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THE ALPERIN WEIGHT CONJECTURE AND UNO'S CONJECTURE FOR THE BABY MONSTER $\mathbb{B}$, p ODD

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Abstract

Suppose that p is 3, 5 or 7. In this paper, faithful permutation representations of maximal p-local subgroups are constructed, and the radical p-chains of the Baby Monster $\mathbb{B}$ are classified. Hence, the Alperin weight conjecture and the Uno reductive conjecture can be verified for $\mathbb{B}$, the latter being a refinement of Dade's reductive conjecture and the Isaacs–Navarro conjecture.

1. *Introduction*

Recently, Isaacs and Navarro [**12**] proposed a new conjecture that is a refinement of the Alperin–McKay conjecture, and Uno [**15**] raised an alternating sum version of the conjecture that is a refinement of the Dade conjecture [**9**].

Dade's reductive conjecture [**9**] has been verified for all of the sporadic simple groups except Fi_{24}', $\mathbb{B}$ and $\mathbb{M}$. The use of computer algebra systems, (namely MAGMA [**6**] and GAP [**10**]) to study permutation (or in some cases matrix) representations of the groups has been a central step of the program. Since the smallest faithful permutation representation of $\mathbb{B}$ has degree 13 571 955 000, it is difficult to verify the conjecture directly. However, from the classification [**16**] of maximal p-local subgroups of $\mathbb{B}$, we know that when p is equal to 3, 5 or 7, the normalizer of each radical p-subgroup of $\mathbb{B}$ is a subgroup of one of precisely thirteen maximal p-local subgroups. Thus we can classify radical chains in these maximal subgroups without performing any calculation in $\mathbb{B}$.

In this paper, we construct a faithful permutation representation for each maximal p-local subgroup. We then classify radical chains, and hence verify the Alperin weight conjecture and Uno's refinement of Dade's reductive conjecture for $\mathbb{B}$.

The paper is organized as follows. In Section 2, we fix the notation, state the conjectures in detail, and state three lemmas. In Section 3, we explain how to construct faithful permutation representations of the thirteen maximal p-local subgroups. In Section 4, we recall the modified local strategy [**4, 5**]; we also explain how we applied it to determine the radical subgroups of each maximal subgroup, and how to determine the fusion of the radical subgroups in $\mathbb{B}$. In Section 5, we classify radical p-subgroups of $\mathbb{B}$, and verify the Alperin weight conjecture. In Section 6, we do some cancellations in the alternating sum of Uno's conjecture, and then determine radical chains (up to conjugacy) and their local structures. In the last section, we verify Uno's projective conjecture for $\mathbb{B}$.

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2. Conjectures and lemmas

Let p be a prime and R a p-subgroup of a finite group G. Then R is *radical* if $O_p(N(R)) = R$, where $O_p(N(R))$ is the largest normal p-subgroup of the normalizer $N(R) = N_G(R)$. Denote by $\mathrm{Irr}(G)$ the set of all irreducible ordinary characters of G, and let $\mathrm{Blk}(G)$ be the set of p-blocks, $B \in \mathrm{Blk}(G)$ and $\varphi \in \mathrm{Irr}(N(R)/R)$. The pair (R, φ) is called a *B-weight* if $\mathrm{d}(\varphi) = 0$ and $B(\varphi)^G = B$ (in the sense of Brauer), where $\mathrm{d}(\varphi) = \log_p(|G|_p) - \log_p(\varphi(1)_p)$ is the p-defect of φ and $B(\varphi)$ is the block of $N(R)$ containing φ. A weight is always identified with its G-conjugates. Let $\mathcal{W}(B)$ be the number of B-weights, and let $\ell(B)$ be the number of irreducible Brauer characters of B. Alperin [1] conjectured that $\mathcal{W}(B) = \ell(B)$ for each $B \in \mathrm{Blk}(G)$.

Given a p-subgroup chain

$$C : P_0 < P_1 < \ldots < P_n \tag{2.1}$$

of G, define $|C| = n$, $C_k : P_0 < P_1 < \ldots < P_k$, and

$$N(C) = N_G(C) = N(P_0) \cap N(P_1) \cap \ldots \cap N(P_n). \tag{2.2}$$

The chain C is said to be *radical* if it satisfies the following two conditions:

(a) $P_0 = O_p(G)$, and

(b) $P_k = O_p(N(C_k))$ for $1 \leqslant k \leqslant n$.

Denote by $\mathcal{R} = \mathcal{R}(G)$ the set of all radical p-chains of G. Let $B \in \mathrm{Blk}(G)$, and let $D(B)$ be a defect group of B. The p-local rank (see [2]) of B is the number

$$\mathrm{plr}(B) = \max\{|C| : C \in \mathcal{R},\ C : P_0 < P_1 < \ldots < P_n \leqslant D(B)\}.$$

Let Z be a cyclic group, $\hat{G} = Z.G$ a central extension of Z by G, and $C \in \mathcal{R}(G)$. Denote by $N_{\hat{G}}(C)$ the preimage $\eta^{-1}(N(C))$ of $N(C)$ in $\hat{G}$, where η is the natural group homomorphism from $\hat{G}$ onto G with kernel Z. Let ρ be a faithful linear character of Z, and let $\hat{B}$ be a block of $\hat{G}$ covering the block $B(\rho)$ of Z containing ρ. Denote by $\mathrm{Irr}(N_{\hat{G}}(C), \hat{B}, d, \rho)$ the irreducible characters ψ of $N_{\hat{G}}(C)$ such that ψ lies over ρ, $d(\psi) = d$ and $B(\psi)^{\hat{G}} = \hat{B}$, and set $\mathrm{k}(N_{\hat{G}}(C), \hat{B}, d, \rho) = |\mathrm{Irr}(N_{\hat{G}}(C), \hat{B}, d, \rho)|$.

Dade's projective conjecture (see [9]). *If $O_p(G) = 1$ and $\hat{B}$ is a p-block of $\hat{G}$ covering $B(\rho)$ with defect group $D(\hat{B}) \neq O_p(Z)$, then*

$$\sum_{C \in \mathcal{R}/G} (-1)^{|C|} \mathrm{k}\big(N_{\hat{G}}(C), \hat{B}, d, \rho\big) = 0, \tag{2.3}$$

where $\mathcal{R}/G$ is a set of representatives for the G-orbits of $\mathcal{R}$.

If $Z = 1$, then $\hat{G} = G$, $\hat{B} = B$ and $\rho = 1$. Set $\mathrm{k}(N(C), B, d) = \mathrm{k}(N_{\hat{G}}(C), \hat{B}, d, \rho)$. The *projective conjecture* is then called the *ordinary conjecture*.

Dade's ordinary conjecture (see [8]). *If $O_p(G) = 1$ and B is a p-block of G with defect group $D(B) \neq 1$, then for any integer $d \geqslant 0$,*

$$\sum_{C \in \mathcal{R}/G} (-1)^{|C|} \mathrm{k}(N(C), B, d) = 0.$$

Let $\hat{H}$ be a subgroup of a finite group $\hat{G}$, let $\varphi \in \operatorname{Irr}(\hat{H})$ and let $r(\varphi) = r_p(\varphi)$ be the integer $0 < r(\varphi) \leqslant (p-1)$ such that the p'-part $(|\hat{H}|/\varphi(1))_{p'}$ of $|\hat{H}|/\varphi(1)$ satisfies

$$\left(\frac{|\hat{H}|}{\varphi(1)}\right)_{p'} \equiv r(\varphi) \pmod{p}.$$

Given an integer $1 \leqslant r \leqslant (p-1)/2$, let $\operatorname{Irr}(\hat{H}, [r])$ be the subset of $\operatorname{Irr}(\hat{H})$ consisting of characters φ such that $r(\varphi) \equiv \pm r (\operatorname{mod} p)$, and let

$$\operatorname{Irr}\left(\hat{H}, \hat{B}, d, \rho, [r]\right) = \operatorname{Irr}\left(\hat{H}, \hat{B}, d, \rho\right) \cap \operatorname{Irr}\left(\hat{H}, [r]\right);$$
$$\mathrm{k}\left(\hat{H}, \hat{B}, d, \rho, [r]\right) = \left|\operatorname{Irr}\left(\hat{H}, \hat{B}, d, \rho, [r]\right)\right|.$$

Suppose that $Z = 1$, and let $\hat{B} = B \in \operatorname{Blk}(G)$ with a defect group $D = D(B)$ and the Brauer correspondent $b \in \operatorname{Blk}(N_G(D))$. Then $\mathrm{k}(N(D), B, d(B), [r])$ is the number of characters $\varphi \in \operatorname{Irr}(b)$ such that φ has height 0 and $r(\varphi) \equiv \pm r \pmod{p}$, where $\mathrm{d}(B)$ is the defect of B.

Isaacs–Navarro conjecture (see [12, Conjecture B]). *In the notation above,*

$$\mathrm{k}(G, B, \mathrm{d}(B), [r]) = \mathrm{k}(N(D), B, \mathrm{d}(B), [r]).$$

The following refinement of Dade's conjecture is due to Uno.

Uno's projective conjecture (see [15, Conjecture 3.2]). *If $O_p(G) = 1$ and if $D(\hat{B}) \neq O_p(Z)$, then for any integer $d \geqslant 0$, faithful $\rho \in \operatorname{Irr}(Z)$ and $1 \leqslant r \leqslant (p-1)/2$,*

$$\sum_{C \in \mathcal{R}/G} (-1)^{|C|} \mathrm{k}\left(N_{\hat{G}}(C), \hat{B}, d, \rho, [r]\right) = 0. \tag{2.4}$$

Similarly, if $Z = 1$, then the projective conjecture is the ordinary conjecture. Note also that if $p = 2$ or $p = 3$, then Uno's conjecture is equivalent to Dade's conjecture.

Let G be the Baby Monster, $\mathbb{B}$. Then its Schur multiplier is cyclic of order 2, and its outer automorphism group is trivial, so Dade's projective conjecture is equivalent to his reductive conjecture (and Uno's reductive conjecture is also equivalent to his reductive conjecture). Thus it suffices to verify:

1. Dade's ordinary conjecture for $\mathbb{B}$;
2. Dade's projective conjecture for the 2 covering group $2{\cdot}\mathbb{B}$ when $p = 3$;
3. Uno's ordinary conjecture for $\mathbb{B}$; and
4. Uno's projective conjecture for $2{\cdot}\mathbb{B}$ when $p \geqslant 5$.

The proofs of the following two lemmas are straightforward.

Lemma 2.1. *Let $\sigma : O_p(G) < P_1 < \ldots < P_{m-1} < Q = P_m < P_{m+1} < \ldots < P_\ell$ be a fixed radical p-chain of a finite group G, where $1 \leqslant m < \ell$. Suppose that*

$$\sigma' : O_p(G) < P_1 < \ldots < P_{m-1} < P_{m+1} < \ldots < P_\ell$$

is also a radical p-chain such that $N_G(\sigma) = N_G(\sigma')$. Let $\mathcal{R}^-(\sigma, Q)$ be the subfamily of $\mathcal{R}(G)$ consisting of chains C whose $(\ell-1)$th subchain, $C_{\ell-1}$, is conjugate to σ' in G. Let $\mathcal{R}^0(\sigma, Q)$ be the subfamily of $\mathcal{R}(G)$ consisting of chains C whose ℓth subchain C_ℓ is conjugate to σ in G. Then the map g sending any $O_p(G) < P_1 < \ldots < P_{m-1} < P_{m+1} < \ldots < P_\ell < \ldots$ in $\mathcal{R}^-(\sigma, Q)$ to $O_p(G) < P_1 < \ldots < P_{m-1} < Q < P_{m+1} < \ldots < P_\ell < \ldots$ induces a bijection, denoted again by g, from $\mathcal{R}^-(\sigma, Q)$ onto $\mathcal{R}^0(\sigma, Q)$. Moreover, for any C in $\mathcal{R}^-(\sigma, Q)$, we have $|C| = |g(C)| - 1$ and $N_G(C) = N_G(g(C))$.

LEMMA 2.2. *Suppose that Q is a p-subgroup of G. Then Q is radical in G if and only if $N_G(Q) \leqslant M$ and Q is radical in M for some maximal p-local subgroup M of G.*

LEMMA 2.3. *If Q is a p-subgroup of a finite group G, then there is a radical p-subgroup R such that*

$$Q \leqslant R \quad \text{and} \quad N_G(Q) \leqslant N_G(R).$$

Proof. This follows by [2, Lemma 2.1]. □

3. *Construction of permutation representations of maximal p-local subgroups*

We will follow the notation of [7]. In particular, if p is odd, then $p^{1+2\gamma} = p_+^{1+2\gamma}$ is an extra-special group of order $p^{1+2\gamma}$ with exponent p, and if $\delta = +$ or $\delta = -$, then $2_\delta^{1+2\gamma}$ is an extra-special group of order $2^{1+2\gamma}$ with type δ. If X and Y are groups, we use $X.Y$, $X{}^{\cdot}Y$ and $X{:}Y$ to denote an extension, a nonsplit extension and a split extension of X by Y, respectively. Given a positive integer n, we use p^n to denote the elementary abelian group of order p^n, n to denote the cyclic group of order n, D_{2n} to denote the dihedral group of order $2n$, and SD_{2n} to denote the semidihedral group of order $2n$.

The subgroups of $2{}^{\cdot}\mathbb{B}$ that we need to construct are given in Table 1.

The general strategy in most cases is to make appropriate subgroups of the Monster first, and to centralize a suitable involution to get the desired subgroup of $2{}^{\cdot}\mathbb{B}$. We can then quotient by the central involution to obtain the corresponding subgroup of $\mathbb{B}$.

Table 1: Maximal p-local subgroups of $\mathbb{B}$ and $2{}^{\cdot}\mathbb{B}$.

Shape in $\mathbb{B}$	Shape in $2{}^{\cdot}\mathbb{B}$	Overgroup in $\mathbb{M}$
$(7{:}3 \times 2{}^{\cdot}L_3(4).2).2$	$(7{:}3 \times 2^2{}^{\cdot}L_3(4).2).2$	$(7{:}3 \times \mathrm{He}){:}2$
$(2^2 \times 7^2{:}(3 \times 2A_4)).2$	$(D_8 \times 7^2{:}(3 \times 2A_4)).2$	$(L_2(7) \times 7^2{:}(3 \times 2A_4)).2$
$5{:}4 \times \mathrm{HS}{:}2$	$(D_{10} \times 2{}^{\cdot}\mathrm{HS}.2).2$	$(D_{10} \times \mathrm{HN}).2$
$5^{1+4}{:}2_-^{1+4}.A_5.4$	$5^{1+4}{:}2.2_-^{1+4}.A_5.4$	$5^{1+6}{:}2{}^{\cdot}J_2.4$
$5^2{:}4S_4 \times S_5$	$(5^2{:}4A_4 \times 2{}^{\cdot}S_5).2$	$(5^2{:}4 \circ Q_8 \times U_3(5)){:}S_3$
$5^3{}^{\cdot}L_3(5)$	$2 \times 5^3{}^{\cdot}L_3(5)$	n/a
$S_3 \times \mathrm{Fi}_{22}{:}2$	$(S_3 \times 2{}^{\cdot}\mathrm{Fi}_{22}).2$	$3{}^{\cdot}\mathrm{Fi}_{24}$
$3^{1+8}{:}2_-^{1+6}{}^{\cdot}U_4(2).2$	$3^{1+8}{:}2.2_-^{1+6}{}^{\cdot}U_4(2).2$	$3^{1+12}{}^{\cdot}2{}^{\cdot}\mathrm{Suz}{:}2$
$(3^2{:}D_8 \times U_4(3).2^2).2$	$(3^2{:}2 \times 2{}^{\cdot}U_4(3).2^2).D_8$	$(3^2{:}2 \times O_8^+(3)){}^{\cdot}S_4$
$3^2.3^3.3^6.(S_4 \times 2{}^{\cdot}S_4)$	$3^2.3^3.3^6{:}(2{}^{\cdot}S_4 \times 2{}^{\cdot}S_4)$	$3^2.3^5.3^{10}(M_{11} \times 2{}^{\cdot}S_4)$
$3^3.3.3^3.3^3(L_3(3) \times 2)$	$2 \times 3^3.3.3^3.3^3(L_3(3) \times 2)$	$2 \times \mathrm{Fi}_{23}$
$3^3.3^6(L_3(3) \times D_8)$	$3^3.3^6(L_3(3) \times SD_{16})$	$(2 \times O_8^+(3)){}^{\cdot}S_4$
$3^6{:}(2 \times L_4(3)).2^2$	$2 \times 3^6{:}(L_4(3).SD_{16})$	$(2 \times O_8^+(3)){}^{\cdot}S_4$

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Many of the groups required are subdirect products, and are thus easy to construct from representations of the constituent groups. For example, to make $(7{:}3 \times \mathrm{He}){:}2$ we first make the Frobenius group 7:6 generated by the permutations $a = (1, 2, 3, 4, 5, 6, 7)$ and $b = (1, 3, 2, 6, 4, 5)$, and He:2 generated by two permutations c and d on 2058 points. Since a and c are in the respective subgroups of index 2, while b and d are not, we see that ac and bd together generate the desired group $(7{:}3 \times \mathrm{He}){:}2$ acting on $7 + 2058 = 2065$ points. Inside this group, we then find the involution centralizer by standard methods, and obtain $(7{:}3 \times 2^{2}{\cdot}L_3(4).2).2$ as the required subgroup of $2{\cdot}\mathbb{B}$. Taking the quotient by the central involution, we obtain the corresponding subgroup $(7{:}3 \times 2{\cdot}L_3(4).2).2$ of $\mathbb{B}$.

Similarly, we construct the affine group $7^2{:}(3\times 2S_4) < 7^2{:}\mathrm{GL}_2(7)$ as a permutation group on forty-nine points, and $L_2(7){:}2$ as a permutation group on eight points. Thus we obtain the direct product acting on fifty-seven points, and its subgroup $(7^2{:}(3\times 2A_4)\times L_2(7)).2$ of index 2 in the same way as above. By centralizing an involution in the $L_2(7)$ subgroup we obtain the required subgroup $(7^2{:}(3 \times 2A_4) \times D_8).2$ of $2{\cdot}\mathbb{B}$, and its quotient $(7^2{:}(3 \times 2A_4) \times 2^2).2$ in $\mathbb{B}$. This completes the construction of the maximal 7-local subgroups of $\mathbb{B}$ and $2{\cdot}\mathbb{B}$.

