THE FOURIER TRANSFORM OF VECTOR-VALUED FUNCTIONS

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For each natural number n, let $u_n(x)=(1-\cos nx)/\pi nx^2$ $(x \in \mathbb{R})$. It is well-known that a bounded continuous function f on the real line $\mathbb{R}$ is the Fourier transform of an integrable function on $\mathbb{R}$ if and only if the functions $\Phi_n(f)$ $(n=1,2,\ldots)$, defined by

$$\Phi_n(f)(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ixt}f(t)\hat{u}_n(t)\,dt, \quad (x \in \mathbb{R}),$$

form a Cauchy sequence in the space $\mathscr{L}_1(\mathbb{R})$ (cf. [**2**]). Such a characterization, which can be extended to functions defined on a locally compact Abelian group more general than $\mathbb{R}$, is based on the fact that the space $\mathscr{L}_1(\mathbb{R})$ is complete with respect to convergence in mean.

However, as noted in [**6**], this criterion cannot be extended to characterize the Fourier transforms of vector-valued, Pettis integrable functions. This is due to the fact that the space of Pettis integrable functions is not complete with respect to convergence in mean (cf. [**8**]).

The integral introduced in [**7**] does not suffer from this defect. It leads to a space of integrable functions which is complete with respect to convergence in mean. Accordingly, the Fourier transforms of such vector-valued, integrable functions can be characterized by a criterion analogous to that which characterizes the Fourier transforms of scalar-valued integrable function. The aim of this note is to present such a criterion.

1. The Archimedes integral. Let λ be a choice of Haar measure in a locally compact Abelian group G. Let X be a complex Banach space and X' its dual space.

Let Y be a locally convex Hausdorff space into which the space X is continuously embedded. A function $F:G\to Y$ is called *Archimedes integrable* with respect to λ in the space X, briefly (X,λ)-integrable, if there exists a function $H:G\to Y$, vectors $c_i \in X$ and Borel subsets E_i of G, $(i=1,2,\ldots)$, such that

(i) the function F is λ-almost everywhere equal to H;

(ii) the sequence $\{c_i\lambda(E_i)\}_{i\in\mathbb{N}}$ is unconditionally summable in the space X; and

(iii) if $y' \in Y'$, then

$$\langle y', H(g)\rangle=\sum_{i=1}^{\infty}\langle y', c_i\rangle \chi_{E_i}(g),$$

for every $g \in G$ for which

$$\sum_{i=1}^{\infty}|\langle y', c_i\rangle|\,\chi_{E_i}(g)<\infty.$$

The X-valued set function $F\lambda$ defined by

$$(F\lambda)(E) = \sum_{i=1}^{\infty} c_i \lambda(E_i \cap E),$$

for every Borel set E, is called the *indefinite integral* of the function F with respect to λ. Since Y' separates points of X, the indefinite integral $F\lambda$ is well-defined (cf. [**7**, Lemma 1, Note 11]). By the Vitali–Hahn–Saks theorem, it is σ-additive. We shall also use the classical notation

$$(F\lambda)(E) = \int_E F\, d\lambda,$$

for each Borel set E.

The vector space of all Y-valued, (X, λ)-integrable functions on G is denoted by $\mathcal{L}(\lambda; X, Y)$. Define the seminorm $\|\cdot\|$ on the space $\mathcal{L}(\lambda; X, Y)$ by

$$\|F\| = \sup\{|\langle x', F\lambda\rangle|(G) : |x'| \leq 1, x' \in X'\},$$

for every $F \in \mathcal{L}(\lambda; X, Y)$, where $|\langle x', F\lambda\rangle|$ denotes the total variation of the complex measure $\langle x', F\lambda\rangle$. The topology on $\mathcal{L}(\lambda; X, Y)$ given by this seminorm is called the topology of *convergence in mean*. It is clear that the space of X-valued, (X, λ)-integrable Borel simple functions on G is dense in the space $\mathcal{L}(\lambda; X, Y)$ (cf. [**7**, Proposition 2]).

There always exists a locally convex Hausdorff space Y, into which X is continuously embedded, such that the space $\mathcal{L}(\lambda, X, Y)$ is complete with respect to convergence in mean. For example, if Θ is a total subset of the dual space X', then X is continuously embedded into the product space $\mathbb{C}^{\Theta}$ and we have the following result.

THEOREM 1.1 ([**7**, Theorem 5]). *The space $\mathcal{L}(\lambda; X, \mathbb{C}^{\Theta})$ is complete with respect to convergence in mean.*

The relationship between the Pettis integral and the Archimedes integral is given by the following result.

