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On the Bound of the C^* Exponential Length

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Abstract. Let X be a compact Hausdorff space. In this paper, we give an example to show that there is $u \in C(X) \otimes M_n$ with $\det(u(x)) = 1$ for all $x \in X$ and $u \sim_h 1$ such that the C^* exponential length of u (denoted by $\mathrm{cel}(u)$) cannot be controlled by π. Moreover, in simple inductive limit C^*-algebras, similar examples also exist.

1 Introduction

Exponential rank was introduced by Phillips and Ringrose [18], and, subsequently, exponential length was introduced by Ringrose [19]. These invariants have been fundamental in the structure and classification of C^*-algebras. Among other things, they have played important roles in factorization and approximation properties for C^*-algebras *e.g.*, the weak FU property [13], Weyl-von Neumann Theorems [10, 11] (which in turn have been important in various generalizations of BDF Theory beyond the Calkin algebra case), and the uniqueness theorems of classification theory [3, 12]. The C^* exponential length and rank have been extensively studied (see [4, 8, 9, 13–19, 21, 23–25], etc., an incomplete list).

In [15], N. C. Phillips calculates the exponential rank of simple C^*-algebra B with representation $B = \lim_{\to} B_i$, where $B_i = \bigoplus_{t=1}^{s(i)} C(X_{it}) \bigotimes M_{n(i,t)}$, and X_{it} are compact metric spaces such that $\sup_{i,t} \dim(X_{it}) < \infty$. He also studies the exponential length for unitary $u \in \bigcup_{i=1}^{\infty} B_i$. In particular, he mentions that (see the paragraph [15, Proposition 7.9, p. 851]), "We believe that suitable modifications of Lemma 5.2 and 5.3 will show that if $u \sim_h 1$ and $\det(u) = 1$, then $\mathrm{cel}(u) \le \pi$ (even though, for general u, $\mathrm{cel}(u)$ can be arbitrarily large)." However, in this paper we provide a method for constructing counterexamples to this conjecture. In fact, for any $\varepsilon > 0$, we can find a simple inductive limit C^*-algebra (simple AH algebra), say A, and a unitary $u \in CU(A)$ with $u \sim_h 1$ and $\mathrm{cel}(u) \ge 2\pi - \varepsilon$ (see Corollary 3.17 and Theorem 3.18). Note that for unital real rank zero C^*-algebras, π is an upper bound for the C^* exponential length (see [8]). In the process of our proof, we show that there are unitaries $u_i \in A_i$ with $u_i \sim_h 1$ and $\det(u_i) = 1$ whose C^* exponential lengths are close to 2π (see Theorem 3.16 and the proof of 3.17). Finally, we conclude that for the C^*-algebra A we constructed, $\mathrm{cel}_{CU}(A) = 2\pi$ (see Theorem 3.19 and Remark 3.20).

H. Lin, in a recent paper [9], gets a result similar to Theorem 3.13 using a different method. He also provides examples that compare to our Theorem 3.18 with lower bound π but not 2π (see [9, 5.12]). Therefore, by our results, 2π is an optimal bound

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of $\mathrm{cel}_{CU}(A)$ for unital separable simple C*-algebra A with tracial rank less than or equal to 1 (*cf.* [9, Lemma 4.5]).

2 Preliminaries

For the convenience of the reader, we recall some definitions and lemmas (see [7, 19] for more details).

Definition 2.1 Let X be a compact metric space and $B = C(X) \otimes M_n$. For $u \in U(B)$(unitary group of B), let $\det(u)$ be a function from X to S^1 whose value at x is $\det(u(x))$.

Definition 2.2 Let A be a unital C*-algebra and let u be a unitary element that lies in the connected component of the identity 1 in A. Define the C* exponential length of u (denoted by $\mathrm{cel}(u)$) as follows:

$$\mathrm{cel}(u) = \inf\Big\{ \sum_{i=1}^{k} \|h_i\| : u = \exp(ih_1)\exp(ih_2)\cdots\exp(ih_k) \Big\}.$$

Definition 2.3 For a unital C*-algebra A, let $U_0(A)$ be the connected component of $U(A)$ containing the identity 1 and $CU(A)$ be the closure of the commutator subgroup of $U_0(A)$. Define

$$\mathrm{cel}_{CU}(A) = \sup\{\mathrm{cel}(u) : u \in CU(A)\}.$$

Remark 2.4 Recall from [19] that if $u \in U_0(A)$, then the C* exponential length $\mathrm{cel}(u)$ is equal to the infimum of the lengths of rectifiable paths from u to 1 in $U(A)$.