Next consider the 5-local subgroups. We can make $(D_{10} \times \mathrm{HN}).2$ as a subdirect product on $5 + 1\,140\,000$ points, and we centralize an involution to get $(D_{10} \times 2{\cdot}\mathrm{HS}{:}2).2$ and its quotient $5{:}4 \times \mathrm{HS}{:}2$. However, these permutations are rather large, so we actually made the group directly as a subdirect product of 5:4 and $4{\cdot}\mathrm{HS}{:}2$. Note that there is an outer element of $(D_{10} \times 2{\cdot}\mathrm{HS}{:}2){\cdot}2$, which acts as the outer automorphism of $2{\cdot}\mathrm{HS}{:}2$ (multiplying elements outside $2{\cdot}\mathrm{HS}$ by the central involution) and squares to the product of an involution in the D_{10} and the central involution of $2{\cdot}\mathrm{HS}$.

Similarly, we make $5^{1+6}{:}2{\cdot}J_2.4$ by following the instructions in [14] for making groups of extraspecial type. Specifically, we make a matrix representation in eight dimensions over GF(5) in which a complementary $2J_2.4$ and the normal 5^{1+6} are represented by matrices of shape

$$\begin{pmatrix} \mu & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 0 & 0 \\ -Sv^T & I_6 & 0 \\ \lambda & v & 1 \end{pmatrix},$$

respectively, where A is a typical element of $2{\cdot}J_2.4$ in its six-dimensional representation over $\mathbb{F}_5$, and S is the matrix of the symplectic form preserved by $2J_2$ in this representation (note that S has inadvertently been replaced by S^{-1} from the bottom of [14, p. 316]), and μ is the scalar such that $ASA^T = \mu S$. We can then convert to a permutation representation on the $5^7 = 78\,125$ images of the vector (0, 0, 0, 0, 0, 0, 0, 1). Centralizing a suitable involution leads to the corresponding subgroups $5^{1+4}{:}2.2^{1+4}_{-}.A_5.4$ of $2{\cdot}\mathbb{B}$ and $5^{1+4}{:}2^{1+4}_{-}.A_5.4$ of $\mathbb{B}$.

The next subgroup can be made as a subdirect product of the affine group $5^2{:}4S_4 < 5^2{:}\mathrm{GL}_2(5)$ (on twenty-five points) and the almost simple group $U_3(5){:}S_3$ (on 126 points) leading to $(5^2{:}4 \circ Q_8 \times U_3(5)){:}S_3$ on 151 points. We then centralize an involution to get $(5^2{:}4A_4 \times 2S_5).2$ in $2{\cdot}\mathbb{B}$ and $5^2{:}4S_4 \times S_5$ in $\mathbb{B}$.

Finally for $p = 5$, we find that $5^3{\cdot}L_3(5)$ is isomorphic to a subgroup of the Lyons group, for which generators are available [17] as 111×111 matrices over GF(5). This can be converted to a permutation representation on 7750 points, by permuting a suitable orbit of vectors in a suitable subquotient of this representation. As there is no double cover of $L_3(5)$, the corresponding subgroup of $2{\cdot}\mathbb{B}$ is a direct product $2 \times 5^3{\cdot}L_3(5)$. This concludes the construction of the maximal 5-local subgroups.

We next take $3{\cdot}\mathrm{Fi}_{24}$ acting on $3 \times 306\,936$ points from [17], and centralize an involution to get $(S_3 \times 2\mathrm{Fi}_{22}).2$. Its quotient $S_3 \times \mathrm{Fi}_{22}{:}2$ is a direct product; it is thus easily constructed as a permutation group on $3 + 3510$ points.

For the next group, we could take $3^{1+12}{:}2{\cdot}\mathrm{Suz}{:}2$ as constructed in [14], and convert it into permutations on $3^{13} = 1594323$ points. Although this is not a subgroup of the Monster, the involution centralizer $3^{1+8}{:}2.2_-^{1+6}U_4(2).2$ is isomorphic to the desired subgroup of $2{\cdot}\mathbb{B}$. However, these permutations are rather large, so instead we take the involution centralizer $2.2_-^{1+6}U_4(2).2$ in $2{\cdot}\mathrm{Suz}.2$, acting on an 8-space and a 4-space over GF(3). We lift the action on the 8-space to an action of $3^{1+8}{:}2_-^{1+6}U_4(2).2$ on a 10-space, and thence to an action on $3^9 = 19\,683$ points. To get the double cover, we adjoin the permutation action on the eighty non-zero vectors of the 4-space.

We take $(3^2{:}2 \times O_8^+(3)).S_4$ as a subdirect product of the affine group $3^2{:}2S_4$ on nine points and $O_8^+(3){:}S_4$ on 3360 points, and then find the involution centralizer $(3^2{:}2 \times 2{\cdot}U_4(3).2^2).D_8$ and its quotient $(3^2{:}D_8 \times U_4(3).2^2).2$. Another involution centralizer $(2 \times O_8^+(3)){\cdot}S_4$ contains the last two 3-locals in Table 1, corresponding to certain maximal parabolic subgroups in $O_8^+(3){:}S_4$. These are therefore straightforward to construct. The third last of the maximal 3-local subgroups listed in Table 1 is contained in Fi_{23}, which becomes $2 \times \mathrm{Fi}_{23}$ in $2{\cdot}\mathbb{B}$. It can therefore be constructed by using information on the character tables of maximal subgroups of Fi_{23}, given in [10].

Finally, we need to construct $3^2.3^3.3^6(S_4 \times 2S_4)$ and its double cover. The latter is a subgroup of the group $3^2.3^5.3^{10}{:}(M_{11} \times 2{\cdot}S_4)$ in the Monster, and so has shape $3^2.3^3.3^6{:}(2{\cdot}S_4 \times 2{\cdot}S_4)$. We first make the corresponding subgroup of $\mathbb{B}$ by explicitly finding words in the standard generators which generate this subgroup. To do this, we start from $3^{1+8}{:}2_-^{1+6}U_4(2).2$, and find a subgroup $3^2.[3^7].2.3^3(S_4 \times 2)$, being the normalizer of a $3B$-pure 3^2 generated by the centre and one other element of 3^{1+8}. We then go to work in the centralizer of an S_4 to find an element swapping the two generators for the 3^2. First we centralize a suitable involution in the S_4, to get a group $(2^2 \times F_4(2)){:}2$, and then we centralize an element of order 4 squaring to this involution, to get a group $4 \times {}^2F_4(2)$. In here, we find by random search, an involution that extends our 3^2 to A_6, in which it is easy to find the involution that we are seeking. This involution now extends the group $3^2.[3^9].(S_4 \times 2 \times S_3)$ to the maximal subgroup $3^2.3^3.3^6.(S_4 \times 2{\cdot}S_4)$ of $\mathbb{B}$, which we are seeking.

To construct a suitable permutation representation, we find a subgroup of order $2^5.3^5$ centralizing an involution, and permute the $2^2.3^8 = 26244$ images of a fixed vector of this subgroup. To construct the double cover, observe that we need only to cover the quotient S_4, which we can do by taking a subdirect product with the group $2{\cdot}S_4 \cong \mathrm{GL}_2(3)$ acting on eight points.

4. *A local subgroup strategy and fusions*

From [16], we know that each radical p-subgroup R of $\mathbb{B}$ is radical in one of the conjugates M of maximal p-local subgroups constructed in Section 3 and, further, $N_{\mathbb{B}}(R) = N_M(R)$.

In [4] and [5], a (modified) local strategy was developed to classify the radical p-subgroups R. We review this method here. Suppose that M is a subgroup of a finite group G satisfying $N_G(R) = N_M(R)$.

Step (1) We first consider the case where M is p-local. Let $Q = O_p(M)$, so that $Q \leqslant R$. Choose a subgroup X of M. We explicitly compute the coset action of M on the cosets of X

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in M; we obtain a group W representing this action, a group homomorphism f from M to W, and the kernel K of f. For a suitable X, we have $K = Q$, and the degree of the action of W on the cosets is much smaller than that of M. We can now directly classify the radical p-subgroup classes of W (or apply Step (2) below to W), and the preimages in M of the radical subgroup classes of W are the radical subgroup classes of M.

Step (2) Now consider the case where M is not p-local. We may be able to find its radical p-subgroup classes directly. Alternatively, we find a (maximal) subgroup K of M such that $N_K(R) = N_M(R)$ for each radical subgroup R of M. If K is p-local, then we apply Step (1) to K. If K is not p-local, we can replace M by K, and repeat Step (2).

Steps (1) and (2) constitute the modified local strategy. After applying the strategy, we list the radical subgroups of each M, and then do the fusions as follows.

Suppose that R is a radical p-subgroup of M. Using the local structure, we can determine whether or not $N_M(R)$ is a subgroup of another maximal subgroup M'. Suppose that $N_M(R)$ is a subgroup of M'. By Lemma 2.3, there is a radical subgroup R' of M' such that $R \leqslant R'$ and $N_M(R) \leqslant N_{M'}(R')$. Using the local structure, we can determine whether or not R is radical in M', and if so, we can identify R with a radical subgroup R' of M'. Some details are given in the proof of Proposition 5.1.

The computations reported in this paper were carried out using MAGMA V2.10-6 on a Sun UltraSPARC Enterprise 4000 server.

5. *Radical subgroups and weights*

Let $\mathcal{R}_0(G, p)$ be a set of representatives for conjugacy classes of radical p-subgroups of G. For $H, K \leqslant G$, we write $H \leqslant_G K$ if $x^{-1}Hx \leqslant K$, and we write $H \in_G \mathcal{R}_0(G, p)$ if $x^{-1}Hx \in \mathcal{R}_0(G, p)$ for some $x \in G$.

Let G be the Baby Monster $\mathbb{B}$. Then

$$|G| = 2^{41} \cdot 3^{13} \cdot 5^6 \cdot 7^2 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23 \cdot 31 \cdot 47,$$

and we may suppose that $p \in \{2, 3, 5, 7\}$, since both conjectures hold for a block with a cyclic defect group, by [**8**, Theorem 7.1] and [**3**, Theorem 5.2]. Suppose that p is odd, so that $p = 3, 5, 7$.

Denote by $\mathrm{Irr}^0(H)$ the set of ordinary irreducible characters of p-defect 0 of a finite group H, and by $\mathrm{d}(H)$ the number $\log_p(|H|_p)$. Given $R \in \mathcal{R}_0(G, p)$, let $C(R) = C_G(R)$, $N = N_G(R)$ and $\hat{N} = N_{2.\mathbb{B}}(R)$, where R is viewed as a subgroup of the covering group $2.\mathbb{B}$. If $B_0 = B_0(G)$ is the principal p-block of G, then (see [**4**, (4.1)])

$$\mathcal{W}(B_0) = \sum_R |\,\mathrm{Irr}^0(N/C(R)R)|, \tag{5.1}$$

where R runs over the set $\mathcal{R}_0(G, p)$ such that $\mathrm{d}(C(R)R/R) = 0$. The character table of $N/C(R)R$ can be calculated by MAGMA, and so we find $|\,\mathrm{Irr}^0(N/C(R)R)|$. Write $\mathcal{W}_N = |\,\mathrm{Irr}^0(N/C(R)R)|$ and $\mathcal{W}_{\hat{N}} = |\,\mathrm{Irr}^0(N_{2.\mathbb{B}}(R)/C(R)R)| - |\,\mathrm{Irr}^0(N/C(R)R)|$; the latter will be used to calculate the number of weights for $2.\mathbb{B}$.

PROPOSITION 5.1. *The non-trivial radical p-subgroups R of $G = \mathbb{B}$ (up to conjugacy) and their local structures are given in Tables 2 and 3, according to whether $p \geqslant 5$ or $p = 3$, where $S \in \mathrm{Syl}_3(G)$ is a Sylow 3-subgroup of G.*

Proof. *Case* (1) Suppose that $p = 7$. By [16, Section 7], $\mathbb{B}$ has two non-trivial radical 7-subgroups, 7 and 7^2, with local structures given by Table 2.

Case (2) Suppose that $p = 5$. By [16, Theorem 6.4], $\mathbb{B}$ has four maximal 5-local subgroups:

$$M_1 = N(5A) = 5{:}4 \times \mathrm{HS}{:}2; \qquad M_2 = N(5B^3) = 5^{3\cdot}L_3(5);$$
$$M_3 = N(5B) = 5_+^{1+4}{:}2_-^{1+4\cdot}A_5.4; \qquad M_4 = N(5A^2) = 5^2{:}4S_4 \times S_5.$$

Let $M = M_3$ or $M = M_4$. Since $|M/O_5(M)|_5 = 5$, it follows that a Sylow subgroup of M is its only radical 5-subgroup other than $O_5(M)$. Thus

$$\mathcal{R}_0(M_i, 5) = \begin{cases} \{5^2, 5^3\} & \text{if } i = 4, \\ \{5_+^{1+4}, 5_+^{1+4}.5\} & \text{if } i = 3; \end{cases} \tag{5.2}$$

in addition, $N_{M_4}(5^3) = 5^2{:}4S_4 \times 5{:}4$, and $N_{M_3}(5_+^{1+4}.5) = 5_+^{1+4}.5.4^2$.

We may take

$$\mathcal{R}_0(M_1, 5) = \{5, 5^2, 5 \times 5_+^{1+2}\},$$

where $5 \times 5_+^{1+2} \in \mathrm{Syl}_5(M_1)$. In addition,

$$N_{M_1}(R) = \begin{cases} 5{:}4 \times 5{:}4 \times S_5 & \text{if } R = 5^2, \\ 5 \times 5_+^{1+2}4.D_8 & \text{if } R = 5 \times 5_+^{1+2}. \end{cases}$$

Note that $5^2 \in_{\mathbb{B}} \mathcal{R}_0(M_4, 5)$, so $N_{\mathbb{B}}(R) \neq N_{M_1}(R)$ for any $R \in \mathcal{R}_0(M_1, 5) \setminus \{5\}$.

We may take

$$\mathcal{R}_0(M_2, 5) = \{5^3, 5_+^{1+4}, 5^2.5^{1+2}, 5_+^{1+4}.5\};$$

in addition,

$$C_{M_2}(5_+^{1+4}) = 5 = Z(5_+^{1+4});$$
$$N_{M_2}(5^2.5^{1+2}) = 5^2.5^{1+2}.\mathrm{GL}_2(5); \qquad N_{M_2}(5_+^{1+4}) = 5_+^{1+4}.\mathrm{GL}_2(5).$$

Table 2: Non-trivial radical p-subgroups of $\mathbb{B}$ with $p \geqslant 5$.

R	$C(R)$	N	$\mathcal{W}_N$	$\mathcal{W}_{\hat{N}}$
7	$7 \times 2^{\cdot}L_3(4){:}2$	$(7{:}3 \times 2^{\cdot}L_3(4){:}2){:}2$		
7^2	$2^2 \times 7^2$	$(2^2 \times 7^2{:}(3 \times 2A_4)){:}2$	24	24
5	$5 \times \mathrm{HS}{:}2$	$5{:}4 \times \mathrm{HS}{:}2$		
5^2	$5^2 \times S_5$	$5^2{:}4S_4 \times S_5$		
5^3	5^3	$5^3.L_3(5)$	1	1
5_+^{1+4}	5	$5_+^{1+4}.2_-^{1+4}.A_5.4$	30	12
$5^2.5_+^{1+2}$	5^2	$5^2.5_+^{1+2}.\mathrm{GL}_2(5)$	4	4
$5_+^{1+4}.5$	5	$5_+^{1+4}.5.4^2$	16	16

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Since $\mathbb{B}$ has exactly two classes, $5A$ and $5B$, of elements of order 5 and $|N(5A)|_5 = 5^4$, it follows that a generator of $C_{M_2}(5_+^{1+4})$ is in a $5B$-class, and so we may suppose that $N_{M_2}(5_+^{1+4}) \leqslant M_3 = N(5B)$. In particular, $N_{M_2}(5_+^{1+4}) \neq N_{\mathbb{B}}(5_+^{1+4})$.

Since $|N_{M_2}(5^2.5^{1+2})|_5 = 5^6$, it follows that M_1 and M_4 have no subgroup conjugate to $N_{M_2}(5^2.5^{1+2})$. Applying Lemma 2.3, we see that $N_{M_2}(5^2.5^{1+2}) \not\leqslant_{\mathbb{B}} M_3$, so by Lemma 2.2, $5^2.5^{1+2}$ is radical in $\mathbb{B}$ with $N_{\mathbb{B}}(5^2.5^{1+2}) = N_{M_2}(5^2.5^{1+2})$. This classifies the radical 5-subgroups of $\mathbb{B}$.

Case (3) Suppose that $p = 3$. By [**16**, Theorem 5.7], $\mathbb{B}$ has seven maximal 3-local subgroups:

$M_1 = N(3B) = 3_+^{1+8}{:}2_-^{1+6}{\cdot}U_4(2){:}2$; $\quad M_2 = N(3B^2) = 3^2.3^3.3^6.(S_4 \times 2S_4)$;

$M_3 = N(3B^3) = 3^3.3^6.(L_3(3) \times D_8)$; $\quad M_4 = N(3B^3) = 3^3.3.3^3.3^3.(L_3(3) \times 2)$;

$M_5 = N(3A) = S_3 \times \mathrm{Fi}_{22}{:}2$; $\quad M_6 = N(3A^2) = (3^2{:}D_8 \times U_4(3){:}2^2).2$;

$M_7 = N(3^6) = 3^6{:}(2 \times L_4(3)){:}2$.

We first classify the radical subgroups of each M_i, applying the modified local strategy, and do the fusions in $\mathbb{B}$ using Lemmas 2.3 and 2.2.

Case (3.1) We may take

$$\mathcal{R}_0(M_1, 3) = \{3^{1+8}, 3^{1+8}.3, 3^{1+8}.3^2, 3^{1+8}.3^3, 3^{1+8}.3^{1+2}, S\}. \tag{5.3}$$

Since $C_{M_1}(R) = 3 = Z(R)$ for each $R \in \mathcal{R}_0(M_1, 3)$, it follows that $N_{\mathbb{B}}(R) \leqslant_{\mathbb{B}} M_1$, so R is radical in $\mathbb{B}$ with $N_{M_1}(R) = N_{\mathbb{B}}(R)$. Thus we may suppose that $\mathcal{R}_0(M_1, 3) \subseteq \mathcal{R}_0(\mathbb{B}, 3)$.

Table 3: Non-trivial radical 3-subgroups of $\mathbb{B}$.

