Canad. Math. Bull. Vol. **41** (4), 1998 pp. 473–477

SEPARATING SINGULARITIES OF HOLOMORPHIC FUNCTIONS

JÜRGEN MÜLLER AND JOCHEN WENGENROTH

ABSTRACT. We present a short proof for a classical result on separating singularities of holomorphic functions. The proof is based on the open mapping theorem and the fusion lemma of Roth, which is a basic tool in complex approximation theory. The same method yields similar separation results for other classes of functions.

1. **Separating singularities in open sets.** There are various elementary results in function theory concerning the separation of singularities. Simple examples are the Laurent decomposition of holomorphic functions in a ring domain or the partial fractions decomposition of rational functions. A general result of this type is due to Aronszajn [1]. It can be viewed as a special case of the Cousin-I-problem (see *e.g.* [7, Theorem 1.4.5]).

THEOREM 1. *Let Ω_1, Ω_2 be open in the extended complex plane $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. Every holomorphic function f on $\Omega = \Omega_1 \cap \Omega_2$ can be written as $f = f_1|_\Omega - f_2|_\Omega$ with f_j holomorphic in Ω_j.*

The usual modern proof [7, Theorem 1.4.5] uses the surjectivity of the $\bar{\partial}$-operator and a partition of unity. We give a different proof based on the

FUSION LEMMA (ROTH [10]). *For every pair K_1, K_2 of disjoint compact sets in $\hat{\mathbb{C}}$ there exists a constant $\alpha = \alpha(K_1, K_2)$ such that for every compact $K \subset \hat{\mathbb{C}}$, every $\varepsilon > 0$ and every pair of rational functions r_1, r_2 with $||r_1 - r_2||_K < \varepsilon$ there is a rational function R with*

$$\|R - r_j\|_{K_j \cup K} < \alpha\varepsilon \quad \text{for } j = 1, 2$$

(where throughout $\|f\|_A = \sup_{z\in A} |f(z)|$).

PROOF OF THEOREM 1. For an open set $G \subset \hat{\mathbb{C}}$ we endow the space

$$H(G) = \big\{f : G \longrightarrow \mathbb{C} \text{ holomorphic}, f(\infty) = 0 \text{ if } \infty \in G\big\}$$

with its usual Fréchet space topology of uniform convergence on compact subsets of G. Since we may assume $f \in H(\Omega)$, we have to show that the continuous linear operator

$$T : H(\Omega_1) \times H(\Omega_2) \longrightarrow H(\Omega), \quad T(f_1, f_2) := f_1|_\Omega - f_2|_\Omega$$

is surjective, which is—by a version of the open mapping theorem which is stated *e.g.* in [8, p. 9]—equivalent to T being almost open, *i.e.*, the closure of $T(U_1 \times U_2)$ is a neighbourhood of 0 in $H(\Omega)$ for all 0-neighbourhoods U_j in $H(\Omega_j)$, $j = 1, 2$.

Received by the editors April 30, 1997; revised December 10, 1997.
AMS subject classification: 30E99, 30E10.

Let U_j be 0-neighbourhoods in $H(\Omega_j)$ which can be assumed to be of the form

$$U_j = \left\{ f \in H(\Omega_j) : \|f\|_{K_j} < \varepsilon \right\}$$

for some $\varepsilon > 0$ and compact sets $K_j \subset \Omega_j$. Let $M \subset \Omega_1 \cup \Omega_2$ be compact with $K_1 \cup K_2 \subset M$ and such that each component of M^c contains a point of $(\Omega_1 \cup \Omega_2)^c$. Choosing an open set $U \supset M \setminus \Omega_1$ with $\overline{U} \subset \Omega_2$, and letting $\tilde{K}_1 = K_1 \cup (M \setminus U)$ and $\tilde{K}_2 = K_2 \cup (M \cap \overline{U})$, we have $K_j \subset \tilde{K}_j \subset \Omega_j$ and $\tilde{K}_1 \cup \tilde{K}_2 = M$. Let now $K \subset \Omega$ be compact with $\tilde{K}_1 \cap \tilde{K}_2 \subset K^\circ$, set $L_j = \tilde{K}_j \setminus K^\circ$ and choose $\alpha = \alpha(L_1, L_2)$ according to the fusion lemma. Then we have

$$V = \left\{ f \in H(\Omega) : \|f\|_K < \frac{\varepsilon}{2\alpha} \right\} \subset \overline{T(U_1 \times U_2)}.$$