The following lemma is an easy example for calculating the C* exponential length.

Lemma 2.5 *Let $\alpha \in \mathbb{R}$ and $u \in C[0,1]$ be defined by $u(t) = \exp(it\alpha)$. Then*

$$\mathrm{cel}(u) = \min_{k\in\mathbb{Z}} \max_{t\in[0,1]} |\alpha t - 2k\pi|.$$

Moreover, if $|\alpha| \le 2\pi$, then $\mathrm{cel}(u) = |\alpha|$.

Proof Since

$$\mathrm{cel}(u) = \inf\Big\{ \text{length}(u_s) : u_s \text{ is a rectifiable path in } U(C[0,1]) \text{ from } u \text{ to } 1 \Big\},$$

let $v_s(t)$ be any rectifiable path from 1 to u, that is, $v_0(t) = 1, v_1(t) = u(t)$. Without loss of generality, we can assume v_s is piecewise smooth. Then $\text{length}(v_s) = \int_0^1 \|\frac{dv}{ds}\| ds$. Since $v_s(t)$ can be considered as a map from $[0,1]\times[0,1]$ to S^1 and $\mathbb{R}$ is a covering space of S^1, there exists a unique map $\widetilde{v}_s(t)$ from $[0,1]\times[0,1]$ to $\mathbb{R}$ such that

$$v_s(t) = \pi(\widetilde{v}_s(t)) \text{ and } \widetilde{v}_0(0) = 0, \tag{2.1}$$

where $\pi(x) = e^{ix}$. Therefore,

$$\frac{dv}{ds} = \pi'(\widetilde{v}_s(t)) \cdot \frac{d\widetilde{v}}{ds},$$

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which implies $\|\frac{dv}{ds}\| = \|\frac{d\widetilde{v}}{ds}\|$.

By (2.1), $\pi(\widetilde{v}_0(t)) = v_0(t) = 1$. Hence, $\widetilde{v}_0(t) \in 2\pi\mathbb{Z}$ for all $t \in [0,1]$. Since $\widetilde{v}_0(0) = 0$ and $\widetilde{v}_0(t)$ is continuous, $\widetilde{v}_0(t) = 0$ for all $t \in [0,1]$. In addition, by (2.1), we can also get $\pi(\widetilde{v}_1(t)) = v_1(t) = \exp(it\alpha)$. Thus, $\widetilde{v}_1(t) - \alpha t \in 2\pi\mathbb{Z}$ for all t. By continuity of $\widetilde{v}_1(t) - \alpha t$, there exists some integer k such that $\widetilde{v}_1(t) - \alpha t = 2k\pi$ for all $t \in [0,1]$. Therefore,

$$\int_0^1 \Big\|\frac{d\widetilde{v}}{ds}\Big\| ds \geq \Big\| \int_0^1 \frac{d\widetilde{v}}{ds} ds \Big\| = \max_{t\in[0,1]} |\widetilde{v}_1(t) - \widetilde{v}_0(t)| \geq \min_{k\in\mathbb{Z}} \max_{t\in[0,1]} |\alpha t - 2k\pi|.$$

Let $L = \min_{k\in\mathbb{Z}} \max_{t\in[0,1]} |\alpha t - 2k\pi|$ and $k_0 \in \mathbb{Z}$ such that $L = \max_{t\in[0,1]} |\alpha t - 2k_0\pi|$. Fix

$$v_s(t) = \exp\{is(\alpha t - 2k_0\pi)\}.$$

Then $v_0(t) = \exp\{0\} = 1$ and $v_1(t) = \exp\{i\alpha t - 2k_0\pi i\} = \exp\{i\alpha t\}$ and

$$\int_0^1 \Big\|\frac{dv}{ds}\Big\| ds = \int_0^1 \|\alpha t - 2k_0\pi\| ds = \int_0^1 \max_{t\in[0,1]} |\alpha t - 2k_0\pi| ds = \int_0^1 L ds = L.$$

Thus $v_s(t)$ is a path in $U(C[0,1])$ connecting 1 and $u(t)$ with length L. Therefore, $\operatorname{cel}(u) = L$.