R	$C(R)$	$N(R)$	$\mathcal{W}_N$	$\mathcal{W}_{\hat{N}}$
3	$3 \times \mathrm{Fi}_{22}{:}2$	$S_3 \times \mathrm{Fi}_{22}{:}2$		
3^2	$3^2 \times U_4(3){:}2^2$	$(3^2{:}D_8 \times U_4(3){:}2^2).2$		
3^6	3^6	$3^6{:}(2 \times L_4(3){:}2){:}2$	5	2
3_+^{1+8}	3	$3_+^{1+8}.2_-^{1+6}{\cdot}U_4(3).2$	11	1
$3^3.3^6$	3^3	$3^3.3^6.(L_3(3) \times D_8)$	5	2
$3_+^{1+8}.3$	3	$3_+^{1+8}.3.2^2.2^4.3^2.D_8$	10	4
$3^3.3.3^3.3^3$	3^3	$3^3.3.3^3.3^3.(L_3(3) \times 2)$	2	2
$3_+^{1+8}.3^2$	3	$3_+^{1+8}.3^2.(D_8 \times 2S_4)$	10	4
$3^2.3^3.3^6$	3^2	$3^2.3^3.3^6.(S_4 \times 2S_4)$	4	0
$3_+^{1+8}.3^3$	3	$3_+^{1+8}.3^3.2^4.S_3$	8	0
$3_+^{1+8}.3_+^{1+2}$	3	$3_+^{1+8}.3_+^{1+2}.(2 \times 2S_4)$	4	4
$3^2.3^3.3^6.3$	3^2	$3^2.3^3.3^6.3.(2 \times 2S_4)$	4	4
S	3	$S.2^3$	8	8

Case (3.2) Applying the local strategy, we find four classes of radical subgroups of M_2; one of them, R, has order 3^{12} satisfying $C_{M_2}(R) = Z(R) = 3$ and $N_{M_2}(R) = R.2.(S_4 \times 2)$. Thus a generator of $Z(R)$ is a $3B$-element as $|N(3A)|_3 = 3^{10}$, and we may suppose that $N_{M_2}(R) \leqslant M_1$. By Lemma 2.3 and (5.3), R is radical in M_1, and by the local structures, $R =_{M_1} 3^{1+8}.3^3$.

Another radical subgroup Q of M_2 also has order 3^{12} and $C_{M_2}(Q) = 3^2 = Z(Q)$. So $N_{\mathbb{B}}(Q) \leqslant_{\mathbb{B}} M_2$ and Q is radical in $\mathbb{B}$. We may take

$$\mathcal{R}_0(M_2, 3) = \{3^2.3^3.3^6, 3^{1+8}.3^3, 3^2.3^3.3^6.3, S\}, \tag{5.4}$$

and $N_{\mathbb{B}}(R) = N_{M_2}(R)$ for each $R \in \mathcal{R}_0(M_2, 3)$, so we may suppose that $\mathcal{R}_0(M_2, 3) \subseteq \mathcal{R}_0(\mathbb{B}, 3)$.

Case (3.3) M_3 has 4 classes of radical subgroups; one of them, R, has order 3^{11} with $C_{M_3}(R) = Z(R) = 3^2$ and $N_{M_3}(R) = R.(2S_4 \times D_8)$. Thus $Z(R)$ is $3B$-pure, and by [**16**, Theorem 4.4] we may suppose that $N_{M_3}(R) \leqslant M_2$. By (5.4) and Lemma 2.2, R is non-radical in $\mathbb{B}$.

Another radical subgroup Q of M_3 also has order 3^{11} and $C_{M_3}(Q) = Z(Q) = 3$, so that $Z(Q)$ is generated by a $3B$-element and we may suppose that $N_{M_3}(Q) \leqslant M_1$. By the local structures, we can identify Q with $3_+^{1+8}.3^2$. We may take

$$\mathcal{R}_0(M_3, 3) = \{3^3.3^6, 3^3.3^6.3^2, 3_+^{1+8}.3^2, 3_+^{1+8}.3^2.3\}, \tag{5.5}$$

and $N_{\mathbb{B}}(R) \neq N_{M_3}(R)$ for $R \in \mathcal{R}_0(M_2, 3) \setminus \{3^3.3^6, 3_+^{1+8}.3^2\}$. Moreover, $C_{M_3}(3^3.3^6.3^2) = 3^2$, $C_{M_3}(3_+^{1+8}.3^2.3) = 3$, and

$$N_{M_3}(R) = \begin{cases} 3^3.3^6.3^2.(2S_4 \times D_8) & \text{if } R = 3^3.3^6.3^2, \\ 3_+^{1+8}.3^2.3.(2^2 \times D_8) & \text{if } R = 3_+^{1+8}.3^2.3. \end{cases}$$

Case (3.4) The fusions in $\mathbb{B}$ of the radical subgroups of M_4 with subgroups in other $\mathcal{R}_0(M_i, 3)$ can be done similarly to that of Case (3.2). We may take

$$\mathcal{R}_0(M_4, 3) = \{3^3.3.3^3.3^3, 3^2.3^3.3^6.3, 3^{1+8}.3^{1+2}, S\}, \tag{5.6}$$

and $N_{\mathbb{B}}(R) = N_{M_4}(R)$ for each $R \in \mathcal{R}_0(M_4, 3)$, so we may suppose that $\mathcal{R}_0(M_4, 3) \subseteq \mathcal{R}_0(\mathbb{B}, 3)$.

Case (3.5) M_7 has six classes of radical subgroups R, three of which have centralizer $C_{M_7}(R) = Z(R) = 3$, and $|N_{M_7}(R)|_3 \geqslant 3^{11}$. So a generator of $Z(R)$ is a $3B$-element, and we may suppose that $N_{M_7}(R) \leqslant M_1$. By (5.3) and the local structures, we find that one of the subgroups R is conjugate to $3^{1+8}.3$, and the other two are not radical in $\mathbb{B}$.

A radical subgroup Q of M_7 has order 3^9 and $C_{M_7}(Q) = Z(Q) = 3^3$. Since we have $|C_{M_7}(x)|_3 \geqslant 3^{11}$ for each $x \in Z(Q)$, $Z(Q)$ is $3B$-pure, and by [**16**, Theorem 4.4], $N_{M_7}(Q) = Q.(2^2 \times L_3(3)) \leqslant_{\mathbb{B}} M_i$ for some $i = 1, 2, 3, 4$. By Lemma 2.2 and equations (5.3)–(5.6), Q is not a radical subgroup of $\mathbb{B}$.

Similarly, M_7 has a radical subgroup W of order 3^{11} with $Z(W) = 3^2$, so that $Z(W)$ is $3B$-pure and $N_{M_7}(W) = W.(2 \times 2S_4{:}2){:}2 \leqslant_{\mathbb{B}} M_i$ for some $i = 1, 2, 3, 4$. This implies that W is not radical in $\mathbb{B}$. Thus we may take

$$\mathcal{R}_0(M_7, 3) = \{3^6, 3^{3+6}, 3_+^{1+8}.3, 3^6.3^{2+3}, 3^2.3^3.3^6, 3^{3+6}.3^{1+2}\}, \tag{5.7}$$

and $N_{M_7}(R) \neq N_{\mathbb{B}}(R)$ for $R \in \mathcal{R}_0(M_7, 3) \setminus \{3^6, 3_+^{1+8}.3\}$.

Moreover,

$$C_{M_7}(3^{3+6}) = 3^3,$$
$$C_{M_7}(3_+^{1+8}.3) \cong C_{M_7}(3^6.3^{2+3}) \cong C_{M_7}(3^{3+6}.3^{1+2}) = 3,$$
$$C_{M_7}(3^2.3^3.3^6) = 3^2,$$

and

$$N_{M_7}(R) = \begin{cases} 3^{3+6}.(2^2 \times L_3(3)) & \text{if } R = 3^{3+6}, \\ 3^6.3^{2+3}.(2^2 \times 2S_4) & \text{if } R = 3^6.3^{2+3}, \\ 3^2.3^3.3^6.(2 \times 2S_4{:}2).2 & \text{if } R = 3^2.3^3.3^6, \\ 3^{3+6}.3^{1+2}.2^3.2^2 & \text{if } R = 3^{3+6}.3^{1+2}. \end{cases} \tag{5.8}$$

Case (3.6) Since $M_5 = S_3 \times \mathrm{Fi}_{22}{:}2$, it follows that each radical subgroup R is of the form $3 \times R_1$ for some $R_1 \in \mathcal{R}_0(\mathrm{Fi}_{22}{:}2, 3)$. The fusion in $\mathbb{B}$ of elements of order 3 in Fi_{22} is given by [**16**, Proposition 3.1]. The radical subgroups of M_5 and their local structures are given in Table 4.

Note that $R = 3^2$ is $3A$-pure, so that $N_{\mathbb{B}}(3^2) \neq N_{M_5}(3^2)$. By [**16**, Proposition 3.2], $\mathbb{B}$ has a unique class of elementary abelian groups of order 3^6 containing $3A$-elements, and so $M_7 =_{\mathbb{B}} N_{\mathbb{B}}(3^6) \neq N_{M_5}(3^6)$.

Since the commutator subgroup $[3 \times 3_+^{1+6}, 3 \times 3_+^{1+6}] = (3 \times 3_+^{1+6})' = 3$ is $3B$-pure, $N_{\mathbb{B}}(3 \times 3_+^{1+6}) \leqslant_{\mathbb{B}} N(3B) = M_1$. But $N_{M_5}(R) \leqslant_{M_5} N_{M_5}(3 \times 3_+^{1+6})$ for $R = 3 \times 3_+^{1+6}.3$ and $R = 3 \times 3^5.3^3$; also, $C_{M_5}(R) \cong 3^2 = C_{M_5}(3 \times 3_+^{1+6})$, so $N_{\mathbb{B}}(R) \leqslant_{\mathbb{B}} M_1$, and by Lemma 2.2 and (5.3), none of them is radical in $\mathbb{B}$.

Suppose that $R = 3 \times 3^5{:}3_+^{1+2}$. Then $[R, R'] = Z(3^5{:}3_+^{1+2}) = 3^2$ is $3B$-pure, $N_{\mathbb{B}}(R) \leqslant_{\mathbb{B}} N(3B^2) = M_2$ and, by (5.4), R is non-radical in $\mathbb{B}$.

Since $(3 \times 3^{3+3})' = 3^3$ is $3B$-pure, $N_{\mathbb{B}}(R) \leqslant_{\mathbb{B}} M_4$ and, by (5.6), R is non-radical in $\mathbb{B}$.

It follows that $N_{M_5}(R) \neq N_{\mathbb{B}}(R)$ for any $R \in \mathcal{R}_0(M_5, 3) \setminus \{3\}$.

Table 4: Radical 3-subgroups of $S_3 \times \mathrm{Fi}_{22}{:}2$.

R	$C(R)$	$N_{M_5}(R)$
3	$3 \times \mathrm{Fi}_{22}{:}2$	$S_3 \times \mathrm{Fi}_{22}{:}2$
3^2	$3^2 \times U_4(3){:}2^2$	$S_3 \times (S_3 \times U_4(3){:}2).2$
3^6	3^6	$S_3 \times 3^5{:}(2 \times U_4(2){:}2)$
$3 \times 3^{3+3}$	3^4	$S_3 \times 3^{3+3}{:}L_3(3)$
$3 \times 3_+^{1+6}$	3^2	$S_3 \times 3_+^{1+6}.2^{3+4}{:}3^2.2^2$
$3 \times 3_+^{1+6}.3$	3^2	$S_3 \times 3_+^{1+6}.3{:}2S_4$
$3 \times 3^5.3^3$	3^2	$S_3 \times 3^5.3^3.(2 \times S_4).2$
$3 \times 3^5.3^{1+2}$	3^3	$S_3 \times 3^5.3^{1+2}.2S_4.2$
$3 \times 3_+^{1+6}.3^2$	3^2	$S_3 \times 3^{1+6}.3^2.2^3$

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Case (3.7) We may take

$$\mathcal{R}_0(M_6, 3) = \{3^2, 3^6, 3^2 \times 3^{1+4}, 3^3.3^2.3^3\}; \tag{5.9}$$

in addition, $C_{M_6}(3^2 \times 3^{1+4}) = 3^3 = C_{M_6}(3^3.3^2.3^3)$ and

$$N_{M_6}(R) = \begin{cases} (3^2{:}D_8 \times 3^4{:}A_6{:}2^2).2 & \text{if } R = 3^6, \\ (3^2{:}D_8 \times 3^{1+4}.2S_4 : 2^2).2 & \text{if } R = 3^2 \times 3^{1+4}, \\ 3^3.3^2.3^3.2^3.2^2.2^3 & \text{if } R = 3^3.3^2.3^3. \end{cases} \tag{5.10}$$

Thus $N_{M_6}(3^6) \neq_{\mathbb{B}} M_7$.

Since $C_{M_6}(3^2) =_{\mathbb{B}} C_{M_5}(3^2)$, we may suppose that $R = 3^2 \times 3^{1+4} \leqslant M_5$. Now $R' = 3$ is $3B$-pure, and so $N_{\mathbb{B}}(R) \leqslant_{\mathbb{B}} N(3B)$ and R is not radical in $\mathbb{B}$.

It follows that $N_{M_6}(R) \neq N_{\mathbb{B}}(R)$ for each $R \in \mathcal{R}_0(M_6, 3) \setminus \{3^2\}$.

Thus the radical 3-subgroups of $\mathbb{B}$ are listed in Table 3, and the centralizers and normalizers are given by MAGMA. □

LEMMA 5.2. *Let $G = \mathbb{B}$ and $B_0 = B_0(G)$, let $\mathrm{Blk}^+(G, p)$ be the set of p-blocks with a non-trivial defect group, and let $\mathrm{Irr}^+(G)$ be the characters of $\mathrm{Irr}(G)$ with positive p-defect. If a defect group $D(B)$ of B is cyclic, then $\mathrm{Irr}(B)$ is given by* [11, p. 387].

(a) *If $p = 7$, then $\mathrm{Blk}^+(G, p) = \{B_i \mid 0 \leqslant i \leqslant 8\}$ such that $D(B_i) \cong 7^2$ when $0 \leqslant i \leqslant 2$ while $D(B_j) \cong 7$ when $3 \leqslant j \leqslant 8$. In the notation of* [7, p. 208],

$$\begin{aligned} \mathrm{Irr}(B_1) = \{&\chi_2, \chi_4, \chi_{10}, \chi_{17}, \chi_{20}, \chi_{21}, \chi_{22}, \chi_{27}, \chi_{42}, \chi_{45}, \chi_{46}, \chi_{55}, \chi_{63}, \chi_{73}, \\ &\chi_{77}, \chi_{85}, \chi_{88}, \chi_{100}, \chi_{103}, \chi_{106}, \chi_{118}, \chi_{121}, \chi_{134}, \chi_{161}, \chi_{166}, \chi_{167}, \chi_{176}\}, \\ \mathrm{Irr}(B_2) = \{&\chi_7, \chi_9, \chi_{11}, \chi_{12}, \chi_{31}, \chi_{32}, \chi_{33}, \chi_{38}, \chi_{43}, \chi_{64}, \chi_{67}, \chi_{69}, \chi_{74}, \chi_{76}, \\ &\chi_{90}, \chi_{91}, \chi_{92}, \chi_{94}, \chi_{96}, \chi_{105}, \chi_{107}, \chi_{112}, \chi_{115}, \chi_{117}, \chi_{119}, \chi_{126}, \chi_{135}\}, \end{aligned}$$

and

$$\mathrm{Irr}(B_0) = \mathrm{Irr}^+(G) \setminus \left(\bigcup_{i=1}^{8} \mathrm{Irr}(B_i) \right).$$

Moreover, $\ell(B_0) = \ell(B_1) = 24$, $\ell(B_2) = 21$, $\ell(B_i) = 6$ for $3 \leqslant i \leqslant 7$ and $\ell(B_8) = 3$.

(b) *If $p = 5$, then $\mathrm{Blk}^+(G, p) = \{B_i \mid 0 \leqslant i \leqslant 6\}$ such that $D(B_i) \cong 5^2$ for $i = 1, 2$, and $D(B_j) \cong 5$ for $3 \leqslant j \leqslant 6$. In the notation of* [7, p. 208],

$$\begin{aligned} \mathrm{Irr}(B_1) = \{&\chi_5, \chi_9, \chi_{14}, \chi_{18}, \chi_{32}, \chi_{40}, \chi_{54}, \chi_{56}, \chi_{81}, \chi_{93}, \\ &\chi_{97}, \chi_{105}, \chi_{115}, \chi_{130}, \chi_{135}, \chi_{144}, \chi_{151}, \chi_{153}, \chi_{159}, \chi_{177}\}, \\ \mathrm{Irr}(B_2) = \{&\chi_{27}, \chi_{29}, \chi_{34}, \chi_{44}, \chi_{55}, \chi_{64}, \chi_{67}, \chi_{74}, \chi_{80}, \chi_{82}, \\ &\chi_{85}, \chi_{91}, \chi_{92}, \chi_{101}, \chi_{109}, \chi_{134}, \chi_{136}, \chi_{156}, \chi_{162}, \chi_{176}\}, \end{aligned}$$

and

$$\mathrm{Irr}(B_0) = \mathrm{Irr}^+(G) \setminus \left(\bigcup_{i=1}^{6} \mathrm{Irr}(B_i) \right).$$

Moreover, $\ell(B_0) = 51$, $\ell(B_1) = \ell(B_2) = 16$, and $\ell(B_j) = 4$ for $3 \leqslant j \leqslant 6$.

(c) *If* $p = 3$, *then* $\mathrm{Blk}^+(G, p) = \{B_i \mid 0 \leqslant i \leqslant 5\}$ *such that* $D(B_i) \cong 3^2$ *for* $i = 1, 2, 3$, *and* $D(B_j) \cong 3$ *for* $4 \leqslant j \leqslant 5$. *In the notation of* [7, p. 208],

$$\mathrm{Irr}(B_i) = \begin{cases} \{\chi_{30}, \chi_{33}, \chi_{79}, \chi_{97}, \chi_{107}, \chi_{123}, \chi_{126}, \chi_{151}, \chi_{164}\} & \text{if } i = 1, \\ \{\chi_{43}, \chi_{64}, \chi_{96}, \chi_{109}, \chi_{111}, \chi_{150}, \chi_{171}, \chi_{173}, \chi_{181}\} & \text{if } i = 2, \\ \{\chi_{63}, \chi_{76}, \chi_{91}, \chi_{100}, \chi_{105}, \chi_{106}, \chi_{154}, \chi_{156}, \chi_{162}\} & \text{if } i = 3. \end{cases}$$

Moreover, $\ell(B_0) = 71$, $\ell(B_1) = \ell(B_2) = 7$, $\ell(B_3) = 5$ *and* $\ell(B_j) = 2$ *for* $j = 4, 5$.

