(\mathcal{X})$.

For any function f, we define the support of f as $\operatorname{supp} f = \{x \in \mathcal{X} : f(x) \neq 0\}$.

If f is a member of $\mathcal{L}^2(\mathcal{X})$, then $\operatorname{supp} f$ is defined up to a null set (*i.e.*, a set of measure zero).

When no confusion is likely to arise, we simply write $\operatorname{supp} f$ for any $f \in \mathcal{L}^2(\mathcal{X})$ to mean $\operatorname{supp} f_0$, where f_0 is a function representing f.

As a first step to understand the geometric characterization of maximal, nonnegative, indecomposable bands of constant finite rank, their structure has to be completely analysed. In this regard, some salient results and definitions are stated in this section, the proofs of which appear in detail in [4].

Definition 1.1 A nonnegative semigroup $\mathcal{S}$ in $\mathcal{B}(\mathcal{L}^2(\mathcal{X}))$ will be called a *full semigroup* if neither $\ker \mathcal{S}$ nor $\ker \mathcal{S}^\star$ has a nonzero, nonnegative vector. A single nonnegative operator is called *full* if the semigroup generated by it is full.

Lemma 1.2 *A nonnegative full band of rank-one operators is indecomposable.*

Theorem 1.3 *Let $\mathcal{S}$ be a band of nonnegative operators in $\mathcal{B}(\mathcal{L}^2(\mathcal{X}))$ with constant finite rank r.*

(i) If $\mathcal{S}$ is full, then there exists a decomposition

$$\mathcal{L}^2(\mathcal{X}) = \mathcal{L}^2(\mathcal{X}_1) \oplus \mathcal{L}^2(\mathcal{X}_2) \oplus \cdots \oplus \mathcal{L}^2(\mathcal{X}_r),$$

with respect to which every member S of $\mathcal{S}$ is of the form

$$\begin{bmatrix} S_1 & & & \\ & S_2 & & \\ & & \ddots & \\ & & & S_r \end{bmatrix},$$

where each $\mathcal{S}_i = \{S_i \in \mathcal{L}^2(\mathcal{X}_i) : S \in \mathcal{S}\}$ is an indecomposable band of rank-one operators.

(ii) In general, there exists a decomposition

$$\mathcal{L}^2(\mathcal{X}) = \mathcal{L}^2(\mathcal{X}_1') \oplus \mathcal{L}^2(\mathcal{X}_2') \oplus \mathcal{L}^2(\mathcal{X}_3')$$

with respect to which every member S of $\mathcal{S}$ is of the form

$$\begin{bmatrix} O & XE & XEY \\ O & E & EY \\ O & O & O \end{bmatrix},$$

where X, Y are nonnegative operators on suitable spaces. Furthermore, the diagonal blocks in $\mathcal{S}_0 = \{E : S \in \mathcal{S}\}$ constitute a band of the form described in Case (i).

Remark 1.4 In Theorem 1.6, the converse of part (i) of the preceding theorem is proved to obtain a characterization of maximal, nonnegative, constant-rank bands which are full.

Lemma 1.5 *Suppose $\mathcal{S}$ is a direct sum of r nonnegative indecomposable semigroups $\mathcal{S}_1, \mathcal{S}_2, \ldots, \mathcal{S}_r$ such that each member of $\mathcal{S}$ has a block diagonal representation*

$$\begin{bmatrix} S_1 & & & \\ & S_2 & & \\ & & \ddots & \\ & & & S_r \end{bmatrix},$$

where $S_i \in \mathcal{S}_i,\ i = 1, \ldots, r$, with respect to some decomposition of $\mathcal{L}^2(\mathcal{X})$, say

$$\mathcal{L}^2(\mathcal{X}) = \mathcal{L}^2(\mathcal{X}_1) \oplus \ldots \oplus \mathcal{L}^2(\mathcal{X}_r).$$

Then every $\mathcal{M} \in \mathcal{L}at'\mathcal{S}$ is of the form $\mathcal{M} = \oplus_{i=1}^{r} \in_i \mathcal{L}^2(\mathcal{X}_i)$ where each $\in_i$ is either 0 or 1.

https://doi.org/10.4153/CMB-2004-025-0 Published online by Cambridge University Press

Theorem 1.6 *A direct sum of r maximal, indecomposable, nonnegative rank-one bands is a maximal band of constant rank r.*

Theorem 1.3 and Remark 1.4 can be combined to give the following characterization of maximal nonnegative bands of constant finite rank.