Let us assume $|\alpha| \leq 2\pi$. For $k = 0$,

$$\max_{t\in[0,1]} |\alpha t - 2k\pi| = \max_{t\in[0,1]} |\alpha t - 0| = |\alpha|.$$

For $k \neq 0$,

$$\max_{t\in[0,1]} |\alpha t - 2k\pi| \geq |0 - 2k\pi| = 2|k|\pi \geq |\alpha|.$$

Hence, $\min_{k\in\mathbb{Z}} \max_{t\in[0,1]} |\alpha t - 2k\pi| = |\alpha|$, that is $\operatorname{cel}(u) = |\alpha|$. ∎

3 Counterexamples

Lemma 3.1 ([13, Lemma 2.4]) *The set of elements in $SU(M_n(\mathbb{C}))$ with at least one repeated eigenvalue is the union of finitely many submanifolds of $SU(M_n(\mathbb{C}))$, all of codimension at least three.*

(Here, $SU(M_n(\mathbb{C}))$ is the set of elements in $U(M_n(\mathbb{C}))$ with determinant 1.*)*

Corollary 3.2 *Let $Z = \{u \in U(M_n(\mathbb{C})) : u$ has repeated eigenvalues$\}$. Then Z is the union of finitely many submanifolds of $U(M_n(\mathbb{C}))$, all of codimension at least three.*

Proof We use the method of proof of Lemma 3.1 (see the proof of [13, Lemma 2.4, pp. 136–137]). For the convenience of the reader, we quote Phillips' proof in [13] below and make suitable modification to fit for our setting. That is, in some places (but not all places) we change $SU(\mathrm{M}_n(\mathbb{C}))$ to $U(\mathrm{M}_n(\mathbb{C}))$. All other notations are the same as [13].

Let P be a partition of n, that is, a sequence $(n_1, \ldots, n_k)$ of positive integers such that $n_1 + \cdots + n_k = n$ and $n_1 \geq n_2 \geq \cdots \geq n_k$. Let M_P be the set of all $u \in U(\mathrm{M}_n(\mathbb{C}))$ having exactly k distinct eigenvalues with multiplicities $n_1, \ldots, n_k$. Let G_P be the set of sequences $(V_1, \ldots, V_k)$ of orthogonal subspaces of $\mathbb{C}^n$ such that $\dim(V_j) = n_j$ for each j. Let W_P be the set of k-tuples of distinct elements $(\lambda_1, \ldots, \lambda_k) \in (S^1)^k$, where

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S^1 is the unit circle on the complex plane. Then W_P and G_P are smooth manifolds. Define $f_P\colon G_P \times W_P \to M_P$ by sending $(V_1, \dots, V_k, \lambda_1, \dots, \lambda_k)$ to the unitary $u \in U(\mathrm{M}_n(\mathbb{C}))$ such that $u\xi = \lambda_j \xi$ for $\xi \in V_j$. Then f_P is a smooth surjective local homeomorphism from $G_P \times W_P$ to M_P.

To show that M_P is a smooth manifold, we must show that f_P is a local diffeomorphism, that is, for each $x \in G_P \times W_P$ there is a smooth map g from a neighborhood of $f_P(x)$ in $U(\mathrm{M}_n(\mathbb{C}))$ to $G_P \times W_P$ such that $g \circ f_P$ is the identity near x and $f_P \circ g$ is the identity on a neighborhood of $f_P(x)$ in M_P. To construct g, let $x = (V_1, \dots, V_k, \lambda_1, \dots, \lambda_k)$ and let $u = f_P(x)$. Choose $\varepsilon > 0$ such that the ε-disks about $\lambda_1, \dots, \lambda_k$ in $\mathbb{C}$ are disjoint. For v close enough to u, let p_j be the spectral projection corresponding to $\{\lambda \in \mathbb{C} : |\lambda - \lambda_j| < \varepsilon\}$ and let W_j be the corresponding subspace. Let $\mu_j = \det(p_j v p_j)^{1/n_j}$, where $p_j v p_j$ is regarded as an operator on W_j and the n_j-th root is the branch going through λ_j. Then $g(v) = (W_1, \dots, W_k, \mu_1, \dots, \mu_k)$ will do. (Note that it is smooth because the projections p_j can be obtained via holomorphic functional calculus.)