In addition, let $\hat{G} = 2.\mathbb{B}$ *and let* ξ *be the faithful linear character of* $\mathrm{Irr}(Z(\hat{G}))$. *If* $p \in \{3, 5, 7\}$, *then there is a unique* p*-block* $\hat{B}$ *of* $\hat{G}$ *such that* $\hat{B}$ *covers the block* $B(\xi)$ *and the defect group* $D(\hat{B})$ *is non-cyclic. In the notation of* [7, p. 218], *let*

$$\Omega = \begin{cases} \{\chi_j : j \in \{189, 192, 193, 197, 202, 205, 206, 207, 208, 222, 225, 234\}\} & \text{if } p = 7, \\ \{\chi_j : j \in \{204, 205, 206, 209\}\} & \text{if } p = 5, \\ \emptyset & \text{if } p = 3. \end{cases}$$

Then

$$\mathrm{Irr}(\hat{B}) = \mathrm{Irr}^+(\hat{G}) \setminus (\Omega \cup \mathrm{Irr}^+(G))$$

and $\ell(\hat{B})$ *is* 24, 33 *or* 31 *when* p *is* 7, 5 *or* 3.

Proof. If $B \in \mathrm{Blk}(G, p)$ is non-principal with $D = D(B)$, then $\mathrm{Irr}^0(C(D)D/D)$ has a non-trivial character θ and $N(\theta)/C(D)D$ is a p'-group, where $N(\theta)$ is the stabilizer of θ in $N(D)$. By [11, p. 387], we may suppose that D is non-cyclic, so that by Proposition 5.1, $D = p^2$.

If $p = 7$, then $C(D) = 2^2 \times 7^2$ and $N(D) = (2^2 \times 7^2{:}(3 \times 2A_4)){:}2$, so that $N(D)$ has 3 orbits on $\mathrm{Irr}^0(C(D)D/D)$ and $\mathbb{B}$ has three blocks with defect group $D = 7^2$.

If $p = 5^2$, then $C(D) = 5^2 \times S_5$ and $N(D) = 5^2{:}4S_4 \times S_5$, so that $N(D)$ has two orbits on $\mathrm{Irr}^0(C(D)D/D)$ and $\mathbb{B}$ has two blocks with defect group $D = 5^2$.

If $p = 3^2$, then $C(D) = 3^2 \times U_4(3){:}2^2$ and $N(D) = (3^2{:}D_8 \times U_4(3){:}2^2).2$, so that $N(D)$ has three orbits on $\mathrm{Irr}^0(C(D)D/D)$ and $\mathbb{B}$ has three blocks with defect group $D = 3^2$.

Using the method of central characters, $\mathrm{Irr}(B)$ is as above. If $D(B)$ is cyclic, then $\ell(B)$ is given by [11, p. 387].

If $p = 7$ and $B \in \{B_1, B_2\}$, then the non-trivial elements of $D(B)$ are conjugate in $\mathbb{B}$, and $C(x) = 7 \times 2{\cdot}L_3(4){:}2$ for any $1 \neq x \in D(B)$. It follows that

$$\mathrm{k}(B) = \ell(B) + \sum_{b \in \mathrm{Blk}(C(x), B)} \ell(b), \tag{5.11}$$

where $\mathrm{Blk}(C(x), B) = \{b \in \mathrm{Blk}(C(x)) : b^G = B\}$. In particular, for $b \in \mathrm{Blk}(C(x), B)$, $b = B_0(7) \times b'$ for some block $b' \in \mathrm{Blk}(2{\cdot}L_3(4){:}2)$ with cyclic defect group 7. Now $C_{2\cdot L_3(4):2}(7) = 7 \times 2^2$, $N_{2\cdot L_3(4):2}(7) = 2^2 \times 7{:}3$, and so $\ell(b) = \ell(b') = \mathcal{W}(b') = 3$. Thus if an outer involution in $(2^2 \times 7^2{:}(3 \times 2A_4)){:}2$ stabilizes a root block of B, then $\ell(B) = 27 - 3 = 24$; otherwise, $\ell(B) = 27 - 6 = 21$. We may suppose that $\ell(B_1) = 24$ and $\ell(B_2) = 21$.

If $p = 5$ and B equals B_1 or B_2, then the non-trivial elements of $D(B)$ are of type $5A$, and $C(5A) = 5 \times \mathrm{HS}{:}2$, so that (5.11) holds with $C(x) = 5 \times \mathrm{HS}{:}2$ and $b = B_0(5) \times b'$, where $b' \in \mathrm{Blk}(\mathrm{HS}{:}2)$ with cyclic defect group 5. Thus $\mathcal{W}(b') = \ell(b') = 4$ as $N_{\mathrm{HS}:2}(5) = 5{:}4 \times S_5$, and $\ell(B) = 20 - 4 = 16$.

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If $p = 3$ and $B \in \{B_1, B_2, B_3\}$, then the non-trivial elements of $D(B)$ are of type $3A$, and $C(3A) = S_3 \times \mathrm{Fi}_{22}{:}2$, so that $b = B_0(S_3) \times b'$ for some $b' \in \mathrm{Blk}(\mathrm{Fi}_{22}{:}2)$ with cyclic defect group 3, where $b \in \mathrm{Blk}(S_3 \times \mathrm{Fi}_{22}{:}2, B)$. Now

$$C_{\mathrm{Fi}_{22}:2}(3) = 3 \times U_4(3){:}2^2;$$
$$N_{\mathrm{Fi}_{22}:2}(3) = (S_3 \times U_4(3){:}2).2; \;\; N_{\mathrm{Fi}_{22}:2}(3)/(3 \times U_4(3)) = 2^3.$$

Thus, if an element of $N_{\mathrm{Fi}_{22}:2}(3) \setminus C_{\mathrm{Fi}_{22}:2}(3)$ stabilizes the root block of b', then $\mathcal{W}(b') = \ell(b') = 2$; otherwise, $\mathcal{W}(b') = \ell(b') = 4$. We may suppose that $\ell(B_i) = k(B_i) - 2 = 9 - 2 = 7$ when $i = 1, 2$ and $\ell(B_3) = k(B_3) - 4 = 9 - 4 = 5$.

If $\ell_p(G)$ is the number of p-regular G-conjugacy classes, then $\ell_7(G) = 165$, $\ell_5(G) = 144$ and $\ell_3(G) = 103$. Thus $\ell(B_0)$ can be calculated by the following equation, due to Brauer:

$$\ell_p(G) = \sum_{B \in \mathrm{Blk}^+(G,p)} \ell(B) + |\,\mathrm{Irr}^0(G)|, \tag{5.12}$$

where $|\,\mathrm{Irr}^0(G)|$ is 63, 45 or 9 when p equals 7, 5 or 3.

The set $\mathrm{Irr}(\hat{B})$ is also determined using the method of central characters, and $\ell(\hat{B})$ is given by (5.12) with G replaced by $\hat{G}$, where $\ell_7(\hat{G}) = 220$, $\ell_5(\hat{G}) = 189$ and $\ell_3(\hat{G}) = 139$. □

Theorem 5.3. *Let $G = 2.\mathbb{B}$, and let B be a p-block of G with a non-cyclic defect group. If $p \geqslant 3$, then the number of B-weights is the number of irreducible Brauer characters of B.*

Proof. We may suppose that p is 3, 5 or 7. If $B = B_0$ or $\hat{B}$, then

$$\mathcal{W}(B) = \sum_R \mathcal{W}_H,$$

where $R \in \mathcal{R}_0(\mathbb{B}, p)$ with $\mathrm{d}(C_{\mathbb{B}}(R)R/R) = 0$, and $H = N = N_{\mathbb{B}}(R)$ or $\hat{N} = N_{2.\mathbb{B}}(R)$, according to whether B equals B_0 or $\hat{B}$. Thus Theorem 5.3 follows by Lemma 5.2 and Tables 2 and 3.

Suppose that $p = 7$. Then $|\,\mathrm{Irr}(N_{\mathbb{B}}(7^2)/7^2)| = 69$, $\mathbb{B}$ has sixty-nine weights of the form $(7^2, \varphi)$, twenty-four of which are B_0-weights. If $b \in \mathrm{Blk}(2^2 \times 7^2)$ is stabilized by an outer involution of $(2^2 \times 7^2{:}(3 \times 2A_4)){:}2$, then its canonical character θ is stabilized by $N_{\mathbb{B}}(7^2)$, so θ has an extension to $N_{\mathbb{B}}(7^2)$ and $b^{N_{\mathbb{B}}(7^2)}$ has twenty-four weight characters, which form all twenty-four B_1-weights. Finally, B_2 has $21 = 69 - 24 - 24$ weights.

If $p = 5$ and B is B_1 or B_2, then $N_{\mathbb{B}}(D)/C_{\mathbb{B}}(D)D = 4S_4$, which has sixteen irreducible characters, so that B has sixteen weights of the form (D, φ).

If $p = 3$ and $B \in \{B_1, B_2\}$, then $N_{\mathbb{B}}(D)/C_{\mathbb{B}}(D)D$ is a semidihedral group SD_{16}, so that B has seven weights of the form (D, φ).

If $p = 3$ and $B = B_3$, then B has five weights of the form (D, φ), since $N_{\mathbb{B}}(D)/D$ has nineteen irreducible characters of defect 0, and fourteen of these are weight characters of B_1 or B_2. □

6. *Radical chains*

Let $G = \mathbb{B}$, $C \in \mathcal{R}(G)$ and $N(C) = N_G(C)$. We will do some cancellations in the alternating sum of Uno's conjecture. We first list some radical p-chains $C(i)$ and their normalizers for certain integers i, and then reduce the proof of the conjecture to the subfamily $\mathcal{R}^0 = \mathcal{R}^0(G)$ of $\mathcal{R}(G)$, where $\mathcal{R}^0(G)$ is the union of G-orbits of all $C(i)$. The subgroups of the p-chains in Tables 5 and 6 are given either by Tables 2 and 3, or in the proofs of Proposition 5.1 and Lemma 6.1.

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LEMMA 6.1. *Let $\mathcal{R}^0(G)$ be the G-invariant subfamily of $\mathcal{R}(G)$ such that*

$$\mathcal{R}^0(G)/G = \begin{cases} \{C(i) : 1 \leqslant i \leqslant 10\} & \text{with } C(i) \text{ given in Table 5 if } p = 5, \\ \{C(i) : 1 \leqslant i \leqslant 36\} & \text{with } C(i) \text{ given in Table 6 if } p = 3. \end{cases}$$

Then

$$\sum_{C \in \mathcal{R}(G)/G} (-1)^{|C|} \mathrm{k}(N(C), B, d, [r]) = \sum_{C \in \mathcal{R}^0(G)/G} (-1)^{|C|} \mathrm{k}(N(C), B, d, [r]). \tag{6.1}$$

Proof. Let $C \in \mathcal{R}(G)$ be given by (2.1), so that we may suppose that $P_1 \in \mathcal{R}_0(G, p)$.

Case (1) Suppose that $p = 5$. If $P_1 = O_5(M_1) = 5$, then $C \in_G \{C(2), C(3), C(4), C(5)\}$, since $\mathcal{R}_0(N_{\mathbb{B}}(5^2), 5) = \{5^2, 5^3\}$.

Let $\sigma : 1 < Q = 5^3 < 5^2.5^{1+2}$, so that $\sigma' : 1 < 5^2.5^{1+2}$. Thus σ and σ' satisfy the conditions of Lemma 2.1, so there is a bijection g from $\mathcal{R}^-(\sigma, 5^3)$ onto $\mathcal{R}^0(\sigma, 5^3)$ such that $N(C') = N(g(C'))$ and $|C'| = |g(C')| - 1$ for each $C' \in \mathcal{R}^-(\sigma, 5^3)$. So

$$\mathrm{k}(N(C'), B, d, [r]) = \mathrm{k}(N(g(C')), B, d, [r]), \tag{6.2}$$

and we may suppose that $C \notin (\mathcal{R}^-(\sigma, 5^3) \cup \mathcal{R}^0(\sigma, 5^3))$. In particular, we may suppose that $P_1 \neq_G 5^2.5^{1+2}$, and if $P_1 = 5^3$, then $P_2 \neq_G 5^2.5^{1+2}$. Similarly, let $C' : 1 < 5^3 < 5^{1+4} < 5^{1+4}.5$ and $g(C') : 1 < 5^3 < 5^{1+4}.5$. Then $N(C') = N(g(C'))$ and (6.2) holds; hence, if $P_1 = 5^3 = O_5(M_2)$, we may suppose that $C \in_G \{C(6), C(7)\}$.

Let $C' : 1 < 5^{1+4} < 5^{1+4}.5$ and $g(C') : 1 < 5^{1+4}.5$, so that $N(C') = N(g(C')) = 5^{1+4}.5.4^2$ and (6.2) holds. If $P_1 = 5^{1+4}$, then we may suppose that $C =_G C(8)$. If $P_1 = 5^2$, then $C =_G C(9)$ or $C(10)$.

Case (2) Suppose that $p = 3$.

Case (2.1) Let $R \in \mathcal{R}_0(M_1, 3) \setminus \{3^{1+8}\}$ and let $\sigma(R) : 1 < Q = 3^{1+8} < R$, so that $\sigma(R)' : 1 < R$, where $\mathcal{R}_0(M_1, 3)$ is given by (5.3). Then $\sigma(R)$ and $\sigma(R)'$ satisfy the conditions of Lemma 2.1.

Table 5: Some radical 5-chains of $\mathbb{B}$.

C		$N(C)$
$C(1)$	1	$\mathbb{B}$
$C(2)$	$1 < 5$	$5{:}4 \times \mathrm{HS}{:}2$
$C(3)$	$1 < 5 < 5^2$	$5{:}4 \times 5{:}4 \times S_5$
$C(4)$	$1 < 5 < 5^2 < 5^3$	$5{:}4 \times 5{:}4 \times 5{:}4$
$C(5)$	$1 < 5 < 5 \times 5_+^{1+2}$	$5{:}4 \times 5_+^{1+2}.4.D_8$
$C(6)$	$1 < 5^3$	$5^3.L_3(5)$
$C(7)$	$1 < 5^3 < 5_+^{1+4}$	$5_+^{1+4}.\mathrm{GL}_2(5)$
$C(8)$	$1 < 5_+^{1+4}$	$5_+^{1+4}.2_-^{1+4}.A_5.4$
$C(9)$	$1 < 5^2 < 5^3$	$5^2{:}4S_4 \times 5{:}4$
$C(10)$	$1 < 5^2$	$5^2{:}4S_4 \times S_5$

A similar proof to that of Case (1) shows that we may suppose that

$$C \notin \bigcup_{R\in\mathcal{R}_0(M_1,3)\setminus\{3^{1+8}\}} (\mathcal{R}^-(\sigma(R), 3^{1+8}) \cup \mathcal{R}^0(\sigma(R), 3^{1+8})).$$

In particular, $P_1 \notin_G \mathcal{R}_0(M_1, 3) \setminus \{3^{1+8}\}$, and if $P_1 = 3^{1+8}$, then $C = C(2)$.

We may suppose that

$$P_1 \in_G \{3, 3^2, 3^6, 3^3.3^6, 3^3.3.3^3.3^3, 3^2.3^3.3^6, 3^2.3^3.3^6.3\}.$$

Case (2.2) Let $\sigma : 1 < Q = 3^2.3^3.3^6 < 3^2.3^3.3^6.3$, so that $\sigma' : 1 < 3^2.3^3.3^6.3$, where $3^2.3^3.3^6, 3^2.3^3.3^6.3 \in \mathcal{R}_0(M_2, 3)$ given by (5.4). Then σ and σ' satisfy the conditions of Lemma 2.1. A similar proof to that of Case (1) shows that we may suppose that

$$C \notin (\mathcal{R}^-(\sigma, 3^2.2^3.3^6) \cup \mathcal{R}^0(\sigma, 3^2.2^3.3^6)).$$

In particular, $P_1 \neq_G 3^2.3^3.3^6.3$ and if $P_1 = 3^2.2^3.3^6$, then $P_2 \neq_G 3^2.3^3.3^6.3$.

Let $C' : 1 < 3^2.2^3.3^6 < S$ and $g(C') : 1 < 3^2.2^3.3^6 < 3^{1+8}.3^3 < S$. Then $N(C') = N(g(C')) = S.2^3$, and so (6.2) holds. Thus if $P_1 = 3^2.3^3.3^6 = O_3(M_2)$, we may suppose that $C \in_G \{C(3), C(4)\}$.

Case (2.3) Let $C' : 1 < 3^3.3^6 < 3^{1+8}.3^2.3$ and $g(C') : 1 < 3^3.3^6 < 3^{1+8}.3^2 < 3^{1+8}.3^2.3$, where $3^{1+8}.3^2, 3^{1+8}.3^2.3 \in \mathcal{R}_0(M_3, 3)$. Then $N(C') = N(g(C'))$, and (6.2) holds. We may suppose that $C \neq_G C'$ and $C \neq_G g(C')$, so that if $P_1 = 3^3.3^6$, we may suppose that $C \in_G \{C(5), C(6), C(7), C(8)\}$.

Case (2.4) Let $C' : 1 < 3^3.3.3^3.3^3 < S$ and $g(C') : 1 < 3^3.3.3^3.3^3 < 3^{1+8}.3^{1+2} < S$, where $3^3.3.3^3.3^3, 3^{1+8}.3^{1+2} \in \mathcal{R}_0(M_4, 3)$. Then $N(C') = N(g(C')) = S.2^3$, and (6.2) holds. We may suppose that $C \neq_G C'$ and $C \neq_G g(C')$, so that if $P_1 = 3^3.3.3^3.3^3$, we may suppose that $C \in_G \{C(9), C(10), C(11), C(12)\}$.

Case (2.5) Let $L = N_{M_5}(3^6) = S_3 \times 3^5{:}(2 \times U_4(2){:}2)$. We may take

$$\mathcal{R}_0(L, 3) = \{3^6, 3 \times 3^5.3^3, 3 \times 3^5.3^{1+2}, 3 \times 3^{1+6}.3^2\} \subseteq \mathcal{R}_0(M_5, 2),$$

and hence $N_{M_5}(R) \leqslant N_{M_5}(3^6)$ for all $R \in \mathcal{R}_0(L, 3)$. Let $R \in \mathcal{R}_0(L, 3) \setminus \{3^6\}$ and $\sigma(R) : 1 < 3 < Q = 3^6 < R$, so that $\sigma(R)' : 1 < 3 < R$. A similar proof to that of Case (1) shows that we may suppose that

$$C \notin \bigcup_{R\in\mathcal{R}_0(L,3)\setminus\{3^6\}} (\mathcal{R}^-(\sigma(R), 3^6) \cup \mathcal{R}^0(\sigma(R), 3^6)).$$

In particular, if $P_1 = 3$, then $P_2 \notin_G \mathcal{R}_0(L, 3)\setminus\{3^6\}$; moreover, if $P_2 = 3^6$, then $C = C(17)$.

