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G_a-ACTION OF THE AFFINE PLANE

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0. Recently, R. Rentschler [4] proved that if k is a field of characteristic 0, any action of the additive group G_a on the affine plane is equivalent to an action of the form ${}^t(x, y) = (x, y + tf(x))$, where $t \in G_a(k)$, $(x, y) \in k^2$ and $f \in k[X]$.

It will be natural to form a conjecture that if k is a field of positive characteristic p, any G_a-action on the affine plane is equivalent to an action of the form ${}^t(x, y) = (x, y + tf_0(x) + t^p f_1(x) + \cdots + t^{p^n} f_n(x))$, where $t \in G_a(k)$, $(x, y) \in k^2$ and $f_0, \cdots, f_n \in k[X]$.

The purpose of this article is to prove this conjecture.

1. Let k be an algebraically closed field of positive characteristic p, $S = \operatorname{Spec}(k[x, y])$ be the affine plane over k and $\sigma : G_a \underset{k}{\otimes} S^2 \longrightarrow S^2$ be a non-trivial action of the additive group G_a on S^2. Let $A = k[x, y]$ and let t be an indeterminate. Then it is well known (cf. [2]) that to give an action σ of G_a on S^2 is equivalent to giving a homomorphism of k-algebras $\Delta : A \longrightarrow A \underset{k}{\otimes} k[t]$ which satisfies the following commutative diagrams;

$$(1)\quad \begin{array}{ccc} A & \xrightarrow{\ \Delta\ } & A \underset{k}{\otimes} k[t] \\ \downarrow \Delta & & \downarrow \Delta \otimes id_{k[t]} \\ A \underset{k}{\otimes} k[t] & \xrightarrow[id_A \otimes \mu]{} & A \underset{k}{\otimes} k[t] \underset{k}{\otimes} k[t] \end{array} \qquad (2)\quad \begin{array}{ccc} A & \xrightarrow{\ \Delta\ } & A \otimes k[t] \\ & \searrow \cong & \downarrow id_A \otimes \varepsilon \\ & & A \underset{k}{\otimes} k \end{array}$$

where $\mu : k[t] \longrightarrow k[t] \underset{k}{\otimes} k[t]$ and $\varepsilon : k[t] \longrightarrow k$ are homomorphisms of k-algebras defined by $\mu(t) = t \otimes 1 + 1 \otimes t$ and $\varepsilon(t) = 0$ respectively. Moreover, if we define k-linear endomorphisms of A by $\Delta(a) = \sum_{i \geqslant 0} D_i(a) \otimes t^i$ for $a \in A$, $D = \{D_i\}_{i \geqslant 0}$ is an iterative infinite higher derivation on A. To give an action σ of G_a on S^2 is equivalent to give an iterative infinite higher deri-

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vation $D = \{D_i\}_{i \geqslant 0}$ on A such that for any $a \in A$, $D_j(a) = 0$ for sufficiently large j (cf. [2]).

We shall begin with

LEMMA 1. (Nagata, Theorem 4.1 of [3]). *Let A_0 be the ring of constants, i.e. $A_0 = \{a \in A \mid D_i(a) = 0$ for $i \geqslant 1\}$. If $a \in A_0$, then each prime factor of a in A belongs to A_0. In particular, A_0 is a unique factorization domain.*

LEMMA 2. (cf. [4]. *A_0 satisfies the following properties.*

(i) *$A_0 = k[f]$, for some irreducible element f in $A_0 - k$. More precisely, $f - \alpha$ is irreducible for any $\alpha \in k$.*

(ii) *A_0 is algebraically closed in A.*

Proof. (i) Since σ is non-trivial, $A_0 \neq A$. On the other hand, $A_0 \neq k$ by virtue of a result in [2]. Let f be an element of $A_0 - k$ such that the total degree with respect to x and y is minimal in $A_0 - k$. Then by Lemma 1, f and $f - \alpha$ are irreducible in A, where $\alpha \in k$. If $A_0 \neq k[f]$, let g be an element of minimal total degree in $A_0 - k[f]$. Then g and $g - \beta$ are irreducible in A, where $\beta \in k$. Let P be a point of S^2 which is not fixed by the action σ and let $\alpha = f(P)$ and $\beta = g(P)$. Then P belongs to the closed sets $V(f - \alpha)$ and $V(g - \beta)$ defined by $f - \alpha$ and $g - \beta$ in S^2 respectively. Hence, $V(f - \alpha) = \overline{O(P)} = V(g - \beta)$, where $\overline{O(P)}$ is the k-closure of the orbit of P. Therefore $g \in k[f]$.