Indeed, by Runge's theorem it is enough to decompose rational functions $R \in V$ as $R = T(f_1, f_2)$ with $f_j \in U_j$ for $j = 1, 2$. Clearly, we can write $R = R_1 - R_2$ with $R_j \in H(\Omega_j)$. Hence, by the fusion lemma, there exists a rational function S with

$$\|S - R_j\|_{K \cup \tilde{K}_j} < \alpha \frac{\varepsilon}{2\alpha} = \varepsilon/2.$$

Applying again Runge's theorem, we find a function $\tilde{S} \in H(\Omega_1 \cup \Omega_2)$ with $\|S - \tilde{S}\|_{\tilde{K}_1 \cup \tilde{K}_2} < \varepsilon/2$. Finally, the functions $f_j = R_j - \tilde{S}$ satisfy $f_j \in U_j$ with $T(f_1, f_2) = R$. ∎

REMARKS. 1. The kernel of the operator T in the previous proof is isomorphic to $H(\Omega_1 \cup \Omega_2)$. In particular, if $\Omega_1 \cup \Omega_2 = \hat{\mathbb{C}}$, then T is injective and $H(\Omega) = H(\Omega_1) \oplus H(\Omega_2)$ is the topological direct sum of the spaces $H(\Omega_j)$. In this case the unique decomposition $f = f_1|_\Omega - f_2|_\Omega$ depends in a continuous linear way on f.

2. Let $P(D)$ be a homogeneous polynomial elliptic partial differential operator with constant coefficients on $\mathbb{R}^n$ and call a C^∞-function on an open set P-holomorphic if it belongs to the kernel of $P(D)$ ($\bar{\partial}$-holomorphic functions are holomorphic in the usual sense and Δ-holomorphic functions are harmonic if Δ denotes the Laplace-operator). Since there are "elliptic analogues" of the tools used in the proof above, namely an elliptic fusion lemma [2] and an elliptic Runge theorem [3], we can extend Theorem 1 to P-holomorphic functions. The harmonic case is already included in [1].

THEOREM 2. *For each pair Ω_1, Ω_2 of open subsets of $\mathbb{R}^n$, every P-holomorphic function f on $\Omega = \Omega_1 \cap \Omega_2$ can be written as $f = f_1|_\Omega - f_2|_\Omega$ where the functions f_j are P-holomorphic on Ω_j.*

2. **Separating singularities in compact sets.** For a compact set $K \subset \hat{\mathbb{C}}$ we denote $A(K) = \{ f \in C(K) : f|_{K^\circ} \in H(K^\circ),\ f(\infty) = 0 \}$. One may ask for an analogue result to Aronszajn's theorem:
If $K = K_1 \cap K_2$ is the intersection of two compact sets, is it true that every $f \in A(K)$ can be decomposed as $f = f_1|_K - f_2|_K$ with $f_j \in A(K_j)$? In general, the answer is no, and in many cases, the decomposability can be characterized by a "fusion property". Let $R(K)$ be the space of all $f \in A(K)$ that can be uniformly approximated by rational functions.

https://doi.org/10.4153/CMB-1998-061-x Published online by Cambridge University Press

THEOREM 3. *Let* $K = K_1 \cap K_2$ *be the intersection of two compact sets* $K_j \subset \hat{\mathbb{C}}$ *with* $R(K) = A(K)$ *and such that every function which is continuous on* $K_1 \cup K_2$ *and holomorphic in* $K_1^\circ \cup K_2^\circ$ *is holomorphic on* $(K_1 \cup K_2)^\circ$.