Let $H = N_{M_5}(3^2) = S_3 \times (S_3 \times U_4(3){:}2).2$. We may take

$$\mathcal{R}_0(H, 3) = \{3^2, 3^6, 3^2 \times 3^{1+4}, 3 \times 3^2.3^2.3^3\};$$

moreover, $C_{M_5}(3^6) = 3^6$, $C_{M_5}(3^2 \times 3^{1+4}) = C_{M_5}(3 \times 3^2.3^2.3^3) = 3^3$, and

$$N_H(R) = \begin{cases} S_3 \times S_3 \times 3^4.2S_6 & \text{if } R = 3^6, \\ S_3 \times S_3 \times 3^{1+4}.2_-^{1+4}.S_3 & \text{if } R = 3^2 \times 3^{1+4}, \\ S_3 \times 3^2.3^2.3^3.2^3.2^2 & \text{if } R = 3 \times 3^2.3^2.3^3. \end{cases}$$

Let $C' : 1 < 3 < 3^2 < 3 \times 3^2.3^2.3^3$ and $g(C') : 1 < 3 < 3^2 < 3^2 \times 3^{1+4} < 3 \times 3^2.3^2.3^3$. Then $N(C') = N(g(C')) = N_H(3 \times 3^2.3^2.3^3)$ and (6.2) holds. It follows that if $P_1 = 3$ and $P_2 = 3^2$, then we may suppose that $C \in_G \{C(13), C(14), C(15), C(16)\}$.

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Table 6: Some radical 3-chains of $\mathbb{B}$.

C		$N(C)$
$C(1)$	1	$\mathbb{B}$
$C(2)$	$1 < 3_+^{1+8}$	$3_+^{1+8}{:}2_-^{1+6}{\cdot}U_4(2).2$
$C(3)$	$1 < 3^2.3^3.3^6 < 3_+^{1+8}.3^3$	$3_+^{1+8}.3^3.2^4.S_3$
$C(4)$	$1 < 3^2.3^3.3^6$	$3^2.3^3.3^6.(S_4 \times 2S_4)$
$C(5)$	$1 < 3^3.3^6 < 3^3.3^6.3^2$	$3^3.3^6.3^2(2S_4 \times D_8)$
$C(6)$	$1 < 3^3.3^6$	$3^3.3^6.(L_3(3) \times D_8)$
$C(7)$	$1 < 3^3.3^6 < 3_+^{1+8}.3^2$	$3_+^{1+8}.3^2(2S_4 \times D_8)$
$C(8)$	$1 < 3^3.3^6 < 3_+^{1+8}.3^2 < 3_+^{1+8}.3^2.3$	$3_+^{1+8}.3^2.3(2^2 \times D_8)$
$C(9)$	$1 < 3^3.3.3^3.3^3 < 3^2.3^3.3^6.3$	$3^2.3^3.3^6.3(2 \times 2S_4)$
$C(10)$	$1 < 3^3.3.3^3.3^3$	$3^3.3.3^3.3^3(L_3(3) \times 2)$
$C(11)$	$1 < 3^3.3.3^3.3^3 < 3_+^{1+8}.3_+^{1+2}$	$3_+^{1+8}.3_+^{1+2}(2 \times 2S_4)$
$C(12)$	$1 < 3^3.3.3^3.3^3 < 3^2.3^3.3^6.3 < S$	$S.2^3$
$C(13)$	$1 < 3 < 3^2$	$S_3 \times (S_3 \times U_4(3){:}2).2$
$C(14)$	$1 < 3 < 3^2 < 3^2 \times 3_+^{1+4}$	$S_3 \times S_3 \times 3_+^{1+4}.2_-^{1+4}.S_3$
$C(15)$	$1 < 3 < 3^2 < 3^6 < 3 \times 3^2.3^2.3^3$	$S_3 \times 3^2.3^2.3^3.2^3.2^2$
$C(16)$	$1 < 3 < 3^2 < 3^6$	$S_3 \times S_3 \times 3^4{:}2S_6$
$C(17)$	$1 < 3 < 3^6$	$S_3 \times 3^5{:}(2 \times U_4(2){:}2)$
$C(18)$	$1 < 3$	$S_3 \times \mathrm{Fi}_{22}{:}2$
$C(19)$	$1 < 3 < 3 \times 3^{3+3}$	$S_3 \times 3^{3+3}{:}L_3(3)$
$C(20)$	$1 < 3 < 3 \times 3^{3+3} < 3 \times 3^5{:}3_+^{1+2}$	$S_3 \times 3^5{:}3_+^{1+2}.2S_4$
$C(21)$	$1 < 3 < 3 \times 3_+^{1+6}$	$S_3 \times 3_+^{1+6}.2^{3+4}{:}3^2.2^2$
$C(22)$	$1 < 3 < 3 \times 3_+^{1+6} < 3 \times 3_+^{1+6}.3$	$S_3 \times 3_+^{1+6}.3{:}2S_4$
$C(23)$	$1 < 3 < 3 \times 3_+^{1+6} < 3 \times 3_+^{1+6}.3 < 3 \times 3_+^{1+6}.3^2$	$S_3 \times 3_+^{1+6}.3^2.2^2$
$C(24)$	$1 < 3 < 3 \times 3_+^{1+6} < 3 \times 3^5.3^3$	$S_3 \times 3^5.3^3.(2 \times S_4).2$
$C(25)$	$1 < 3^2 < 3^6$	$(3^2{:}D_8 \times 3^4{:}A_6{:}2^2).2$
$C(26)$	$1 < 3^2 < 3^6 < 3^3.3^2.3^3$	$3^3.3^2.3^3.2^3.2^2.2^3$
$C(27)$	$1 < 3^2 < 3^2 \times 3_+^{1+4}$	$(3^2{:}D_8 \times 3_+^{1+4}.2S_4{:}2^2).2$
$C(28)$	$1 < 3^2$	$(3^2{:}D_8 \times U_4(3){:}2^2).2$
$C(29)$	$1 < 3^6 < 3^{3+6}$	$3^{3+6}.(2^2 \times L_3(3))$
$C(30)$	$1 < 3^6 < 3^{3+6} < 3^2.3^3.3^6$	$3^2.3^3.3^6.(2^2 \times 2S_4)$
$C(31)$	$1 < 3^6 < 3^{3+6} < 3^2.3^3.3^6 < 3^{3+6}.3_+^{1+2}$	$3^{3+6}.3_+^{1+2}.2^4$
$C(32)$	$1 < 3^6 < 3^{3+6} < 3^6.3^{2+3}$	$3^6.3^{2+3}.(2^2 \times 2S_4)$
$C(33)$	$1 < 3^6 < 3_+^{1+8}.3$	$3_+^{1+8}.3.2^2.2^4.3^2.D_8$
$C(34)$	$1 < 3^6$	$3^6{:}(2 \times L_4(3){:}2){:}2$
$C(35)$	$1 < 3^6 < 3^2.3^3.3^6$	$3^2.3^3.3^6.(2 \times 2S_4{:}2){:}2$
$C(36)$	$1 < 3^6 < 3^2.3^3.3^6 < 3^{3+6}.3_+^{1+2}$	$3^{3+6}.3_+^{1+2}.2^3.2^2$

Let $J = N_{M_5}(3 \times 3^{3+3}) = S_3 \times 3^{3+3}{:}L_3(3)$. We may take

$$\mathcal{R}_0(J, 3) = \{3 \times 3^{3+3}, 3 \times 3^{1+6}.3, 3 \times 3^5{:}3^{1+2}, 3 \times 3^{1+6}.3^2\} \subseteq \mathcal{R}_0(M_5, 3);$$

moreover, $N_J(3 \times 3^{1+6}.3) = N_{M_5}(3 \times 3^{1+6}.3)$, and

$$N_J(R) = \begin{cases} S_3 \times 3^5{:}3^{1+2}.2S_4 & \text{if } R = 3 \times 3^5{:}3^{1+2}, \\ S_3 \times 3^{1+6}.3^2.2^2 & \text{if } R = 3 \times 3^{1+6}.3^2. \end{cases}$$

Let $\sigma : 1 < 3 < Q = 3 \times 3^{3+3} < 3 \times 3^{1+6}.3$, so that $\sigma' : 1 < 3 < 3 \times 3^{1+6}.3$. A similar proof to that of Case (1) shows that we may suppose that

$$C \notin (\mathcal{R}^-(\sigma, 3 \times 3^{3+3}) \cup \mathcal{R}^0(\sigma, 3 \times 3^{3+3})).$$

Let $C' : 1 < 3 < 3 \times 3^{3+3} < 3 \times 3^{1+6}.3^2$ and $g(C') : 1 < 3 < 3 \times 3^{3+3} < 3 \times 3^5{:}3^{1+2} < 3 \times 3^{1+6}.3^2$. Then $N(C') = N(g(C')) = N_J(3 \times 3^{1+6}.3^2)$ and (6.2) holds. It follows that if $P_1 = 3$ and $P_2 = 3 \times 3^{3+3}$, then we may suppose that $C \in_G \{C(19), C(20)\}$.

Let $K = N_{M_5}(3 \times 3^{1+6}) = S_3 \times 3^{1+6}.2^{3+4}{:}3^2.2^2$. We may take

$$\mathcal{R}_0(K, 3) = \{3 \times 3^{1+6}, 3 \times 3^{1+6}.3, 3 \times 3^5.3^3, 3 \times 3^{1+6}.3^2\} \subseteq \mathcal{R}_0(M_5, 3);$$

moreover, $N_K(R) = N_{M_5}(R)$ for each $R \in \mathcal{R}_0(K, 3)$.

Let $C' : 1 < 3 < 3 \times 3^{1+6} < 3 \times 3^{1+6}.3^2$ and $g(C') : 1 < 3 < 3 \times 3^{1+6} < 3 \times 3^5.3^3 < 3 \times 3^{1+6}.3^2$. Then $N(C') = N(g(C')) = N_K(3 \times 3^{1+6}.3^2)$ and (6.2) holds. If $P_1 = 3$ and $P_2 = 3 \times 3^{1+6}$, then we may suppose that $C \in_G \{C(21), C(22), C(23), C(24)\}$.

It follows that if $P_1 = 3$, then $C =_G C(i)$ for $13 \leqslant i \leqslant 24$

Case (2.6) Let $\sigma : 1 < 3^2 < Q = 3^2 \times 3^{1+4} < 3^3.3^2.3^3$, so that $\sigma' : 1 < 3^2 < 3^3.3^2.3^3$, where $3^3.3^2.3^3$, $3^2 \times 3^{1+4} \in \mathcal{R}_0(M_6, 3)$. A similar proof to that of Case (1) shows that we may suppose that

$$C \notin (\mathcal{R}^-(\sigma, 3^2 \times 3^{1+4}) \cup \mathcal{R}^0(\sigma, 3^2 \times 3^{1+4})).$$

In particular, if $P_1 = 3^2$, then $P_2 \neq_G 3^3.3^2.3^3$; if, moreover, $P_2 = 3^2 \times 3^{1+4}$, then $P_3 \neq_G 3^3.3^2.3^3$. Thus we may suppose that $C \in_G \{C(25), C(26), C(27), C(28)\}$.

Case (2.7) Let $T = N_{M_7}(3^{1+8}.3) = N(3^{1+8}.3) = 3^{1+8}.3.2^2.2^4.3^2.D_8$. We may take

$$\mathcal{R}_0(T, 3) = \{3^{1+8}.3, 3^6.3^{2+3}, 3^{3+6}.3^{1+2}\} \subseteq \mathcal{R}_0(M_7, 3);$$

moreover, $N_T(R) = N_{M_7}(R)$ for each $R \in \mathcal{R}_0(T, 3)$.

Let $R \in \mathcal{R}_0(T, 3) \setminus \{3^{1+8}.3\}$, $\sigma(R) : 1 < 3^6 < Q = 3^{1+8}.3 < R$, so that $\sigma(R)' : 1 < 3^6 < R$. A similar proof to that of Case (1) shows that we may suppose that

$$C \notin \bigcup_{R \in \mathcal{R}_0(T,3) \setminus \{3^{1+8}.3\}} (\mathcal{R}^-(\sigma(R), 3^{1+8}.3) \cup \mathcal{R}^0(\sigma, 3^{1+8}.3)).$$

In particular, if $P_1 = 3^6$, then $P_2 \notin \mathcal{R}_0(T, 3) \setminus \{3^{1+8}.3\}$; if, moreover, $P_2 = 3^{1+8}.3$, then $P_3 \notin \mathcal{R}_0(T, 3)$.

Let $V = N_{M_7}(3^{3+6}) = 3^{3+6}.(2^2 \times L_3(3))$. We may take

$$\mathcal{R}_0(V, 3) = \{3^{3+6}, 3^2.3^3.3^6, 3^6.3^{2+3}, 3^{3+6}.3^{1+2}\} \subseteq \mathcal{R}_0(M_7, 3);$$

moreover,

$$N_V(R) = \begin{cases} 3^2.3^3.3^6.(2^2 \times 2S_4) & \text{if } R = 3^2.3^3.3^6, \\ 3^6.3^{2+3}.(2^2 \times 2S_4) & \text{if } R = 3^6.3^{2+3}, \\ 3^{3+6}.3^{1+2}.2^4 & \text{if } R = 3^{3+6}.3^{1+2}. \end{cases}$$

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Let $C' : 1 < 3^6 < 3^{3+6} < 3^6.3^{2+3} < 3^{3+6}.3^{1+2}$ and $g(C') : 1 < 3^6 < 3^{3+6} < 3^{3+6}.3^{1+2}$. Then $N(C') = N(g(C')) = N_V(3^{3+6}.3^{1+2})$ and (6.2) holds. If $P_1 = 3^6$ and $P_2 = 3^{3+6}$, then we may suppose that $C \in_G \{C(29), C(30), C(31), C(32)\}$.

Since Sylow 3-subgroups of $N_{M_7}(3^2.3^3.3^6) = 3^2.3^3.3^6.(2\times 2S_4{:}2){:}2$ are the only radical subgroups of $N_{M_7}(3^2.3^3.3^6)$ other than $3^2.3^3.3^6$, it follows that if $P_1 = 3^6$, then $C =_G C(i)$ for some $29 \leqslant i \leqslant 36$. □

Remark. *Let $\hat{G} = 2.\mathbb{B}$ and let ξ be the faithful character of* $\mathrm{Irr}(Z(\hat{G}))$ *and $\hat{B} \in \mathrm{Blk}(\hat{G})$ covering the block of $B(\xi)$. If $D(\hat{B}) \neq 1$ and $p = 3, 5$, then*

$$\sum_{C\in\mathcal{R}(G)/G} (-1)^{|C|}\mathrm{k}(N_{\hat{G}}(C), \hat{B}, d, \xi, [r]) = \sum_{C\in\mathcal{R}^0(G)/G} (-1)^{|C|}\mathrm{k}(N_{\hat{G}}(C), \hat{B}, d, \xi, [r]).$$

The proof of the remark is the same as that of Lemma 6.1, since $N(C') = N(g(C'))$ implies that $N_{\hat{G}}(C') = N_{\hat{G}}(g(C'))$.

7. *The proof of Uno's projective conjecture*

Lemma 7.1. *Let G be a finite group, and let $B \in \mathrm{Blk}(G)$ with $\mathrm{plr}(B) = 2$ and abelian defect group $D = D(B)$. Let $O_p(G) \neq R < D$ be radical, and let $b \in Blk(N_G(R))$ with $b^G = B$. Then*

$$\mathrm{k}(N_G(R) \cap N_G(D), b, d, \rho, [r]) = \mathrm{k}(N_G(R), b, d, \rho, [r]). \tag{7.1}$$

Proof. Since R is radical and D is abelian, D is a defect group of b, $\mathrm{plr}(b) \neq 0$ and $\mathrm{plr}(b) = 1$. By [3], we see that [3, Conjecture 1.3] holds for b; that is,

$$\begin{aligned}\mathrm{k}(N_G(R), b, d, \rho, [r]) &- \mathrm{k}(N_G(R) \cap N_G(D), b, d, \rho, [r]) \\ &= w(N_G(R), b, d, \rho, [r], R) - w(N_G(R) \cap N_G(D), b, d, \rho, [r], R),\end{aligned}$$

where $w(H, Q)$ is the number of irreducible characters of H afforded by a Q-projective $\mathcal{O}H$-module. By [13], if

$$w(N_G(R), b, d, \rho, [r], R) \neq 0 \quad \text{or} \quad w(N_G(R) \cap N_G(D), b, d, \rho, [r], R) \neq 0,$$

then $C_D(R) \leqslant R$, which is impossible. Thus $w(N_G(R) \cap N_G(D), b, d, \rho, [r], R) = w(N_G(R), b, d, \rho, [r], R) = 0$ and (7.1) holds. □

Suppose that $\hat{G} = 2.\mathbb{B}$, and that $B \in \mathrm{Blk}(\hat{G})$ with $D(B) \cong p^2$, so that $\mathrm{plr}(B) = 2$. Thus Uno's projective conjecture for B is equivalent to the equation

$$\mathrm{k}(2.\mathbb{B}, B, d, \rho, [r]) = \mathrm{k}(N_{2.\mathbb{B}}(D(B)), B, d, \rho, [r]). \tag{7.2}$$

Note that if ρ is the trivial character, then B is a block of $\mathbb{B}$.

The tables listing the degrees of the irreducible characters referred to in the proof of Theorems 7.2 and 7.3 are available in Appendix A.

Theorem 7.2. *Let B be a p-block of the Baby Monster $G = \mathbb{B}$ with a positive defect. If p is odd, then B satisfies Uno's ordinary conjecture.*

Proof. We may suppose that $D(B)$ is non-cyclic; by Lemma 5.2, $B \in \{B_0, B_1, B_2\}$ when $p = 7, 5$ and $B \in \{B_0, B_1, B_2, B_3\}$ when $p = 3$.

Case (1) If $p = 7$, then $D(B) \cong 7^2$, $N_{\mathbb{B}}(D(B)) \cong (2^2 \times 7^2{:}(3 \times 2A_4)){:}2$ and

$$\mathrm{k}(\mathbb{B}, B, d, [r]) = \mathrm{k}(N(D(B)), B, d, [r]) = \begin{cases} 9 & \text{if } d = 2 \text{ and } r = 1, \\ 9 & \text{if } d = 2 \text{ and } r = 2, \\ 9 & \text{if } d = 2 \text{ and } r = 3, \\ 0 & \text{otherwise.} \end{cases} \quad (7.3)$$

Thus (7.2) holds.