PROPOSITION 1.2 ([**7**, Proposition 14]). *A function $F: G \to X$ is (X, λ)-integrable if and only if it is strongly measurable and Pettis integrable with respect to λ. Moreover, the indefinite integral $F\lambda$ of each function $F \in \mathcal{L}(\lambda; X, X)$ is equal to the indefinite Pettis integral of F.*

2. The Fourier transform. Let G be a locally compact Abelian group and Γ its dual group. Let λ be a fixed Haar measure in the group G. Let ν be the Haar measure in Γ which is so normalized that the inversion formula is valid (cf. [**9**, Theorem I.5.1]).

Throughout this section, X is a Banach space and Y is a locally convex Hausdorff space into which X is continuously embedded.

For each function $F \in \mathcal{L}(\lambda; X, Y)$, let

$$\hat{F}(\gamma) = \int_G F(g)(-g, \gamma)\, dg \quad (\gamma \in \Gamma).$$

The X-valued function $\hat{F}$ so defined is called the *Fourier transform* of the function F. It exists by [**7**, Corollary 4].

According to [**4**, Theorem 33.12], the Banach algebra $\mathscr{L}_1(G)$ of all complex-valued, λ-integrable functions on G admits an approximate unit $\{u_\alpha\}_{\alpha \in A}$ such that

(i) for every $\alpha \in A$, the function u_α is non-negative and $(u_\alpha \lambda)(G) = 1$;

(ii) for every $\alpha \in A$, the Fourier transform $\hat{u}_\alpha$ of u_α is a continuous function on Γ with compact support;

(iii) for every $\alpha \in A$, the function u_α is equal to the inverse Fourier transform of $\hat{u}_\alpha$;

(iv) $\lim_{\alpha \in A} \hat{u}_\alpha = 1$ on Γ, the convergence being uniform on all compact subsets of Γ.

Let $\alpha \in A$. Suppose that $f : \Gamma \to X$ is a weakly continuous map. Then, for every $g \in G$, the X-valued function $\gamma \mapsto f(\gamma)\hat{u}_\alpha(\gamma)(g, \gamma)$, where $\gamma \in \Gamma$, is Pettis ν-integrable since it has weakly compact range (cf. [**5**, Lemma 4] or [**3**, p. 88]). Let

$$\Phi_\alpha(f)(g) = \int_\Gamma f(\gamma)\hat{u}_\alpha(\gamma)(g, \gamma)\, d\gamma \quad (g \in G), \tag{1}$$

where the right hand side of (1) is the Pettis integral over Γ.

THEOREM 2.1. *Let $f : \Gamma \to X$ be the Fourier transform of an element of $\mathscr{L}(\lambda; X, Y)$. Then f is a bounded continuous function vanishing at infinity, and the X-valued functions $\Phi_\alpha(f)$, where $\alpha \in A$, given by* (1) *are (X, λ)-integrable and form a Cauchy net in the space $\mathscr{L}(\lambda; X, X)$ with respect to convergence in mean.*

Proof. Take a function $F \in \mathscr{L}(\lambda; X, Y)$ such that $\hat{F} = f$. Then there exist (X, λ)-integrable, Borel simple functions $H_n : G \to X$ $(n = 1, 2, \ldots)$, which are convergent in mean to the function F. It follows that, for every natural number n,

$$|f(\gamma) - \hat{H}_n(\gamma)| \leq \|F - H_n\| \quad (\gamma \in \Gamma). \tag{2}$$

For every natural number n, the function $\hat{H}_n$ is continuous and vanishes at infinity since H_n is a simple function (cf. [**9**, Theorem I.2.4]). It follows from (2) that the function f is also continuous and vanishes at infinity.

Let $\alpha \in A$. By [**6**, Lemma 7], the function $\Phi_\alpha(f)$ is Pettis λ-integrable. We claim that $\Phi_\alpha(f)$ has relatively compact range in the space X. Indeed, the bounded function $f\hat{u}_\alpha$ with compact support is Bochner ν-integrable and its indefinite Bochner integral has relatively compact range R in X (cf. [**3**, Theorem VIII.1.5]). Consequently, the range of $\Phi_\alpha(f)$ is included in the compact set $4\overline{\text{bco}}\, R$, where $\overline{\text{bco}}\, R$ denotes the closed balanced convex hull of the set R.

Since the function $\Phi_\alpha(f)$ is scalarly λ-integrable and has separable range, it vanishes λ-almost everywhere outside some Borel set of σ-finite measure. By the Pettis measurability theorem, the function $\Phi_\alpha(f)$ is strongly λ-measurable. It follows from Proposition 1.2 that the function $\Phi_\alpha(f)$ is (X, λ)-integrable.