(ii) Let h be an element of A which satisfies an equation of the form, $a_0 h^n + a_1 h^{n-1} + \cdots + a_n = 0$, where $a_0, a_1, \cdots, a_n \in A_0$. Then the equation $a_0 X^n + a_1 X^{n-1} + \cdots + a_n = 0$ has only a finite number of solutions in A. If $t \in G_a(k)$, then $h \rightsquigarrow t_h$ is a permutation of these solutions. However, G_a is connected. Hence h is fixed by the action of G_a, i.e. $h \in A_0$. q.e.d.

We shall next define a function λ on A with values in the ring of integers $\boldsymbol{Z}$ by $\lambda(a) = n$ if $\Delta(a) = a \otimes 1 + a_1 \otimes t + \cdots + a_n \otimes t^n$ and $a_n \neq 0$. Then A_0 consists of elements such that $\lambda(a) = 0$. Let n_0 be the minimal value of $\lambda(a)$ for $a \in A - A_0$. Then $n_0 \geqslant 1$. Let q be an element such that $\lambda(q) = n_0$ and that q is of minimal total degree in the set of elements a whose values of λ are n_0. Then we have the following result.

LEMMA 3. $$\Delta(q) - q \otimes 1 \in A_0 \otimes_k k[t].$$

Proof. Let $\Delta(q) = q\otimes 1 + q_1\otimes t + \cdots + q_{n_0}\otimes t^{n_0}$, where $q_1, \cdots q_{n_0}\in A$. Since $(\Delta\otimes id_{k[t]})\cdot\Delta(q) = (id_A\otimes\mu)\cdot\Delta(q)$ from the commutative diagram (1), it is easy to deduce an equality, $\Delta(q_i) = q_i\otimes 1 + \binom{i+1}{1}q_{i+1}\otimes t + \cdots + \binom{i+j}{j}q_{i+j}\otimes t^j + \cdots + \binom{n_0}{n_0-i}q_{n_0}\otimes t^{n_0-i}$ for $1\leqslant i\leqslant n_0$. Therefore by the choice of q, $\binom{i+1}{1}q_{i+1} = \cdots = \binom{n_0}{n_0-i}q_{n_0} = 0$ and $q_i\in A_0$. q.e.d.

Since q is also of the minimal total degree in $A - A_0$, $q-\beta$ is irreducible for any $\beta\in k$. In particular, $q-\beta$ is not divisible by any linear factor $f-\alpha$, for $\alpha\in k$.

Let $B = k[f, q]$. Then the restriction Δ on B sends B to $B\underset{k}{\otimes} k[t]$ and commutes the diagrams (1) and (2) where A is replaced by B. Namely Δ induces an action σ' of G_a on $X = \operatorname{Spec}(B)$ and the inclusion $B\subset A$ implies that there exists a $\operatorname{Spec}(k[f])$-morphism $\varphi : S^2 = \operatorname{Spec}(A)\longrightarrow X = \operatorname{Spec}(B)$ which commutes with the actions σ and σ' on S^2 and X respectively.

We now see that φ is isomorphic on the fibres over the generic point of $\operatorname{Spec}(k[f])$. In fact, let K be the quotient field of $k[f]$ and $A_K = A\underset{A_0}{\otimes}K$. Then the higher derivation $D = \{D_i\}_{i\geqslant 0}$ in A can be canonically extended into A_K and it is easy to see that K is the ring of constant elements in A_K. Therefore $\operatorname{Spec}(A_K)$ is a G_a-homogeneous space defined over K. The argument as in [5] shows that A_K is isomorphic to $K[u]$, the polynomial ring of one variable over K. Since $\lambda(u)$ must be minimal in $K[u]$, we can take q for u. Therefore $B\underset{A_0}{\otimes}K\cong A\underset{A_0}{\otimes}K$. In particular, φ is birational.