Theorem 1.7 *Let $\mathcal{S}$ be a nonnegative band in $\mathcal{B}(\mathcal{L}^2(\mathcal{X}))$ of constant finite rank r.*

(i) If $\mathcal{S}$ is full, then $\mathcal{S}$ is maximal if and only if

$$\mathcal{S} = \left\{ \begin{bmatrix} S_1 & & & \\ & S_2 & & \\ & & \ddots & \\ & & & S_r \end{bmatrix} : S_i \in \mathcal{S}_i, i = 1, 2, \dots, r \right\},$$

where $\mathcal{S}_i$ is a maximal rank-one indecomposable band for each i.
(ii) In general, if $\mathcal{S}$ is maximal, then

$$\mathcal{S} = \left\{ \begin{pmatrix} O & XE & XEY \\ O & E & EY \\ O & O & O \end{pmatrix} : E \in \mathcal{S}_0, X \in \mathcal{X}, Y \in \mathcal{Y} \right\},$$

where $\mathcal{S}_0$ is a direct sum as in part (i) and $\mathcal{X}, \mathcal{Y}$ are the entire sets of nonnegative operators on appropriate spaces.

1.2 A Geometric Characterization

In Theorem 1.7, it is proved that every maximal, indecomposable, nonnegative band with constant finite rank *r*, say, which is full is the direct sum of maximal, indecomposable, nonnegative rank-one bands. Thus the structure of such bands is completely determined if the structure of maximal, constant rank-one bands is known.

We know that a nonzero, nonnegative rank-one operator in $\mathcal{B}(\mathcal{L}^2(\mathcal{X}))$ is of the form $u \otimes v$, where u, v are nonzero, nonnegative functions in $\mathcal{L}^2(\mathcal{X})$ and $(u \otimes v)f = \langle f, v \rangle u$ for all $f \in \mathcal{L}^2(\mathcal{X})$. Further, for $u \otimes v$ to be an idempotent, u, v must satisfy the equation $\langle u, v \rangle = 1$.

Thus, if $\mathcal{S}$ is a nonnegative band of rank-one operators in $\mathcal{B}(\mathcal{L}^2(\mathcal{X}))$, then we can find sets $\mathcal{U}, \mathcal{V}$ in the nonnegative cone of $\mathcal{L}^2(\mathcal{X})$, *viz.* $\mathcal{K}$ such that $\mathcal{S} \supseteq \mathcal{U} \otimes \mathcal{V}$, where

$$\mathcal{U} \otimes \mathcal{V} = \{u \otimes v : u \in \mathcal{U}, v \in \mathcal{V}\}$$

and

$$\langle u, v \rangle = 1 \text{ for all } u \in \mathcal{U} \text{ and for all } v \in \mathcal{V}.$$

(By the nonnegative cone of $\mathcal{L}^2(\mathcal{X})$, we mean the set $\mathcal{K} = \{f \in \mathcal{L}^2(\mathcal{X}) : f \geq 0\}$).