Case (2) Suppose that $p = 5$ and $B = B_1$ or B_2. Then $D(B) \cong 5^2$, $N(D(B)) = N(C(10)) \cong 5^2{:}4S_4 \times S_5$ and Theorem 7.2 follows because

$$\mathrm{k}(\mathbb{B}, B, d, [r]) = \mathrm{k}(N(C(10)), B, d, [r]) = \begin{cases} 10 & \text{if } d = 2 \text{ and } r = 1, \\ 10 & \text{if } d = 2 \text{ and } r = 2, \\ 0 & \text{otherwise.} \end{cases}$$

Suppose that $B = B_0$. Since $N_{\mathbb{B}}(C(3)) \cong 5{:}4 \times 5{:}4 \times S_5$, its principal block $b_0 = B_0(N(C(3)))$ has p-local rank one and $N(C(4)) = N(D(b_0))$. Thus (7.1) holds for $b = b_0$ (with $\rho = 1$). Similarly, (7.1) holds for $b_0 = B_0(N(C(10)))$ with $N(C(9)) = N(D(b_0))$, since $N(C(10)) \cong 5^2{:}4S_4 \times S_5$.

We set $\mathrm{k}(i, d, r) = \mathrm{k}(N(C(i)), B, d, [r])$ for integers i, d and r. The values $\mathrm{k}(i, d, r)$ are given in Table 7.

It follows that

$$\sum_{i=1}^{10} (-1)^{|C(i)|} \mathrm{k}(N(C(i)), B_0, d, [r]) = 0.$$

Case (3) Suppose that $p = 3$, so that Uno's projective conjecture is equivalent to Dade's projective conjecture.

If $B = B_1$, B_2 or B_3, then $N(D(B)) = N(C(28)) \cong (3^2{:}D_8 \times U_4(3){:}2^2).2$ and Theorem 7.2 follows because

$$\mathrm{k}(\mathbb{B}, B, d) = \mathrm{k}(N(C(28)), B, d) = \begin{cases} 9 & \text{if } d = 2, \\ 0 & \text{otherwise.} \end{cases}$$

Table 7: Values of $\mathrm{k}(i, d, r)$ when $p = 5$ and $B = B_0$.

Defect d	6	5	5	4	4	3	3	otherwise
Value r	1	1	2	1	2	1	2	otherwise
$\mathrm{k}(1, d, r)$	25	22	22	1	0	6	3	0
$\mathrm{k}(2, d, r) = \mathrm{k}(5, d, r)$	0	0	0	50	50	5	20	0
$\mathrm{k}(6, d, r)$	25	8	4	1	0	1	0	0
$\mathrm{k}(7, d, r)$	25	8	4	4	4	1	0	0
$\mathrm{k}(8, d, r)$	25	22	22	4	4	6	3	0

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Suppose that $B = B_0$; suppose also that $C \in \mathcal{R}^0$ with $\mathrm{d}(N(C)) = 8$, so that $C =_G C(i)$ for $13 \leqslant i \leqslant 16$ or $25 \leqslant i \leqslant 28$. Set $\mathrm{k}(i, d) = \mathrm{k}(N(C(i)), B, d)$. The values $\mathrm{k}(i, d)$ are given in Table 8.

It follows that

$$\sum_{\mathrm{d}(N(C))=8} (-1)^{|C|}\mathrm{k}(N(C), B_0, d) = 0.$$

Suppose that $C \in \mathcal{R}^0$ with $\mathrm{d}(N(C)) = 10$, so that $C =_G C(i)$ for $17 \leqslant i \leqslant 24$. The values $\mathrm{k}(i, d)$ are given in Table 9.

It follows that

$$\sum_{\mathrm{d}(N(C))=10} (-1)^{|C|}\mathrm{k}(N(C), B_0, d) = 0.$$

Table 8: Values of $\mathrm{k}(i, d)$ when $p = 3$ and $\mathrm{d}(N(C(i))) = 8$.

Defect d	8	7	6	5	otherwise
$\mathrm{k}(13, d)$	243	108	108	36	0
$\mathrm{k}(14, d)$	243	108	135	36	0
$\mathrm{k}(15, d)$	243	108	135	0	0
$\mathrm{k}(16, d)$	243	108	108	0	0
$\mathrm{k}(25, d)$	162	54	63	0	0
$\mathrm{k}(26, d)$	162	54	81	0	0
$\mathrm{k}(27, d)$	162	81	81	27	0
$\mathrm{k}(28, d)$	162	81	63	27	0

Table 9: Values of $\mathrm{k}(i, d)$ when $p = 3$ and $\mathrm{d}(N(C(i))) = 10$.

Defect d	10	9	8	7	6	5	otherwise
$\mathrm{k}(17, d)$	81	117	114	9	36	0	0
$\mathrm{k}(18, d)$	81	117	39	9	36	12	0
$\mathrm{k}(19, d)$	54	36	39	9	0	0	0
$\mathrm{k}(20, d)$	54	36	138	9	0	0	0
$\mathrm{k}(21, d)$	81	144	39	54	45	12	0
$\mathrm{k}(22, d)$	54	63	39	54	9	0	0
$\mathrm{k}(23, d)$	54	63	138	54	0	0	0
$\mathrm{k}(24, d)$	81	144	114	54	36	0	0

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Suppose that $C \in \mathcal{R}^0$ with $\mathrm{d}(N(C)) = 12$, so that $C =_G C(i)$ for $5 \leqslant i \leqslant 8$ or $29 \leqslant i \leqslant 36$. The values $\mathrm{k}(i, d)$ are given in Table 10.

It follows that

$$\sum_{\mathrm{d}(N(C))=12} (-1)^{|C|}\mathrm{k}(N(C), B_0, d) = \begin{cases} 9 & \text{if } d = 7, \\ 0 & \text{otherwise.} \end{cases}$$

Suppose that $C \in \mathcal{R}^0$ with $\mathrm{d}(N(C)) = 13$, so $C =_G C(i)$ for $1 \leqslant i \leqslant 4$ or $9 \leqslant i \leqslant 12$. The values $\mathrm{k}(i, d)$ are given in Table 11.

Table 10: Values of $\mathrm{k}(i, d)$ when $p = 3$ and $\mathrm{d}(N(C(i))) = 12$.

Defect d	12	11	10	9	8	7	6	otherwise
$\mathrm{k}(5, d)$	81	27	72	81	2	0	0	0
$\mathrm{k}(6, d)$	81	27	36	24	2	0	0	0
$\mathrm{k}(7, d)$	81	72	36	24	36	9	0	0
$\mathrm{k}(8, d)$	81	72	72	81	36	0	0	0
$\mathrm{k}(29, d)$	81	27	36	18	1	0	0	0
$\mathrm{k}(30, d)$	81	27	81	69	1	0	0	0
$\mathrm{k}(31, d)$	81	87	81	69	30	0	0	0
$\mathrm{k}(32, d)$	81	87	36	18	30	9	0	0
$\mathrm{k}(33, d)$	81	72	45	31	36	9	9	0
$\mathrm{k}(34, d)$	81	27	45	31	2	0	9	0
$\mathrm{k}(35, d)$	81	27	72	81	2	0	0	0
$\mathrm{k}(36, d)$	81	72	72	81	36	0	0	0

Table 11: Values of $\mathrm{k}(i, d)$ when $p = 3$ and $\mathrm{d}(N(C(i))) = 13$.

Defect d	13	12	11	10	9	8	7	6	5	otherwise
$\mathrm{k}(1, d)$	27	39	13	30	16	0	1	9	7	0
$\mathrm{k}(2, d)$	27	39	38	30	34	16	13	9	7	0
$\mathrm{k}(3, d)$	27	48	38	90	51	16	0	0	0	0
$\mathrm{k}(4, d)$	27	48	13	90	36	0	0	0	0	0
$\mathrm{k}(9, d)$	27	33	8	90	60	2	0	0	0	0
$\mathrm{k}(10, d)$	27	24	8	48	18	2	1	0	0	0
$\mathrm{k}(11, d)$	27	24	41	48	36	26	4	0	0	0
$\mathrm{k}(12, d)$	27	33	41	90	75	26	0	0	0	0

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It follows that

$$\sum_{\mathrm{d}(N(C))=13} (-1)^{|C|}\mathrm{k}(N(C), B_0, d) = \begin{cases} -9 & \text{if } d = 7, \\ 0 & \text{otherwise.} \end{cases}$$

Theorem 7.2 follows for B_0. □

THEOREM 7.3. *Let B be a p-block of the covering group $\hat{G} = 2.\mathbb{B}$ of the Baby Monster $\mathbb{B}$ with a positive defect. If p is odd, then B satisfies Uno's projective conjecture.*

Proof. We may suppose that $D(B)$ is non-cyclic, and that $\mathrm{Irr}(B) \not\subseteq \mathrm{Irr}(\mathbb{B})$, so that $B = \hat{B}$.

Case (1) If $p = 7$, then $N_{2.\mathbb{B}}(D(B)) \cong 2.(2^2 \times 7^2{:}(3 \times 2A_4)){:}2$. The values of $\mathrm{k}(2.\mathbb{B}, B, d, \xi, [r])$ and $\mathrm{k}(N_{2.\mathbb{B}}(D(B)), B, d, \xi, [r])$ are the same, and are given by (7.3).

Case (2) Suppose that $p = 5$, so that $N_{2.\mathbb{B}}(C(i))$ has a unique block $\hat{b}(i)$ such that $\hat{b}(i)^{N_{2.\mathbb{B}}(C(i))} = \hat{B}$. By Lemma 7.1, equation (7.1) holds for $b = \hat{b}(10)$ and $\hat{b}(3)$ (with $\rho = \xi$). Set $\mathrm{k}(i, d, r) = \mathrm{k}(N_{2.\mathbb{B}}(C(i)), \hat{B}, d, \xi, [r])$ for integers i, d and r. The values $\mathrm{k}(i, d, r)$ are given in Table 12.

Table 12: Values of $\mathrm{k}(i, d, r)$ when $p = 5$ and $B = \hat{B}$.

Defect d	6	5	5	4	4	3	3	otherwise
Value r	2	1	2	1	2	1	2	otherwise
$\mathrm{k}(1, d, r)$	25	5	12	0	1	5	1	0
$\mathrm{k}(2, d, r) = \mathrm{k}(5, d, r)$	0	0	0	11	26	4	9	0
$\mathrm{k}(6, d, r)$	25	4	8	0	1	0	1	0
$\mathrm{k}(7, d, r)$	25	4	8	4	4	0	1	0
$\mathrm{k}(8, d, r)$	25	5	12	4	4	5	1	0

Table 13: Values of $\mathrm{k}(i, d)$ when $p = 3$ and $\mathrm{d}(N_{2.\mathbb{B}}(C(i))) = 8$.

Defect d	8	7	6	5	otherwise
$\mathrm{k}(13, d)$	87	18	27	12	0
$\mathrm{k}(14, d)$	87	18	63	12	0
$\mathrm{k}(15, d)$	87	42	63	0	0
$\mathrm{k}(16, d)$	87	42	27	0	0
$\mathrm{k}(25, d)$	57	18	18	0	0
$\mathrm{k}(26, d)$	57	18	30	0	0
$\mathrm{k}(27, d)$	57	12	30	6	0
$\mathrm{k}(28, d)$	57	12	18	6	0

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It follows that

$$\sum_{i=1}^{10}(-1)^{|C(i)|}\mathrm{k}(N_{2.\mathbb{B}}(C(i)), \hat{B}, d, \xi, [r]) = 0.$$

Case (3) Suppose that $p = 3$, so that Uno's projective conjecture is equivalent to Dade's projective conjecture. Set $\mathrm{k}(i, d) = \mathrm{k}(N_{2.\mathbb{B}}(C(i)), \hat{B}, d, \xi)$.

Suppose that $C = C(i)$ is a chain with $\mathrm{d}(N_{2.\mathbb{B}}(C)) = 8$. Then $13 \leqslant i \leqslant 16$ or $25 \leqslant i \leqslant 28$, and the values $\mathrm{k}(i, d)$ are given in Table 13.

It follows that

$$\sum_{\mathrm{d}(N_{2.\mathbb{B}}(C))=8}(-1)^{|C|}\mathrm{k}(N_{2.\mathbb{B}}(C), \hat{B}, d, \xi) = 0.$$

Suppose that $C = C(i)$ is a chain with $\mathrm{d}(N_{2.\mathbb{B}}(C)) = 10$. Then $17 \leqslant i \leqslant 24$ and the values $\mathrm{k}(i, d)$ are given in Table 14.

It follows that

$$\sum_{\mathrm{d}(N_{2.\mathbb{B}}(C))=10}(-1)^{|C|}\mathrm{k}(N_{2.\mathbb{B}}(C), \hat{B}, d, \xi) = 0.$$

Suppose that $C = C(i)$ is a chain with $\mathrm{d}(N_{2.\mathbb{B}}(C)) = 12$. Then $5 \leqslant i \leqslant 8$ or $29 \leqslant i \leqslant 36$, and the values $\mathrm{k}(i, d)$ are given in Table 15.

It follows that

$$\sum_{\mathrm{d}(N_{2.\mathbb{B}}(C))=12}(-1)^{|C|}\mathrm{k}(N_{2.\mathbb{B}}(C), \hat{B}, d, \xi) = \begin{cases} 6 & \text{if } d = 7, \\ 0 & \text{otherwise.} \end{cases}$$

Suppose that $C = C(i)$ is a chain with $\mathrm{d}(N_{2.\mathbb{B}}(C)) = 13$. Then $1 \leqslant i \leqslant 4$ or $9 \leqslant i \leqslant 12$, and the values $\mathrm{k}(i, d)$ are given in Table 16.

It follows that

$$\sum_{\mathrm{d}(N_{2.\mathbb{B}}(C))=13}(-1)^{|C|}\mathrm{k}(N_{2.\mathbb{B}}(C), \hat{B}, d, \xi) = \begin{cases} -6 & \text{if } d = 7, \\ 0 & \text{otherwise.} \end{cases}$$

Theorem 7.3 follows for $\hat{B}$. □

Table 14: Values of $\mathrm{k}(i, d)$ when $p = 3$ and $\mathrm{d}(N_{2.\mathbb{B}}(C(i))) = 10$.

Defect d	10	9	8	7	6	otherwise
$\mathrm{k}(17, d, r)$	33	36	48	6	9	0
$\mathrm{k}(18, d, r)$	33	36	3	6	9	0
$\mathrm{k}(19, d, r)$	54	36	39	9	0	0
$\mathrm{k}(20, d, r)$	54	36	138	9	0	0
$\mathrm{k}(21, d, r)$	33	51	3	27	18	0
$\mathrm{k}(22, d, r)$	54	63	39	54	9	0
$\mathrm{k}(23, d, r)$	54	63	138	54	0	0
$\mathrm{k}(24, d, r)$	33	51	48	27	9	0

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Table 15: Values of $k(i,d)$ when $p = 3$ and $d(N_{2.\mathbb{B}}(C(i))) = 12$.

Defect d	12	11	10	9	8	7	6	otherwise
$k(5,d,r)$	36	12	36	54	2	0	0	0
$k(6,d,r)$	36	12	18	18	2	0	0	0
$k(7,d,r)$	36	36	18	18	24	6	0	0
$k(8,d,r)$	36	36	36	54	24	0	0	0
$k(29,d,r)$	36	12	27	12	1	0	0	0
$k(30,d,r)$	36	12	54	51	1	0	0	0
$k(31,d,r)$	36	51	54	51	24	0	0	0
$k(32,d,r)$	36	51	27	12	24	6	0	0
$k(33,d,r)$	36	36	18	10	24	6	3	0
$k(34,d,r)$	36	12	18	10	2	0	3	0
$k(35,d,r)$	36	12	36	54	2	0	0	0
$k(36,d,r)$	36	36	36	54	24	0	0	0

Table 16: Values of $k(i,d)$ when $p = 3$ and $d(N_{2.\mathbb{B}}(C(i))) = 13$.

Defect d	13	12	11	10	9	8	7	6	5	otherwise
$k(1,d)$	27	9	3	12	1	0	1	3	2	0
$k(2,d)$	27	9	14	12	19	6	10	3	2	0
$k(3,d)$	27	18	14	63	30	6	0	0	0	0
$k(4,d)$	27	18	3	63	15	0	0	0	0	0
$k(9,d)$	27	33	8	90	60	2	0	0	0	0
$k(10,d)$	27	24	8	48	18	2	1	0	0	0
$k(11,d)$	27	24	41	48	36	26	4	0	0	0
$k(12,d)$	27	33	41	90	75	26	0	0	0	0

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Appendix A. *Degrees of character tables for chain normalisers of* $\mathbb{B}$ *and* $2\mathbb{B}$

Let ξ be the faithful linear character of $Z(2.\mathbb{B})$, let $Z(2.\mathbb{B}) \leqslant H \leqslant 2.\mathbb{B}$, and let $\mathrm{Irr}(H \mid \xi)$ be the character of $\mathrm{Irr}(H)$ covering ξ.

Table A.1: The degrees of characters in $\mathrm{Irr}(3^{1+8}.2_-^{1+6}.U_4(2).2)$.

Degree	1	6	8	10	15	20	24	27	30
Number	2	2	2	1	4	3	2	2	2
Degree	36	48	60	64	80	81	90	108	120
Number	4	2	3	2	2	2	1	2	4
Degree	135	160	162	180	192	240	270	324	360
Number	4	3	2	8	2	2	5	4	4
Degree	405	480	512	540	576	640	648	720	810
Number	2	3	2	2	2	1	2	1	2
Degree	972	1296	1440	1620	2430	2880	3240	3888	4374
Number	1	1	4	2	2	4	1	1	1
Degree	4860	5120	5760	5832	6480	7290	7776	8640	9720
Number	3	4	5	2	4	2	1	6	1
Degree	10240	10368	11520	12960	13122	15360	17280	17496	19440
Number	2	1	4	6	3	4	7	1	2
Degree	21870	23040	25920	29160	30720	31104	34560	38880	40960
Number	4	1	1	4	4	1	2	3	2
Degree	43740	51840	52488	58320	61440	65610	77760	81920	82944
Number	2	3	2	4	5	1	1	1	1
Degree	87480	93312	104976						
Number	1	1	1						

Table A.2: The degrees of characters in $\mathrm{Irr}(2.3^{1+8}.2_-^{1+6}.U_4(2).2 \mid \xi)$.

Degree	8	20	40	60	64	72	80	108	120
Number	1	2	2	2	3	1	2	2	1
Degree	160	288	320	432	480	512	540	576	640
Number	2	2	2	2	2	2	2	3	2
Degree	648	720	960	3240	5184	5760	5832	9720	10240
Number	3	2	1	5	2	2	2	3	4
Degree	10368	11520	12960	17280	17496	20480	23328	25920	34560
Number	1	4	1	2	1	2	4	5	3
Degree	40960	46656	52488	61440	69120	69984	77760	81920	82944
Number	2	2	2	2	1	1	3	1	1
Degree	87480	93312	103680	116640	122880				
Number	1	1	1	1	1				

Table A.3: The degrees of characters in $\mathrm{Irr}(3^{1+8}.3^3.2^4.S_3)$.