(a) The following conditions are equivalent:

(1) Every $f \in A(K)$ *can be decomposed as* $f = f_1|_K - f_2|_K$ *with* $f_j \in A(K_j)$.

(2) There is a constant $a > 0$ *such that for every pair* $u_j \in A(K_j)$, $j = 1, 2$, *there exists a function* $f \in A(K_1 \cup K_2)$ *with*

$$\| f - u_j\|_{K_j} \leq a\|u_1 - u_2\|_K.$$

(b) If, moreover, $A(K_1 \cup K_2) = R(K_1 \cup K_2)$, *then condition (2) is equivalent to:*

(3) There is a constant $b > 0$ *such that for every* $\varepsilon > 0$ *and every pair* r_1, r_2 *of rational functions with* $\|r_1 - r_2\|_K < \varepsilon$ *there exists a rational function* R *with*

$$\|R - r_j\|_{K_j} < b\varepsilon.$$

PROOF. Assume that condition (1) holds, which means, that the continuous linear operator

$$T: A(K_1) \times A(K_2) \longrightarrow A(K), \quad T(f_1, f_2) = f_1|_K - f_2|_K$$

is surjective (where we endow $A(K_j)$ and $A(K)$ with the Banach space topologies given by the sup-norm). By the open mapping theorem, T is open. Given $u_j \in A(K_j)$ we therefore can decompose $h = u_1|_K - u_2|_K$ as $h = f_1|_K - f_2|_K$ with $f_j \in A(K_j)$ and $\| f_j\|_{K_j} \leq a\|h\|_K$ for some constant $a > 0$ depending only on K_1 and K_2.

Since $f_1 - f_2 = u_1 - u_2$ on K, the function $f = u_1 - f_1 = u_2 - f_2$ is consistently defined on $K_1 \cup K_2$, continuous there and holomorphic in $K_1^\circ \cup K_2^\circ$ and thus, $f \in A(K_1 \cup K_2)$ with

$$\| f - u_j\|_{K_j} = \| f_j\|_{K_j} \leq a\|u_1 - u_2\|_K.$$

To prove that (2) implies (1), we have to show that the operator T above is surjective or—equivalently—almost open. Given $h \in A(K)$ with $\|h\|_K < 1$ and $\varepsilon \in (0, 1)$, there is a rational $R \in A(K)$ with $\|h - R\|_K < \varepsilon$. Clearly $R = R_1 - R_2$ with $R_j \in A(K_j)$. Applying (2), we find $f \in A(K_1 \cup K_2)$ such that

$$\| f - R_j\|_{K_j} \leq a\|R_1 - R_2\|_K = a\|R\|_K \leq 2a.$$

With $f_j = R_j - f \in A(K_j)$ we get $\| f_j\|_{K_j} \leq 2a$ and

$$\|h - T(f_1, f_2)\|_K = \|h - R\|_K < \varepsilon.$$

This proves that T is almost open.

The same proof yields that (3) implies (1). To show that (2) implies (3) we first exclude exactly as in [4, p. 247] the case that the r_j have poles in K_j. Hence we may assume $r_j \in A(K_j)$. Now, we just approximate the function f given by (2) by a rational function R on $K_1 \cup K_2$ to obtain

$$\|R - r_j\|_{K_j} \leq \|R - f\|_{K_1 \cup K_2} + \| f - r_j\|_{K_j} < a\varepsilon + a\|r_1 - r_2\|_K \leq 2a\varepsilon.$$

Thus we may take $b = 2a$. ∎

https://doi.org/10.4153/CMB-1998-061-x Published online by Cambridge University Press

REMARKS. 1. For compact sets $K_1, K_2 \subset \hat{\mathbb{C}}$ with $\infty \notin \partial K_j$, the condition that every function which is continuous on $K_1 \cup K_2$ and holomorphic in $K_1^\circ \cup K_2^\circ$ is holomorphic in $(K_1 \cup K_2)^\circ$ is satisfied if and only if $\alpha\big(\partial K_1 \cap \partial K_2 \cap (K_1 \cup K_2)^\circ\big) = 0$, where $\alpha(E)$ denotes the continuous analytic capacity of a set $E \subset \mathbb{C}$ (see [6, Chap. I]). For example, this is the case if $\partial K_1 \cap \partial K_2 \cap (K_1 \cup K_2)^\circ$ is a countable union of sets of finite one-dimensional Hausdorff measure [6, Corollary 2.4] and thus, in particular, if K_1 and K_2 have smooth boundary. Sufficient conditions for $R(K) = A(K)$ are well known (see for example [5]). In particular, $A(K) = R(K)$ holds if K^c has only finitely many components.