Then $U(\mathrm{M}_n(\mathbb{C}))$ is the disjoint union of the manifolds M_P as P runs through all partitions. So the corollary is proved if we can show that $codim(M_P) \geq 3$ for $P \neq (1, \dots, 1)$. Using the notations above, since $v[(f_P \circ g)(v)]^* = \sum_{i=1}^{k} \mu_i^{-1} p_i v p_i$, it is easily seen that the map g above extends to a local diffeomorphism

$$v \mapsto (W_1, \dots, W_k, \mu_1, \dots, \mu_k, \mu_1^{-1} p_1 v p_1, \dots, \mu_k^{-1} p_k v p_k)$$

to a manifold locally diffeomorphic to $G_P \times W_P \times SU(\mathrm{M}_{n_1}(\mathbb{C})) \times \cdots \times SU\,(\mathrm{M}_{n_k}(\mathbb{C}))$, and the dimension of the last part is at least 3 if some $n_j \neq 1$. ■

Lemma 3.3 *Let $f(s,t)\colon X \triangleq [0,1] \times [0,1] \to U(M_n(\mathbb{C}))$ be a smooth map. For any $\delta > 0$, there is a smooth map $g(s,t) : [0,1] \times [0,1] \to U(M_n(\mathbb{C}))$ such that*

(i) $\|f - g\| < \delta$, $\|\frac{\partial f}{\partial s}(s,t) - \frac{\partial g}{\partial s}(s,t)\| < \delta$, $\|\frac{\partial f}{\partial t}(s,t) - \frac{\partial g}{\partial t}(s,t)\| < \delta$;
(ii) *$g(s,t)$ has no repeated eigenvalues for all $(s,t) \in [0,1] \times [0,1]$.*

Proof This is a standard transversal argument. See, for example, [6, pp. 70–71]. We note that although the statement in [6] (see the Transversality Homotopy Theorem on page 70) does not assert that the derivatives are close, the proof shows it nonetheless. For the convenience of the reader, we repeat the construction here for our special case.

By smoothly extending f to an open neighborhood of $[0,1] \times [0,1]$, we can assume f is defined on an open manifold without boundary. Let Z be a subspace of $U(\mathrm{M}_n(\mathbb{C}))$ defined by

$$Z = \{u \in U(\mathrm{M}_n(\mathbb{C})) : \ u \text{ has repeated eigenvalues}\}.$$

Since $U(\mathrm{M}_n(\mathbb{C}))$ is a subspace of $\mathrm{M}_n(\mathbb{C})$ and the latter can be identified with $\mathbb{R}^{2n^2}$ as a topological space, f is a smooth map from X to $\mathbb{R}^{2n^2}$. Let B be the open unit ball of $\mathbb{R}^{2n^2}$ (with Euclidean metric). Then B corresponds to some open ball (contained in the unit ball of $\mathrm{M}_n(\mathbb{C})$) in $\mathrm{M}_n(\mathbb{C})$ with the matrix norm, for which we still use the notation B. Let $0 < \varepsilon < 1/2$. For $x \in X$, $r \in B$, define

$$F(x,r) = \pi[f(x) + \varepsilon r],$$

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where $\pi\colon Gl_n(\mathbb{C}) \to U(\mathrm{M}_n(\mathbb{C}))$ is defined by the polar decomposition, which serves as the map π in [6, p. 69] from the tubular neighborhood of $U(\mathrm{M}_n(\mathbb{C}))$ (which is Y in the notation of [6, p. 70]) to $U(\mathrm{M}_n(\mathbb{C}))$. Notice that $Gl_n(\mathbb{C})$ is an open submanifold of $\mathrm{M}_n(\mathbb{C})$. Thus, π can be considered as a map from an open submanifold of $\mathbb{R}^{2n^2}$ to $\mathbb{R}^{2n^2}$. Even though all the scalars here are complex, the objects are being viewed as real manifolds. Note that for $A \in Gl_n(\mathbb{C})$, $\pi(A) = A \cdot (A^*A)^{-\frac{1}{2}}$. The matrix square root on positive definite Hermitian matrices can be defined using the holomorphic functional calculus of the branch of the square root on the right open half plane of $\mathbb{C}$. Thus, $\pi\colon Gl_n(\mathbb{C})(\subseteq \mathbb{R}^{2n^2}) \to \mathbb{R}^{2n^2}$ is C^∞, since π is a composition of smooth maps. Hence

$$F\colon [0,1] \times [0,1] \times B\,(\subseteq \mathbb{R}^{2n^2+2}) \to \mathbb{R}^{2n^2}$$

is a C^∞ map.