Degree	1	2	3	4	6	8	12	16	18	24	32	36	48	54
Number	8	8	8	2	12	4	14	4	8	9	1	10	4	12
Degree	72	96	108	144	162	216	324	432	486	648	864	972	1296	1944
Number	16	1	26	4	8	34	16	16	4	21	2	8	6	4

Table A.4: The degrees of characters in $\mathrm{Irr}(2.3^{1+8}.3^3.2^4.S_3 \mid \xi)$.

Degree	2	4	8	12	16	24	32	36	48	72
Number	8	8	6	4	4	8	1	4	5	2
Degree	96	108	144	216	324	432	648	864	1296	1944
Number	1	8	8	34	4	19	18	2	8	6

Table A.5: The degrees of characters in $\mathrm{Irr}(3^2.3^3.3^6(S_4 \times 2S_4))$.

Degree	1	2	3	4	6	8	9	12	24	32	48
Number	4	8	8	5	8	1	4	2	8	4	14
Degree	54	64	96	108	128	144	162	192	216	288	324
Number	4	4	5	13	1	4	4	2	21	5	5
Degree	384	432	648	864	1296	1728	2592	3456	5184		
Number	1	19	6	21	11	10	8	2	2		

Table A.6: The degrees of characters in $\mathrm{Irr}(2.3^2.3^3.3^6(S_4 \times 2S_4) \mid \xi)$.

Degree	2	4	6	8	12	16	32	48	64
Number	4	8	4	5	2	1	4	4	4
Degree	96	108	128	192	216	288	324	384	432
Number	4	2	1	3	12	2	2	1	17
Degree	576	648	864	1296	1728	2592	3456	5184	
Number	1	3	17	1	13	6	2	3	

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Table A.7: The degrees of characters in $\mathrm{Irr}(3^3.3^6.3^2.(D_8 \times 2S_4))$.

Degree	1	2	3	4	6	8	16	18	32
Number	8	14	8	7	2	9	22	4	17
Degree	36	48	54	64	72	96	108	144	192
Number	17	4	4	4	19	9	9	11	4
Degree	216	288	432	576	648	864	1152	1728	3456
Number	16	14	20	6	2	21	1	10	1

Table A.8: The degrees of characters in $\mathrm{Irr}(2.3^3.3^6.3^2.(D_8 \times 2S_4) \mid \xi)$.

Degree	2	4	6	8	16	32	36	64	72	96	108
Number	4	6	4	2	4	14	2	6	11	2	2
Degree	144	192	216	288	432	576	648	864	1152	1728	3456
Number	10	6	10	3	10	9	2	20	1	11	1

Table A.9: The degrees of characters in $\mathrm{Irr}(3^3.3^6.(L_3(3) \times D_8))$.

Degree	1	2	12	13	16	24	26	27	32
Number	4	1	4	4	16	1	13	4	4
Degree	39	52	54	78	104	156	208	234	416
Number	4	11	1	1	12	8	4	4	8
Degree	468	624	702	832	936	1248	1404	1872	2496
Number	9	4	4	4	7	1	1	7	4
Degree	2808	3744	5616	7488	8424	14976			
Number	8	6	6	2	2	1			

Table A.10: The degrees of characters in $\mathrm{Irr}(2.3^3.3^6.(L_3(3) \times D_8) \mid \xi)$.

Degree	2	24	26	32	52	54	78	104	208	312	468	832
Number	2	2	2	8	6	2	2	6	6	2	2	6
Degree	936	1248	1404	1872	2496	2808	3744	5616	7488	8424	14976	
Number	5	2	2	4	4	8	3	6	3	2	1	

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Table A.11: The degrees of characters in $\mathrm{Irr}(3^{1+8}.3^2.(2S_4 \times D_8))$.

Degree	1	2	3	4	6	8	12	16	32	48
Number	8	14	8	15	2	21	8	10	9	16
Degree	64	72	96	144	162	192	288	324	384	432
Number	4	8	24	14	4	12	9	9	2	8
Degree	486	576	648	864	972	1152	1296	1728	1944	2592
Number	4	4	13	14	1	1	9	2	4	1

Table A.12: The degrees of characters in $\mathrm{Irr}(2.3^{1+8}.3^2.(2S_4 \times D_8) \mid \xi)$.

Degree	2	4	6	8	16	24	32	64	96	144	192	288
Number	4	6	4	8	10	2	2	6	12	4	16	8
Degree	324	384	576	648	864	972	1152	1296	1728	1944	2592	
Number	2	2	5	11	16	2	1	10	2	4	1	

Table A.13: The degrees of characters in $\mathrm{Irr}(3^{1+8}.3^2.3.(2^2 \times D_8))$.

Degree	1	2	4	6	8	12	16	18	24	36	48	54
Number	16	20	24	8	17	18	4	8	32	30	12	8
Degree	72	96	108	144	162	216	288	324	432	648	864	1296
Number	25	2	18	8	8	42	1	10	12	16	1	2

Table A.14: The degrees of characters in $\mathrm{Irr}(2.3^{1+8}.3^2.3.(2^2 \times D_8) \mid \xi)$.

Degree	2	4	8	12	16	24	36	48	72	96
Number	8	8	14	4	6	12	4	18	20	2
Degree	108	144	216	288	324	432	648	864	1296	
Number	4	11	34	1	4	15	18	1	2	

Table A.15: The degrees of characters in $\mathrm{Irr}(3^2.3^3.3^6.3.(2 \times 2S_4))$.

Degree	1	2	3	4	6	8	16	24	32	48	54	96
Number	4	8	4	5	2	5	4	16	1	10	20	1
Degree	108	144	162	216	324	432	486	648	864	1296	2592	
Number	24	8	14	16	7	26	2	24	4	14	1	

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Table A.16: The degrees of characters in $\mathrm{Irr}(2 \times 3^2.3^3.3^6.3.(2 \times 2S_4) \mid \xi)$.

Degree	1	2	3	4	6	8	16	24	32	48	54	96
Number	4	8	4	5	2	5	4	16	1	10	20	1
Degree	108	144	162	216	324	432	486	648	864	1296	2592	
Number	24	8	14	16	7	26	2	24	4	14	1	

Table A.17: The degrees of characters in $\mathrm{Irr}(3^3.3.3^3.3^3.(L_3(3) \times 2))$.

Degree	1	12	13	16	26	27	39	52	54
Number	2	2	2	8	8	2	2	3	1
Degree	78	104	156	208	234	416	468	624	648
Number	8	1	6	2	2	1	3	6	1
Degree	702	864	936	1404	1458	2106	4212	5616	6318
Number	13	4	3	16	1	11	6	12	2

Table A.18: The degrees of characters in $\mathrm{Irr}(2 \times 3^3.3.3^3.3^3.(L_3(3) \times 2) \mid \xi)$.

Degree	1	12	13	16	26	27	39	52	54
Number	2	2	2	8	8	2	2	3	1
Degree	78	104	156	208	234	416	468	624	648
Number	8	1	6	2	2	1	3	6	1
Degree	702	864	936	1404	1458	2106	4212	5616	6318
Number	13	4	3	16	1	11	6	12	2

Table A.19: The degrees of characters in $\mathrm{Irr}(3^{1+8}.3^{1+2}.(2S_4 \times 2))$.

Degree	1	2	3	4	6	8	12	16	18	32	36	48	54	72
Number	4	6	4	2	6	8	6	6	4	1	6	8	4	16
Degree	108	144	162	216	288	324	432	486	648	864	972	1296	1458	
Number	4	12	8	25	3	6	14	14	12	1	12	10	4	

Table A.20: The degrees of characters in $\mathrm{Irr}(2 \times 3^{1+8}.3^{1+2}.(2S_4 \times 2) \mid \xi)$.

Degree	1	2	3	4	6	8	12	16	18	32	36	48	54	72
Number	4	6	4	2	6	8	6	6	4	1	6	8	4	16
Degree	108	144	162	216	288	324	432	486	648	864	972	1296	1458	
Number	4	12	8	25	3	6	14	14	12	1	12	10	4	

Table A.21: The degrees of characters in $\mathrm{Irr}(3^{1+8}.3^{1+2}.3.2^3)$.

Degree	1	2	4	6	8	12	18	24	36
Number	8	12	6	20	1	12	16	1	22
Degree	54	72	108	162	216	324	486	648	972
Number	52	3	34	48	4	26	24	1	2

Table A.22: The degrees of characters in $\mathrm{Irr}(2 \times 3^{1+8}.3^{1+2}.3.2^3 \mid \xi)$.

Degree	1	2	4	6	8	12	18	24	36
Number	8	12	6	20	1	12	16	1	22
Degree	54	72	108	162	216	324	486	648	972
Number	52	3	34	48	4	26	24	1	2

Table A.23: The degrees of characters in $\mathrm{Irr}(S_3 \times (S_3 \times U_4(3){:}2).2)$.

Degree	1	2	4	21	35	42	70	84	90	140	180	189
Number	16	16	4	16	32	16	32	4	16	24	16	16
Degree	210	280	315	360	378	420	560	630	729	756	840	896
Number	16	16	32	4	16	32	20	32	16	4	20	16
Degree	1120	1260	1280	1458	1680	1792	2240	2560	2916	3584	4480	5120
Number	20	8	8	16	4	16	8	8	4	4	1	2

Table A.24: The degrees of characters in $\mathrm{Irr}(2.(S_3 \times (S_3 \times U_4(3){:}2).2 \mid \xi)$.

Degree	40	80	112	140	224	240	280	448	480	560
Number	4	8	8	12	8	4	14	2	8	7
Degree	840	1008	1080	1120	1260	1680	1792	2016	2160	2240
Number	2	8	4	4	4	4	4	8	8	4
Degree	2520	2560	3584	4032	4480	5040	5120	7168		
Number	4	2	4	2	1	1	4	1		

Table A.25: The degrees of characters in $\mathrm{Irr}(S_3 \times S_3 \times 3^{1+4}2_-^{1+4}.S_3)$.

Degree	1	2	3	4	6	8	12	16	18	32	36
Number	32	48	32	32	32	16	8	38	48	49	48
Degree	48	54	64	72	96	108	128	144	192	216	288
Number	16	16	24	24	16	16	4	12	4	4	3

Table A.26: The degrees of characters in $\mathrm{Irr}(2.(S_3 \times S_3 \times 3^{1+4}2_-^{1+4}.S_3) \mid \xi)$.

Degree	4	8	12	16	24	32	36	64
Number	14	24	2	6	4	17	12	18
Degree	72	96	108	128	144	192	216	288
Number	36	4	4	8	12	8	8	3

Table A.27: The degrees of characters in $\mathrm{Irr}(S_3 \times 3^2.3^2.3^3.2^3.2^2)$.

Degree	1	2	4	8	12	16	18	24	32	36	48	72	96	144
Number	32	56	68	58	32	25	48	48	4	60	24	24	4	3

Table A.28: The degrees of characters in $\mathrm{Irr}(2.(S_3 \times 3^2.3^2.3^3.2^3.2^2) \mid \xi)$.

Degree	2	4	8	16	24	32	36	48	72	96	144
Number	8	18	34	19	16	8	24	18	36	8	3

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Table A.29: The degrees of characters in $\mathrm{Irr}(S_3 \times S_3 \times 3^4{:}2S_6)$.

Degree	1	2	4	5	9	10	16	18	20	30	32	36
Number	16	16	4	32	16	48	8	16	40	32	8	4
Degree	40	60	64	80	90	120	160	180	240	320	360	
Number	24	48	2	24	32	24	17	32	4	4	8	

Table A.30: The degrees of characters in $\mathrm{Irr}(2.(S_3 \times S_3 \times 3^4{:}2S_6) \mid \xi)$.

Degree	2	4	8	10	18	20	32	36	40	60
Number	4	4	1	8	4	12	2	4	14	16
Degree	64	80	120	160	180	240	320	360	720	
Number	4	1	18	11	8	8	6	8	2	

Table A.31: The degrees of characters in $\mathrm{Irr}(S_3 \times 3^5{:}(2 \times U_4(2){:}2))$.

Degree	1	2	6	10	12	15	20	24	30	40	48
Number	8	4	8	4	4	16	14	8	16	6	4
Degree	60	64	72	80	81	90	120	128	144	160	162
Number	16	8	8	12	8	12	6	4	4	10	4
Degree	180	240	320	360	480	540	640	648	720	810	960
Number	14	8	2	30	12	4	4	8	29	8	14
Degree	1080	1152	1280	1296	1440	1620	1920	2160	2304	2560	2880
Number	4	4	4	4	10	4	5	1	2	1	1

Table A.32: The degrees of characters in $\mathrm{Irr}(2.(S_3 \times 3^5{:}(2 \times U_4(2){:}2)) \mid \xi)$.

Degree	2	4	12	20	24	30	40	48	60	80
Number	2	1	2	3	1	4	5	2	4	1
Degree	96	120	128	144	160	162	180	240	256	288
Number	1	6	2	2	9	2	9	1	1	1
Degree	320	324	360	480	720	960	1080	1280	1296	1440
Number	3	1	8	2	11	9	5	5	2	12
Degree	1620	1920	2160	2304	2560	2592	2880	3240	3840	
Number	2	3	1	3	1	1	2	1	1	

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Table A.33: The degrees of characters in $\mathrm{Irr}(S_3 \times \mathrm{Fi}_{22}{:}2)$.

Degree	1	2	78	156	429	858	1001
Number	4	2	4	2	4	2	4
Degree	1430	2002	2860	3003	3080	6006	6160
Number	4	2	2	4	4	2	2
Degree	10725	13650	21450	27300	30030	32032	43680
Number	4	4	2	2	4	4	4
Degree	45045	48048	50050	60060	64064	75075	81081
Number	4	4	8	2	2	12	4
Degree	87360	90090	96096	100100	114400	150150	162162
Number	2	2	2	4	4	10	2
Degree	205920	228800	277200	289575	300300	320320	360855
Number	4	2	2	4	6	4	4
Degree	370656	411840	450450	554400	576576	577368	579150
Number	4	2	8	1	4	4	6
Degree	600600	640640	675675	720720	721710	741312	800800
Number	6	2	4	4	2	2	8
Degree	852930	900900	938223	972972	1153152	1154736	1158300
Number	4	4	4	4	2	2	2
Degree	1164800	1201200	1351350	1360800	1372800	1441440	1441792
Number	2	8	2	4	4	2	4
Degree	1601600	1705860	1791153	1876446	1945944	1965600	2027025
Number	4	2	4	6	2	2	4
Degree	2050048	2316600	2329600	2402400	2555904	2721600	2729376
Number	4	4	1	7	8	2	4
Degree	2745600	2883584	3582306	3752892	3931200	4054050	4100096
Number	2	2	2	2	1	2	2
Degree	4633200	4804800	5111808	5458752			
Number	2	2	4	2			

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Table A.34: The degrees of characters in $\mathrm{Irr}((S_3 \times 2.\mathrm{Fi}_{22}){:}2) \mid \xi)$.

Degree	704	4160	8320	11648	23296	27456	54912
Number	3	2	1	4	2	2	4
Degree	96096	192192	211200	246400	266112	292864	457600
Number	2	1	6	3	3	2	3
Degree	471744	585728	640640	800800	873600	943488	960960
Number	2	1	2	2	2	1	2
Degree	1281280	1372800	1601600	1747200	1830400	1921920	2059200
Number	1	2	1	1	2	1	3
Degree	2402400	2594592	2745600	2883584	3326400	3660800	4392960
Number	2	2	1	3	2	1	2
Degree	4717440	4804800	5111808	5189184	6652800	8785920	9434880
Number	2	3	2	1	1	1	1
Degree	9609600	10223616					
Number	1	1					

Table A.35: The degrees of characters in $\mathrm{Irr}(S_3 \times 3^{3+3}{:}L_3(3))$.

Degree	1	2	12	13	16	24	26	27	32
Number	2	1	2	2	8	1	13	2	4
Degree	39	52	54	78	104	156	208	234	312
Number	2	12	1	9	3	10	6	14	3
Degree	416	468	624	702	936	1248	1404		
Number	3	19	6	4	6	3	2		

Table A.36: The degrees of characters in $\mathrm{Irr}(2 \times S_3 \times 3^{3+3}{:}L_3(3) \mid \xi)$.

Degree	1	2	12	13	16	24	26	27	32
Number	2	1	2	2	8	1	13	2	4
Degree	39	52	54	78	104	156	208	234	312
Number	2	12	1	9	3	10	6	14	3
Degree	416	468	624	702	936	1248	1404		
Number	3	19	6	4	6	3	2		

Table A.37: The degrees of characters in $\mathrm{Irr}(S_3 \times 3^5.3^{1+2}.2S_4)$.

Degree	1	2	3	4	6	8	12	16	18	24
Number	4	8	4	5	8	17	9	16	20	3
Degree	32	36	48	54	72	96	108	144	288	
Number	4	28	8	6	45	4	3	36	9	

Table A.38: The degrees of characters in $\mathrm{Irr}(2.(S_3 \times 3^5.3^{1+2}.2S_4) \mid \xi)$.

Degree	1	2	3	4	6	8	12	16	18	24
Number	4	8	4	5	8	17	9	16	20	3
Degree	32	36	48	54	72	96	108	144	288	
Number	4	28	8	6	45	4	3	36	9	

Table A.39: The degrees of characters in $\mathrm{Irr}(S_3 \times 3^{1+6}2^{3+4}{:}3^2.2^2)$.

Degree	1	2	4	6	8	9	12	16	18
Number	8	12	14	12	13	8	20	6	4
Degree	24	32	36	48	54	72	96	108	128
Number	27	1	2	18	12	17	8	12	8
Degree	144	192	216	256	324	384	432	486	512
Number	8	20	15	12	18	25	12	4	6
Degree	648	768	864	972	1024	1296	1536	1944	
Number	21	12	3	6	1	6	2	2	

Table A.40: The degrees of characters in $\mathrm{Irr}(2.(S_3 \times 3^{1+6}2^{3+4}{:}3^2.2^2) \mid \xi)$.

Degree	4	8	12	16	24	48	96	192	216	256
Number	4	10	4	4	8	5	14	4	18	6
Degree	288	384	432	512	648	768	1024	1296	1536	2592
Number	3	5	9	8	6	6	1	9	5	3

Table A.41: The degrees of characters in $\mathrm{Irr}(S_3 \times 3^{1+6}.3{:}2S_4)$.