On the other hand, we now show that $B\underset{A_0}{\otimes}A_0/(f-\alpha)\longrightarrow A\underset{A_0}{\otimes}A_0/(f-\alpha)$ is injective for all $\alpha\in k$. In fact, suppose that an element $\alpha_0q^n + \alpha_1q^{n-1} + \cdots + \alpha_n$ in $B\underset{A_0}{\otimes}A_0(f-\alpha)\cong k[q]$ is sent to 0 in $A\underset{A_0}{\otimes}A_0/(f-\alpha) = k[x, y]/(f-\alpha)$. Then $\alpha_0q^n + \alpha_1q^{n-1} + \cdots + \alpha_n\in(f-\alpha)A$. Since k is algebraically closed, $\alpha_0q^n + \alpha_1q^{n-1} + \cdots + \alpha_n = \alpha_0(q-\beta_1)\cdots(q-\beta_n)$ for $\beta_1, \cdots, \beta_n\in k$. However since $(f-\alpha)A$ is prime ideal of A, some factor, say $(q-\beta_1)$, belongs to $(f-\alpha)A$. Hence, there exists an element h in A such that $(q-\beta_1) = (f-\alpha)h$. But this is not possible by virtue of the choice of q.

Since q is not algebraic over $k[f]$ and since $k[x, y]/(f-\alpha)$ is the affine algebra of an irreducible curve $f = \alpha$ in S^2, φ restricted to the fibres over a closed point $f = \alpha$ of $\operatorname{Spec}(k[f])$ is quasi-finite. Moreover, note that X is normal.

Now applying Zariski's Main Theorem (cf. *EGA* (III, 4.4.9)), φ is an open immersion. Let $V = X - \varphi(S^2)$. Then it is easy to see that $\operatorname{codim}(V)\geqslant 2$, noting that the structural morphism $S^2 \longrightarrow \mathrm{Spec}\,(k[f])$ is surjective. Hence $\mathrm{Prof}_V(\mathcal{O}_X) \geqslant 2$ because X is normal. Therefore $\Gamma(X-V, \mathcal{O}_X) \cong \Gamma(X, \mathcal{O}_X)$ by virtue of *EGA*, (IV, 5, 10, 5). Since $\varphi(S^2)$ is an affine open set, $\varphi(S^2) = \mathrm{Spec}\,(\Gamma(X-V, \mathcal{O}_X)) \cong \mathrm{Spec}\,(\Gamma(X, \mathcal{O}_X)) = X$. Thus we have proven that $k[f, q] = k[x, y]$.

Now we shall note that the action σ of G_a on S^2 is determined by a homorphism of group schemes Ψ: $G_a \longrightarrow \mathrm{Aut}\,(S^2/S^1)$, where $\mathrm{Aut}\,(S^2/S^1)$ is the $k[f]$-automorphism group of the affine line $S^2 = \mathrm{Spec}\,(k[f, q])$ over $S^1 = \mathrm{Spec}\,(k[f])$ and is isomorphic to a sub-group scheme $G = \left\{\begin{pmatrix} a, b \\ 0, 1 \end{pmatrix};\ a \in k^* = k - 0,\ b \in k[f]\right\}$ of $GL\,(2, k[f])$. The action of an element $\begin{pmatrix} a, b \\ 0, 1 \end{pmatrix}$ of G on S^2 is given by sending $\begin{pmatrix} q \\ 1 \end{pmatrix}$ to $\begin{pmatrix} a, b \\ 0, 1 \end{pmatrix}\begin{pmatrix} q \\ 1 \end{pmatrix} = \begin{pmatrix} aq + b \\ 1 \end{pmatrix}$. Then the correspondence $t \in G_a \approx\!\longrightarrow a(t)$ is a multiplicative character of G_a hence $a(t) = 1$. Then $t \approx\!\longrightarrow \begin{pmatrix} 1, b \\ 0, 1 \end{pmatrix}$ defines an endomorphism of G_a. Then b is a p-polynomial with respect to t with coefficients in $k[f]$, $b = a_0(f)t + \cdots + a_n(f)t^{p^n}$. Then ${}^t q = q + a_0(f)t + \cdots + a_n(f)t^{p^n}$.

Thus we have proven

THEOREM. *Let k be an algebraically closed field of positive characteristic p. Then any action of the additive group G_a on the affine plane is equivalent to the action of the form,*

$$ {}^t(x, y) \rightsquigarrow (x, y + tf_0(x) + t^p f_1(x) + \cdots + t^{p^n} f_n(x)) $$

where $t \in G_a(k)$, $(x, y) \in k^2$ and $f_0(x), \cdots, f_n(x) \in k[X]$.

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