Define $f_r\colon X \to U(\mathrm{M}_n(\mathbb{C}))$ by

$$f_r(x) = F(x,r).$$

Since π restricts to the identity on $U(\mathrm{M}_n(\mathbb{C}))$,

$$f_0(x) = F(x,0) = \pi(f(x)) = f(x).$$

For fixed x, $r \to f(x) + \varepsilon r$ is certainly a submersion of B into $U(\mathrm{M}_n(\mathbb{C}))$. As the composition of two submersions is another, $r \to F(x,r)$ is a submersion. By Corollary 3.2, Z is a finite union of submanifolds of $U(\mathrm{M}_n(\mathbb{C}))$, say $\{N_1,\dots,N_L\}$. So F is transversal to each N_j for $1 \leq j \leq L$. Then by applying the Transversality Theorem (see [6, p. 68]), we have that f_r is transversal to N_j for all $j = 1,\dots,L$ and for almost all $r \in B$. Since each N_j is of codimension at least 3,

$$\dim(X) + \dim(N_j) < \dim\big(U(\mathrm{M}_n(\mathbb{C}))$$

So f_r transversal to N_j implies $\operatorname{Im} f_r \bigcap N_j = \varnothing$. Therefore, $\operatorname{Im} f_r \bigcap Z = \varnothing$ for almost all $r \in B$.

Since F is smooth, $\frac{\partial F}{\partial s}$ and $\frac{\partial F}{\partial t}$ are continuous with respect to r. Therefore, for any $\delta > 0$, there exists $\eta > 0$ such that for all $r \in B$ with $\|r\| \leq \eta$, we have

$$\Big\|\frac{\partial F}{\partial s}(s,t,r) - \frac{\partial F}{\partial s}(s,t,0)\Big\| \leq \delta, \quad \Big\|\frac{\partial F}{\partial t}(s,t,r) - \frac{\partial F}{\partial t}(s,t,0)\Big\| \leq \delta.$$

Thus,

$$\begin{aligned}\Big\|\frac{\partial f_r}{\partial s}(s,t) - \frac{\partial f}{\partial s}(s,t)\Big\| &= \Big\|\frac{\partial F}{\partial s}(s,t,r) - \frac{\partial F}{\partial s}(s,t,0)\Big| \leq \delta,\\ \Big\|\frac{\partial f_r}{\partial t}(s,t) - \frac{\partial f}{\partial t}(s,t)\Big\| &= \Big\|\frac{\partial F}{\partial t}(s,t,r) - \frac{\partial F}{\partial t}(s,t,0)\Big\| \leq \delta.\end{aligned}$$

Finally, by taking r appropriately, we can get that f_r satisfies properties (i) and (ii). Setting $g = f_r$ completes the proof. ■

Remark 3.4 In [13, Lemma 2.5], N. C. Phillips proves that any continuous map, say f, from a 2-dimensional space X to $SU(\mathrm{M}_n(\mathbb{C}))$ can be approximated arbitrarily well by a continuous map, say g, from X to $SU(\mathrm{M}_n(\mathbb{C}))$ such that $g(x)$ has no repeated eigenvalues for all $x \in X$. The similar case for self-adjoint matrices (instead of unitaries) has been proved in [2, p. 77].

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Corollary 3.5 *Let $\widetilde{F}_s$ be a rectifiable path in $U(M_k(C[0,1]))$. For any $\varepsilon > 0$, there exists a path F_s in $U(M_k(C[0,1]))$ such that*

(i) $\|F - \widetilde{F}\| < \varepsilon$;
(ii) *$F_s(t)$ has no repeated eigenvalues for all $(s,t) \in [0,1] \times [0,1]$;*
(iii) $|\operatorname{length}(\widetilde{F}) - \operatorname{length}(F)| < \varepsilon$.