2. If $K = K_1 \cap K_2$ is infinite, then the condition (3) is equivalent to:

(3′) There is a constant $c > 0$ such that for every pair r_j of rational functions there exists a rational function R with

$$\|R - r_j\|_{K_j} \leq c\|r_1 - r_2\|_K.$$

3. The question under which condition (3) holds was first investigated by Gauthier (*cf.* [4] or [5]), who gave an example where (3) fails (in this example, $A(K_1 \cup K_2) = R(K_1 \cup K_2)$ is violated). Much simpler examples where obtained by Gaier in [4], where he asked for conditions under which the strong version (3) of the fusion lemma is true. Several results can be found in [9], [12].

4. The kernel of T in the situation of Theorem 3 is isomorphic to $A(K_1 \cup K_2)$. In particular, if $K_1 \cup K_2 = \hat{\mathbb{C}}$, then T is injective, hence a topological isomorphism, and this implies $A(K) = A(K_1) \oplus A(K_2)$ if condition (1) holds. For $K_1 = \{z \in \mathbb{C} : |z| \leq 1\}$ and $K_2 = \{z \in \mathbb{C} : |z| \geq \varrho\} \cup \{\infty\}$, where $\varrho \leq 1$, we easily see by Laurent decomposition that $A(K) = A(K_1) \oplus A(K_2)$ holds true if $\varrho < 1$. On the other hand, for $\varrho = 1$, this would imply that the disc algebra $A(K_1)$ is complemented in the space of continuous functions on $\{|z| = 1\}$, which is not true [11, Example 5.19]. Hence we have another simple situation where the strong form of the fusion lemma fails.

REFERENCES

1. N. Aronszajn, *Sur les décompositions des fonctions analytiques uniformes et sur leur applications*. Acta Math. **65**(1935), 1–156.
2. A. Bonilla and J. C. Fariña, *Elliptic fusion lemma*. Math. Japon. **41**(1995), 441–445.
3. A. Dufresnoy, P. M. Gauthier and W. H. Ow, *Uniform approximation on closed sets by solutions of elliptic partial differential equations*. Complex Variables Theory Appl. **6**(1986), 235–247.
4. D. Gaier, *Remarks on Alice Roth's fusion lemma*. J. Approx. Theory **37**(1983), 246–250.
5. ———, *Lectures on Complex Approximation*. Birkhäuser, Boston, 1987.
6. J. Garnett, *Analytic Capacity and Measure*. Springer, Berlin, 1972.
7. L. Hörmander, *An Introduction to Complex Analysis in Several Variables*. 3rd edn, North-Holland, Amsterdam, 1990.
8. N. J. Kalton, N. T. Peck and J. W. Roberts, *An F-Space Sampler*. Cambridge University Press, Cambridge, 1984.
9. A. Nersesjan, *Alice Roth's fusion lemma*. Soviet J. Contemporary Math. Anal. **23**(1988), 34–47.
10. A. Roth, *Uniform and tangential approximation by meromorphic functions on closed sets*. Canad. J. Math. **28**(1976), 104–111.

11. W. Rudin, *Functional Analysis*. McGraw-Hill, New York, 1973.
12. G. Schmieder, *Fusion lemma and boundary structure*. J. Approx. Theory **71**(1992), 305–311.

Fachbereich IV, Mathematik
Universität Trier
D-54286 Trier
Germany
email: jmueller@uni-trier.de
wengen@uni-trier.de