For every $\alpha \in A$ and every natural number n,

$$\|\Phi_\alpha(f) - \Phi_\alpha(\hat{H}_n)\| \leq \|F - H_n\|. \tag{3}$$

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In fact, the strongly λ-measurable functions $\Phi_\alpha(f)$ and $\Phi_\alpha(\hat{H}_n)$ both vanish λ-almost everywhere outside a Borel set E of σ-finite measure. Let $\mu=(F-H_n)\lambda$. Since $\hat{u}_\alpha$ has compact support, the Fubini theorem ensures that, for every $x'\in X'$,

$$\begin{aligned}\int_G |\langle x', \Phi_\alpha(f)-\Phi_\alpha(\hat{H}_n)\rangle|\, d\lambda &= \int_E \left|\int_G d\langle x', \mu\rangle(h)\int_\Gamma \hat{u}_\alpha(\gamma)(g-h, \gamma)\, d\gamma\right| dg\\ &= \int_E \left|\int_G u_\alpha(g-h)\, d\langle x', \mu\rangle(h)\right| dg\\ &\le \int_G d\,|\langle x', \mu\rangle|\,(h)\int_G u_\alpha(g-h)\, dg \le |x'|\,\|F-H_n\|.\end{aligned}$$

Thus, the inequality (3) is valid.

Let n be a natural number. Since H_n is a simple function, it follows from [**10**, Theorem 2(a)] that the net $\{\Phi_\alpha(\hat{H}_n)\}_{\alpha\in A}$ is Cauchy in the space $\mathcal{L}(\lambda; X, X)$. By (3), the net $\{\Phi_\alpha(f)\}_{\alpha\in A}$ is also Cauchy in the space $\mathcal{L}(\lambda; X, X)$.

The statement in Theorem 2.1 that the Fourier transform of an Archimedes integrable function vanishes at infinity is a vector version of the classical Riemann–Lebesgue lemma.

A sufficient condition for an X-valued function on Γ to be the Fourier transform of an Archimedes integrable function on G is given by the following result.

THEOREM 2.2. *Let Θ be a total subset of the dual space X' and $Y=\mathbb{C}^\Theta$. Let $f:\Gamma\to X$ be a bounded weakly continuous function such that, for every $\alpha\in A$, the function $\Phi_\alpha(f): G\to X$, given by* (1), *is (X,λ)-integrable and the net $\{\Phi_\alpha(f)\}_{\alpha\in A}$ is Cauchy in the space $\mathcal{L}(\lambda; X, X)$ with respect to convergence in mean.*

Then the function f is the Fourier transform of a function belonging to the space $\mathcal{L}(\lambda; X, Y)$.

Proof. The Cauchy net $\{\Phi_\alpha(f)\}_{\alpha\in A}$ has a limit, F, in the complete space $\mathcal{L}(\lambda; X, Y)$ (cf. Theorem 1.1). It follows from [**10**, Theorem 2(a)] that, for every $\theta\in\Theta$,

$$\langle\theta, f(\gamma)\rangle=\int_G \langle\theta, F(g)\rangle(-g, \gamma)\, dg=\langle\theta, \hat{F}(\gamma)\rangle \quad (\gamma\in\Gamma).$$

Since Θ is a total subset of X', we have $f=\hat{F}$.

Theorems 2.1 and 2.2 can be used to characterize those X-valued functions on Γ which are Fourier transforms of X-valued Bochner λ-integrable functions on G. This gives a slight extension of the characterization in [**1**], where it was assumed that G is a metrizable locally compact Abelian group.

COROLLARY 2.3. *A function $f:\Gamma\to X$ is the Fourier transform of an X-valued, Bochner*

λ-integrable function on G if and only if the following conditions hold:

(i) *the function f is bounded and continuous;*

(ii) *for every $\alpha \in A$, the function $\Phi_\alpha(f): G \to X$, given by* (1), *is Bochner λ-integrable, and the net $\{\Phi_\alpha(f)\}_{\alpha \in A}$ is Cauchy with respect to the Bochner seminorm; that is,*

$$\lim_{\alpha,\beta \in A} \int_G |\Phi_\alpha(f) - \Phi_\beta(f)|\, d\lambda = 0.$$

Proof. To prove the 'only if' part, take a Bochner λ-integrable function $F: G \to X$ such that $\hat{F} = f$. Statement (i) is a consequence of Theorem 2.1. Now, given $\varepsilon > 0$, choose a Bochner λ-integrable, Borel simple function $H: G \to X$ for which

$$\int_G |F - H|\, d\lambda < \varepsilon.$$

By the argument in the proof of Theorem 2.1,

$$\int_G |\Phi_\alpha(f) - \Phi_\alpha(\hat{H})|\, d\lambda \leq \int_G |F - H|\, d\lambda,$$

which implies statement (ii), since H is a simple function.

To prove the 'if' part, take a Bochner λ-integrable function $F: G \to X$ such that

$$\lim_{\alpha \in A} \int_G |F - \Phi_\alpha(f)|\, d\lambda = 0.$$

It then follows from Theorem 2.2 that $f = \hat{F}$.

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