Further, if $\mathcal{S}$ is maximal, then we must have $\mathcal{S} = \mathcal{U} \otimes \mathcal{V}$ for some pair $\mathcal{U}, \mathcal{V}$ of the kind mentioned above. We wish to find the general form of $\mathcal{U}$ and $\mathcal{V}$ for a maximal, nonnegative, indecomposable band $\mathcal{S}$ of rank-one operators in $\mathcal{B}(\mathcal{L}^2(\mathcal{X}))$.

https://doi.org/10.4153/CMB-2004-025-0 Published online by Cambridge University Press

We observe that if $u_1, u_2 \in \mathcal{U}$, then $\langle tu_1 + (1-t)u_2, v\rangle = 1$ for $0 \le t \le 1$ and for all $v \in \mathcal{V}$. Thus for a maximal $\mathcal{U} \otimes \mathcal{V}$, $\mathcal{U}$ must contain all the convex combinations of its members too. Furthermore, it is clear that $\mathcal{U}$ is closed (in norm). Also, we cannot have every member of $\mathcal{U}$ equal to zero a.e. on any Borel subset of $\mathcal{X}$ with positive measure, for if, there were such a set, say $\mathcal{W} \subseteq \mathcal{X}$ such that $u = 0$ a.e. in $\mathcal{W}$ for every $u \in \mathcal{U}$, then for any $u \in \mathcal{U}$ and $v \in \mathcal{V}$

$$\begin{aligned}\langle (u \otimes v)f, \chi_{\mathcal{W}}\rangle &= \int_{\mathcal{W}} (u \otimes v) f(x)\mu(dx)\\ &= \int_{\mathcal{W}} \langle f, v\rangle u(x)\mu(dx)\\ &= \langle f, v\rangle \int_{\mathcal{W}} u(x)\mu(dx)\\ &= 0 \quad \text{for all } f \in \mathcal{L}^2(\mathcal{X})\end{aligned}$$

which by Lemma 2.5 [4] gives that $\mathcal{U} \otimes \mathcal{V}$ is decomposable. This together with the fact that $\mathcal{U}$ is closed and convex allows us to assume with no loss of generality that $\mathcal{U}$ has a positive element. Let us pick one such element in $\mathcal{U}$, say, u_0, *i.e.*, $u_0 > 0$ a.e. on $\mathcal{X}$.

Now, any $u \in \mathcal{U}$ satisfies $\langle u, v\rangle = 1$ for all $v \in \mathcal{V}$. In particular, $\langle u_0, v\rangle = 1$ for all $v \in \mathcal{V}$. Thus, for any $u \in \mathcal{U}$,

$$\begin{aligned}\langle u, v\rangle = \langle u_0, v\rangle \text{ for all } v \in \mathcal{V} &\Rightarrow \langle u - u_0, v\rangle = 0 \text{ for all } v \in \mathcal{V}\\ &\Rightarrow u - u_0 \in \mathcal{V}^\perp\\ &\Rightarrow u \in u_0 + \mathcal{V}^\perp \text{ for all } u \in \mathcal{U}\\ &\Rightarrow \mathcal{U} \subseteq u_0 + \mathcal{V}^\perp.\end{aligned}$$

Also, if $v' \in \mathcal{V}^\perp$, then for any $v \in \mathcal{V}$,

$$\langle u_0 + v', v\rangle = \langle u_0, v\rangle = 1.$$

Thus, by the maximality of $\mathcal{S}$, we obtain

$$\mathcal{U} = \{u_0 + \mathcal{V}^\perp\} \cap \mathcal{K}. \tag{1}$$

By the same reasoning, we can find a positive vector v_0 in $\mathcal{V}$ and obtain

$$\mathcal{V} = \{v_0 + \mathcal{U}^\perp\} \cap \mathcal{K}. \tag{2}$$

Next, we show that if $\mathcal{U}$ and $\mathcal{V}$ are given as in (1) and (2) respectively, for some positive u_0, v_0 and subspaces $\mathcal{W}, \mathcal{Z}$ *i.e.*,

$$\mathcal{U} = \{u_0 + \mathcal{W}\} \cap \mathcal{K} \tag{3}$$

$$\mathcal{V} = \{v_0 + \mathcal{Z}\} \cap \mathcal{K} \tag{4}$$

https://doi.org/10.4153/CMB-2004-025-0 Published online by Cambridge University Press