Degree	1	2	3	4	6	8	12	16	24	32
Number	4	14	4	14	4	8	1	10	24	4
Degree	48	54	72	96	108	144	162	216	288	324
Number	24	18	24	6	27	14	6	9	1	3

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Table A.42: The degrees of characters in $\mathrm{Irr}(2.(S_3 \times 3^{1+6}.3{:}2S_4) \mid \xi)$.

Degree	1	2	3	4	6	8	12	16	24	32
Number	4	14	4	14	4	8	1	10	24	4
Degree	48	54	72	96	108	144	162	216	288	324
Number	24	18	24	6	27	14	6	9	1	3

Table A.43: The degrees of characters in $\mathrm{Irr}(S_3 \times 3^{1+6}.3^2.2^2)$.

Degree	1	2	4	6	8	12	18	24	36	54	72	108
Number	8	24	18	28	4	28	72	7	56	36	10	18

Table A.44: The degrees of characters in $\mathrm{Irr}(2.(S_3 \times 3^{1+6}.3^2.2^2) \mid \xi)$.

Degree	1	2	4	6	8	12	18	24	36	54	72	108
Number	8	24	18	28	4	28	72	7	56	36	10	18

Table A.45: The degrees of characters in $\mathrm{Irr}(S_3 \times 3^5.3^3.(2 \times S_4).2)$.

Degree	1	2	3	4	6	8	12	16
Number	16	24	16	12	32	10	40	12
Degree	18	24	32	36	48	54	64	72
Number	16	32	6	28	17	24	1	42
Degree	96	108	144	162	192	216	288	324
Number	6	24	24	24	1	6	4	12

Table A.46: The degrees of characters in $\mathrm{Irr}(2.(S_3 \times 3^5.3^3.(2 \times S_4).2) \mid \xi)$.

Degree	2	4	6	8	12	16	24	32	36	48
Number	4	10	4	4	8	6	18	8	4	11
Degree	64	72	96	108	144	192	216	288	324	648
Number	1	17	9	18	20	1	9	7	6	3

Table A.47: The degrees of characters in $\mathrm{Irr}((3^2{:}D_8 \times 3^4{:}A_6{:}2^2).2)$.

Degree	1	2	4	8	9	10	16	18	20
Number	8	10	1	8	8	24	8	10	22
Degree	32	36	40	60	72	80	120	128	160
Number	6	1	31	16	8	12	12	4	16
Degree	180	240	320	360	480	640	720		
Number	16	18	4	4	8	8	16		

Table A.48: The degrees of characters in $\mathrm{Irr}(2.((3^2{:}D_8 \times 3^4{:}A_6{:}2^2).2) \mid \xi)$.

Degree	2	4	16	18	20	32	36	40	64	80	120	144
Number	4	3	2	4	8	4	3	8	1	11	4	2
Degree	160	240	256	320	360	480	640	720	960	1280	1440	
Number	2	5	1	8	4	8	4	1	1	1	4	

Table A.49: The degrees of characters in $\mathrm{Irr}(3^3.3^2.3^3.2^3.2^2.2^3)$.

Degree	1	2	4	8	16	18	24	32
Number	16	36	22	36	24	8	16	20
Degree	36	48	64	72	96	144	192	288
Number	30	12	8	12	18	25	8	6

Table A.50: The degrees of characters in $\mathrm{Irr}(2.(3^3.3^2.3^3.2^3.2^2.2^3) \mid \xi)$.

Degree	2	4	8	16	32	36	48	64	72	96	128	144	192	288	384
Number	8	14	8	8	10	4	4	8	11	5	1	3	8	12	1

Table A.51: The degrees of characters in $\mathrm{Irr}((3^2{:}D_8 \times 3^{1+4}{:}2S_4{:}2^2.).2)$.

Degree	1	2	3	4	6	8	12	16	18
Number	16	28	16	20	20	27	2	11	8
Degree	24	32	36	48	54	64	72	96	108
Number	16	20	26	8	8	14	9	10	10
Degree	128	144	192	216	256	288	384	432	576
Number	18	31	1	1	8	1	8	8	6

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Table A.52: The degrees of characters in $\mathrm{Irr}(2.((3^2{:}D_8 \times 3^{1+4}{:}2S_4{:}2^2.).2) \mid \xi)$.

Degree	4	8	12	16	24	32	64	72	96	128
Number	12	15	4	3	1	7	6	8	1	5
Degree	144	192	216	256	288	512	576	768	864	
Number	9	4	4	8	7	1	6	2	2	

Table A.53: The degrees of characters in $\mathrm{Irr}((3^2{:}D_8 \times U_4(3){:}2^2.).2)$.

Degree	1	2	4	8	21	42	70	84	90
Number	8	10	1	8	8	10	16	1	8
Degree	140	168	180	189	210	280	360	378	420
Number	12	8	10	8	8	26	1	10	18
Degree	560	630	720	729	756	840	896	1120	1260
Number	1	16	8	8	1	11	8	20	4
Degree	1280	1458	1512	1680	1792	2240	2520	2560	2916
Number	8	10	8	9	10	5	16	6	1
Degree	3360	3584	4480	5832	7168	8960	10240		
Number	8	1	8	8	8	2	4		

Table A.54: The degrees of characters in $\mathrm{Irr}(2.((3^2{:}D_8 \times U_4(3){:}2^2.).2) \mid \xi)$.

Degree	80	140	224	280	320	448	480	560	840	896
Number	4	4	4	3	2	1	4	4	4	4
Degree	1120	1260	1680	1792	1920	2016	2160	2240	2520	2560
Number	10	4	1	4	2	4	4	4	3	4
Degree	3584	4032	5120	6720	8064	8640	8960	10080	14336	20480
Number	3	1	1	1	4	2	2	2	2	1

Table A.55: The degrees of characters in $\mathrm{Irr}(3^{3+6}.(2^2 \times L_3(3)))$.

Degree	1	12	13	16	26	27	39	52	78	104
Number	4	4	4	16	20	4	4	14	8	7
Degree	156	208	234	416	468	624	702	832	936	1248
Number	2	9	4	6	10	4	4	1	8	4
Degree	1404	1872	2496	2808	3744	5616	7488	8424		
Number	4	6	1	3	6	3	2	1		

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Table A.56: The degrees of characters in $\mathrm{Irr}(2.3^{3+6}.(2^2 \times L_3(3)) \mid \xi)$.

Degree	2	24	26	32	52	54	78	104	156	208	416	468
Number	1	1	1	4	11	1	1	9	4	1	8	7
Degree	832	936	1248	1404	1872	2496	2808	3744	5616	7488	8424	
Number	1	9	5	5	2	1	3	7	3	2	1	

Table A.57: The degrees of characters in $\mathrm{Irr}(3^2.3^3.3^6.(2^2 \times 2S_4))$.

Degree	1	2	3	4	8	16	18	32	36	48	54	64
Number	8	12	8	4	24	24	12	8	24	12	12	1
Degree	72	96	108	144	192	216	288	432	576	648	864	1728
Number	15	6	6	15	1	11	12	19	3	1	16	5

Table A.58: The degrees of characters in $\mathrm{Irr}(2.3^2.3^3.3^6.(2^2 \times 2S_4)) \mid \xi)$.

Degree	2	4	6	8	16	32	36	64	72	96
Number	2	3	2	1	18	11	15	1	18	9
Degree	108	144	192	216	288	432	576	648	864	1728
Number	9	3	1	3	15	17	3	1	17	5

Table A.59: The degrees of characters in $\mathrm{Irr}(3^{3+6}.3^{1+2}.2^4)$.

Degree	1	2	4	6	8	12	16	18	24	36
Number	16	32	24	24	8	36	1	24	20	36
Degree	48	54	72	108	144	162	216	324	432	648
Number	7	8	18	28	3	8	28	12	5	10

Table A.60: The degrees of characters in $\mathrm{Irr}(2.3^{3+6}.3^{1+2}.2^4) \mid \xi)$.

Degree	2	4	8	12	16	24	36	48	72	108	144	216	324	432	648
Number	4	20	11	18	1	26	30	7	21	14	3	32	14	5	10

Table A.61: The degrees of characters in $\mathrm{Irr}(3^6.3^{2+3}.(2^2 \times 2S_4))$.

Degree	1	2	3	4	6	8	12	16	24
Number	8	20	8	18	8	15	2	13	16
Degree	32	48	64	72	96	144	162	192	288
Number	6	28	1	8	18	16	4	7	10
Degree	324	432	486	576	648	864	972	1296	1944
Number	10	8	4	2	11	10	4	5	1

Table A.62: The degrees of characters in $\mathrm{Irr}(2.3^6.3^{2+3}.(2^2 \times 2S_4)) \mid \xi)$.

Degree	2	4	6	8	12	16	32	48	64	96
Number	2	11	2	10	4	3	9	16	1	22
Degree	144	192	288	324	576	648	864	972	1296	1944
Number	14	7	11	7	2	12	12	5	5	1

Table A.63: The degrees of characters in $\mathrm{Irr}(3^{1+8}.3.2^2.2^4.3^2.D_8)$.

Degree	1	2	4	6	8	9	12	16	18	24
Number	8	6	17	8	10	8	8	9	6	2
Degree	32	36	48	64	96	128	144	162	192	256
Number	12	1	8	6	16	9	16	4	4	4
Degree	288	324	384	432	648	768	864	972	1152	1296
Number	8	5	16	8	12	8	8	4	4	6
Degree	1458	1536	1728	1944	2304	2592	2916	3456	3888	5184
Number	4	2	14	4	2	8	5	1	1	1

Table A.64: The degrees of characters in $\mathrm{Irr}(2.3^{1+8}.3.2^2.2^4.3^2.D_8) \mid \xi)$.

Degree	2	4	8	12	16	18	24	32	36	64	96
Number	4	2	10	2	2	4	4	6	2	4	2
Degree	128	192	256	288	324	384	576	648	768	864	1152
Number	2	8	6	4	2	8	2	9	10	2	4
Degree	1296	1536	1728	1944	2304	2592	2916	3456	3888	5184	5832
Number	3	2	4	5	2	9	2	4	1	1	1

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Table A.65: The degrees of characters in $\mathrm{Irr}(3^6{:}(2 \times L_4(3){:}2){:}2)$.

Degree	1	2	39	52	78	90	104	130	180
Number	4	1	4	8	1	4	1	4	1
Degree	260	351	390	416	468	520	702	729	780
Number	8	4	4	8	12	6	1	4	5
Degree	832	936	1040	1170	1280	1458	1560	2080	2340
Number	6	1	12	4	8	1	5	2	4
Degree	2808	3120	4160	4680	7020	8320	9360	11232	12480
Number	4	2	4	3	8	4	6	4	4
Degree	14040	16640	24960	28080	29952	33280	37440	37908	42120
Number	4	1	2	6	4	4	6	4	2

Table A.66: The degrees of characters in $\mathrm{Irr}(3^6{:}(2 \times L_4(3){:}2){:}2) \mid \xi)$.

Degree	2	78	104	180	260	520	702	780	832
Number	2	2	3	2	1	4	2	2	4
Degree	936	1040	1458	1560	1664	2080	2340	2560	3120
Number	4	5	2	2	1	4	1	2	3
Degree	4160	4680	5616	9360	14040	16640	18720	22464	24960
Number	4	4	1	2	2	2	1	1	3
Degree	28080	33280	37440	42120	56160	59904	74880	75816	
Number	3	4	2	2	1	1	1	1	

Table A.67: The degrees of characters in $\mathrm{Irr}(3^6.3^{1+4}.(2 \times 2S_4{:}2).2)$.

Degree	1	2	3	4	6	8	16	18	32
Number	8	14	8	7	2	9	22	4	17
Degree	36	48	54	64	72	96	108	144	192
Number	17	4	4	4	19	9	9	11	4
Degree	216	288	432	576	648	864	1152	1728	3456
Number	16	14	20	6	2	21	1	10	1

Table A.68: The degrees of characters in $\mathrm{Irr}(2.3^6.3^{1+4}.(2 \times 2S_4{:}2).2) \mid \xi)$.

Degree	2	4	6	8	16	32	36	64	72	96	108
Number	4	6	4	2	4	14	2	6	11	2	2
Degree	144	192	216	288	432	576	648	864	1152	1728	3456
Number	10	6	10	3	10	9	2	20	1	11	1

Table A.69: The degrees of characters in $\mathrm{Irr}(3^{3+6}.3^{1+2}.2^3.2^2)$.

Degree	1	2	4	6	8	12	16	18	24	36	48	54
Number	16	20	24	8	17	18	4	8	32	30	12	8
Degree	72	96	108	144	162	216	288	324	432	648	864	1296
Number	25	2	18	8	8	42	1	10	12	16	1	2

Table A.70: The degrees of characters in $\mathrm{Irr}(2.3^{3+6}.3^{1+2}.2^3.3^2) \mid \xi)$.

Degree	2	4	8	12	16	24	36	48	72	96
Number	8	8	14	4	6	12	4	18	20	2
Degree	108	144	216	288	324	432	648	864	1296	
Number	4	11	34	1	4	15	18	1	2	

Table A.71: The degrees of characters in $\mathrm{Irr}(5{:}4 \times \mathrm{HS}{:}2)$.

Degree	1	4	22	77	88	154	175	231	308	616
Number	8	2	8	8	2	8	8	8	6	2
Degree	693	700	770	825	924	1056	1232	1386	1408	1540
Number	8	2	8	8	2	8	1	8	8	4
Degree	1750	1792	1925	2520	2750	2772	3080	3200	3300	4224
Number	8	4	16	8	8	2	2	8	2	2
Degree	5544	5632	6160	7000	7168	7700	10080	11000	12800	
Number	2	2	1	2	1	4	2	2	2	

Table A.72: The degrees of characters in $\mathrm{Irr}((5{:}4 \times 2.\mathrm{HS}).2 \mid \xi)$.

Degree	112	224	352	1232	1408	1848	2000	2464	3584	3696	3960
Number	2	2	4	4	1	4	2	4	2	2	4
Degree	4000	4608	4928	5040	7168	7392	9856	10080	15840	18432	
Number	2	4	1	4	2	3	1	4	1	1	

Table A.73: The degrees of characters in $\mathrm{Irr}(5{:}4 \times 5{:}4 \times S_5)$.

Degree	1	4	5	6	16	20	24	64	80	96
Number	32	48	32	16	18	16	8	2	2	1

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Table A.74: The degrees of characters in $\mathrm{Irr}(2.(5{:}4 \times 5{:}4 \times S_5) \mid \xi)$.

Degree	4	8	12	16	32	48	64	96
Number	16	8	8	8	4	4	3	2

Table A.75: The degrees of characters in $\mathrm{Irr}(5{:}4 \times 5{:}4 \times 5{:}4)$.

Degree	1	4	16	64
Number	64	48	12	1

Table A.76: The degrees of characters in $\mathrm{Irr}(2.(5{:}4 \times 5{:}4 \times 5{:}4) \mid \xi)$.

Degree	2	4	8	16	64
Number	16	16	8	12	1

Table A.77: The degrees of characters in $\mathrm{Irr}(5{:}4 \times 5^{1+2}.4.D_8)$.

Degree	1	2	4	8	16	20	32	40	64	80	160
Number	32	24	8	22	8	16	4	4	2	4	1

Table A.78: The degrees of characters in $\mathrm{Irr}(2.(5{:}4 \times 5^{1+2}.4.D_8) \mid \xi)$.

Degree	2	4	8	16	32	40	64	80	160
Number	16	4	4	5	6	8	2	4	1

Table A.79: The degrees of characters in $\mathrm{Irr}(5^3{\cdot}L_3(5))$.

Degree	1	30	31	96	124	125	155	186	620	1240	1860	2480	3100	3720
Number	1	1	3	10	10	1	3	1	1	2	2	2	1	1

Table A.80: The degrees of characters in $\mathrm{Irr}(2 \times 5^3{\cdot}L_3(5) \mid \xi)$.

Degree	1	30	31	96	124	125	155	186	620	1240	1860	2480	3100	3720
Number	1	1	3	10	10	1	3	1	1	2	2	2	1	1

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Table A.81: The degrees of characters in $\mathrm{Irr}(5_+^{1+4}.\mathrm{GL}_2(5))$.

Degree	1	4	5	6	24	96	100	120	200	240	300	400	500	600
Number	4	10	4	6	4	1	1	4	2	4	2	2	1	1

Table A.82: The degrees of characters in $\mathrm{Irr}(2 \times 5_+^{1+4}.\mathrm{GL}_2(5) \mid \xi)$.

Degree	1	4	5	6	24	96	100	120	200	240	300	400	500	600
Number	4	10	4	6	4	1	1	4	2	4	2	2	1	1

Table A.83: The degrees of characters in $\mathrm{Irr}(5_+^{1+4}.2_-^{1+4}.A_5.4)$.

Degree	1	4	5	6	10	15	16	20	24	100	240
Number	4	8	8	2	10	4	4	8	2	1	8
Degree	300	384	400	480	500	1000	1200	1500	1536	1600	2000
Number	2	4	2	6	4	2	2	1	1	1	2

Table A.84: The degrees of characters in $\mathrm{Irr}(2.(5_+^{1+4}.2_-^{1+4}.A_5.4) \mid \xi)$.

Degree	4	6	10	16	20	24	200	384	400
Number	6	4	4	6	8	4	2	4	1
Degree	480	600	800	960	1000	1536	1600	2000	2400
Number	4	1	2	1	5	1	1	1	1

Table A.85: The degrees of characters in $\mathrm{Irr}(5^2{:}4S_4 \times S_5)$.

Degree	1	2	3	4	5	6	8	10	12	15	16	18	20	24	96	120	144
Number	8	12	8	12	8	4	12	12	14	8	4	4	4	10	8	8	4

Table A.86: The degrees of characters in $\mathrm{Irr}(2.(5^2{:}4S_4 \times S_5) \mid \xi)$.

Degree	4	8	12	16	24	36	96	192	288
Number	4	12	10	8	8	2	4	2	2

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Table A.87: The degrees of characters in $\mathrm{Irr}(5^2{:}4S_4 \times 5{:}4)$.

Degree	1	2	3	4	8	12	16	24	96
Number	16	24	16	12	6	4	2	16	4

Table A.88: The degrees of characters in $\mathrm{Irr}(2.(5^2{:}4S_4 \times 5{:}4) \mid \xi)$.

Degree	2	4	6	8	12	16	48	96
Number	12	16	4	6	4	2	4	4

Table A.89: The degrees of characters in $\mathrm{Irr}((2^2 \times 7^2{:}(3 \times 2A_4)){:}2)$.

Degree	1	2	3	4	6	48
Number	12	27	12	15	3	12

Table A.90: The degrees of characters in $\mathrm{Irr}(2.(2^2 \times 7^2{:}(3 \times 2A_4)){:}2 \mid \xi)$.

Degree	2	4	6	8	96
Number	6	9	6	3	3

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