where $\langle u_0, v_0\rangle = 1$, $\mathcal{W} = \{v_0 + \mathcal{Z}\}^{\perp}$ and $\mathcal{Z} = \{u_0 + \mathcal{W}\}^{\perp}$, then $\mathcal{S} = \mathcal{U} \otimes \mathcal{V}$ is a maximal band of nonnegative rank-one operators in $\mathcal{B}(\mathcal{L}^2(\mathcal{X}))$. It is easy to see that $\mathcal{S}$ forms a nonnegative band of rank-one operators. Suppose $\mathcal{S}$ is contained in a band $\mathcal{S}_0$ of rank-one operators, where

$$\mathcal{S}_0 \subset \mathcal{U}' \otimes \mathcal{V}' = \{u' \otimes v' : u' \in \mathcal{U}', v' \in \mathcal{V}'\},$$

for some sets $\mathcal{U}', \mathcal{V}' \subseteq \mathcal{K}$ and $\langle u', v'\rangle = 1$ for all $u' \in \mathcal{U}'$ and $v' \in \mathcal{V}'$.

Let $\mathcal{S} = p \otimes q \in \mathcal{S}_0$. Since $\mathcal{S}_0$ is a semigroup, $(u \otimes v) \cdot (p \otimes q) \in \mathcal{S}_0$ for all $u \otimes v \in \mathcal{S}$. Therefore, for any $f \in \mathcal{L}^2(\mathcal{X})$,

$$\begin{aligned}(u \otimes v)(p \otimes q)f &= (u \otimes v)\langle f, q\rangle p \\ &= \langle f, q\rangle \big((u \otimes v)p\big) \\ &= \langle f, q\rangle \langle p, v\rangle u \\ &= \langle p, v\rangle \big(\langle f, q\rangle u\big) \\ &= \langle p, v\rangle (u \otimes q) f \\ &= \big(\langle p, v\rangle u \otimes q\big) f,\end{aligned}$$

i.e., $(u \otimes v)(p \otimes q) = \langle p, v\rangle u \otimes q$. Thus $(u \otimes v)(p \otimes q)$ is an idempotent if and only if $\langle \langle p, v, \rangle u, q\rangle = 1$, *i.e.*, if and only if $\langle p, v\rangle \langle u, q\rangle = 1$. With no loss of generality, we can assume that $\langle p, v\rangle = 1$ and $\langle u, q\rangle = 1$ (for if, $\langle p, v\rangle = \alpha(\neq 1)$), then $\langle u, q\rangle = \frac{1}{\alpha}$, so that we can write $s = \frac{1}{\alpha} p \otimes \alpha q = p' \otimes q'$ where $p' = \frac{1}{\alpha} p$, $q' = \alpha q$ and $\langle p', v\rangle = 1$, $\langle u, q'\rangle = 1$). Now

$$\begin{aligned}\langle u, q\rangle = 1 \text{ for all } u \in \mathcal{U} &\Rightarrow \langle u_0, q\rangle = 1 \text{ and } \langle u_0 + w, q\rangle = 1 \;\; \forall\, w \in \mathcal{W} \\ &\Rightarrow \langle u_0, q\rangle = 1 \text{ and } \langle w, q\rangle = 0 \;\; \forall\, w \in \mathcal{W} \\ &\Rightarrow \langle u_0 + w, q - v_0\rangle = 0 \;\; \forall\, w \in \mathcal{W} \\ &\Rightarrow q - v_0 \in \{u_0 + \mathcal{W}\}^{\perp} = \mathcal{Z} \\ &\Rightarrow q \in v_0 + \mathcal{Z} = \mathcal{V}.\end{aligned}$$

Similarly, we can show that $p \in \mathcal{U}$. Thus $p \otimes q \in \mathcal{U} \otimes \mathcal{V} = \mathcal{S}$ which implies that $\mathcal{S}_0 \subseteq \mathcal{S}$. Hence $\mathcal{S}$ is maximal.

Next, we would like to see which subspaces $\mathcal{W}$ and $\mathcal{Z}$ give rise to maximal indecomposable bands as in (3) and (4). Suppose there is some $w \in \mathcal{W}$ such that $w \geq 0$ or $w \leq 0$. Consider the case when $w \geq 0$ and the support of w is a Borel subset of positive measure. Then

$$\begin{aligned}\langle w, v\rangle = 0 \;\; \forall\, v \in \mathcal{V} &\Rightarrow \int_{\mathcal{X}} w(x)v(x)\mu(dx) = 0 \;\; \forall\, v \in \mathcal{V} \\ &\Rightarrow w(x)v(x) = 0 \text{ a.e. on } \mathcal{X} \;\; \forall\, v \in \mathcal{V} \;\; (\text{as } w, v \geq 0)\end{aligned}$$

https://doi.org/10.4153/CMB-2004-025-0 Published online by Cambridge University Press

Let $\mathcal{N} = \operatorname{supp} w$; then $v = 0$ a.e. on $\mathcal{N}$ $\forall\, v \in \mathcal{V}$. By the same argument given once before, this will yield decomposability of $\mathcal{S}$, which is not true. Similarly, if $w \leq 0$ with positive-measured support, we shall find $\mathcal{S}$ to be decomposable. This shows that every vector of $\mathcal{W}$ must necessarily be a "mixed" vector, *i.e.*, a vector having positive and negative parts with supports of positive measure. In other words, the space $\mathcal{W}$ intersects $\mathcal{K}$ trivially. Following the same argument, we conclude that $\mathcal{Z} \cap \mathcal{K} = \{0\}$, (We shall call such a space a *mixed space*).

We summarize the discussion above in the following theorem.

Theorem 2.1 *Let $\mathcal{S}$ be a maximal, nonnegative, indecomposable band of rank-one operators in $\mathcal{B}(\mathcal{L}^2(\mathcal{X}))$. Denote the positive cone of $\mathcal{L}^2(\mathcal{X})$ by $\mathcal{K}$. Then there exist positive vectors u_0, v_0 in $\mathcal{K}$ with $\langle u_0, v_0\rangle = 1$ and there exist mixed subspaces $\mathcal{W}, \mathcal{Z}$ of $\mathcal{L}^2(\mathcal{X})$ with $\mathcal{W} = \{v_0 + \mathcal{Z}\}^{\perp}$, $\mathcal{Z} = \{u_0 + \mathcal{W}\}^{\perp}$ such that $\mathcal{S} = \mathcal{U} \otimes \mathcal{V}$, where*

$$\mathcal{U} = \{u_0 + \mathcal{W}\} \cap \mathcal{K},$$
$$\mathcal{V} = \{u_0 + \mathcal{Z}\} \cap \mathcal{K}.$$

References

[1] M. D. Choi, E. A. Nordgren, H. Radjavi, P. Rosenthal and Y. Zhong, *Triangularizing semigroups of quasinilpotent operators with nonnegative entries*. Indiana Univ. Math. J. **42**(1993), 15–25.

[2] P. Fillmore, G. MacDonald, M. Radjabalipour, and H. Radjavi, *Towards a classification of maximal unicellular bands*. Semigroup Forum **49**(1994), 195–215.

[3] ———, *On principal-ideal bands*. Semigroup Forum, to appear.

[4] A. Marwaha, *Decomposability and structure of nonnegative bands in infinite dimensions*. J. Operator Theory **47**(2002), 37–61.

[5] H. Radjavi, *On reducibility of semigroups of compact operators*. Indiana Univ. Math. J. **39**(1990), 499–515.

Alka Marwaha
Department of Mathematics
Jesus and Mary College (University of Delhi)
Chanakyapuri
New Delhi - 110 021
India
email: alkasamta@yahoo.co.uk

https://doi.org/10.4153/CMB-2004-025-0 Published online by Cambridge University Press