Moreover, if for each $t \in [0,1]$ $\widetilde{F}_1(t)$ has no repeated eigenvalues, then F can be chosen to be such that $F_1(t) = \widetilde{F}_1(t)$ for all $t \in [0,1]$.

Proof Let ε_1 be a small number to be determined later, and let δ_0 be such that $|1 - e^{i\theta}| \le \delta_0$ implies $|\theta| \le (1+\varepsilon_1)|1 - e^{i\theta}|$ for $\theta \in (-\pi, \pi)$. For $\varepsilon > 0$, let $\delta = \min\{\delta_0, \varepsilon/6, 1/2\}$. By the definition of the length, there exist $0 = s_0 < s_1 < s_2 < \cdots < s_n = 1$ such that

$$\|\widetilde{F}_{s_{j+1}} - \widetilde{F}_{s_j}\| < \delta/2, \text{ for } j = 0, 1, \ldots, n-1,$$

and

$$\sum_{j=0}^{n-1} \|\widetilde{F}_{s_{j+1}} - \widetilde{F}_{s_j}\| \le \operatorname{length}(\widetilde{F}_s) \le \sum_{j=0}^{n-1} \|\widetilde{F}_{s_{j+1}} - \widetilde{F}_{s_j}\| + \varepsilon/4.$$

Note that for each j, $\widetilde{F}_{s_j}(t)$ is a continuous map from $[0,1]$ to $U(\mathrm{M}_k(\mathbb{C}))$. There exist smooth maps $G_{s_j}(t)\colon [0,1] \to U(\mathrm{M}_k(\mathbb{C}))$, such that

$$\|G_{s_j} - \widetilde{F}_{s_j}\| = \sup_{t \in [0,1]} \|G_{s_j}(t) - \widetilde{F}_{s_j}(t)\| < \frac{\delta}{4n}, \quad j = 0, 1, \ldots, n.$$

Then

$$\begin{aligned} \|G_{s_{j+1}} - G_{s_j}\| &= \|G_{s_{j+1}} - \widetilde{F}_{s_{j+1}} + \widetilde{F}_{s_{j+1}} - \widetilde{F}_{s_j} + \widetilde{F}_{s_j} - G_{s_j}\| \\ &\le \|\widetilde{F}_{s_{j+1}} - \widetilde{F}_{s_j}\| + \frac{\delta}{2n} \le \delta. \end{aligned}$$

And by the first equality, we can also get

$$\|G_{s_{j+1}} - G_{s_j}\| \ge \|\widetilde{F}_{s_{j+1}} - \widetilde{F}_{s_j}\| - \frac{\delta}{2n}.$$

Therefore,

$$\Big| \|G_{s_{j+1}} - G_{s_j}\| - \|\widetilde{F}_{s_{j+1}} - \widetilde{F}_{s_j}\| \Big| \le \frac{\delta}{2n},$$
$$\Big| \sum_{j=0}^{n-1} \|G_{s_{j+1}} - G_{s_j}\| - \sum_{j=0}^{n-1} \|\widetilde{F}_{s_{j+1}} - \widetilde{F}_{s_j}\| \Big| \le \frac{\delta}{2}.$$

Thus,

$$\sum_{j=0}^{n-1} \|G_{s_{j+1}} - G_{s_j}\| - \frac{\delta}{2} \le \operatorname{length}(\widetilde{F}_s) \le \sum_{j=0}^{n-1} \|G_{s_{j+1}} - G_{s_j}\| + \frac{\delta}{2} + \varepsilon/4,$$

$$\sum_{j=0}^{n-1} \|G_{s_{j+1}} - G_{s_j}\| - \frac{\varepsilon}{8} \le \operatorname{length}(\widetilde{F}_s) \le \sum_{j=0}^{n-1} \|G_{s_{j+1}} - G_{s_j}\| + \varepsilon/2. \tag{3.1}$$

Now we want to define a smooth function

$$\widetilde{G}_s(t)\colon [0,1] \times [0,1] \to U(\mathrm{M}_k(\mathbb{C}))$$

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such that $\widetilde{G}_{s_j}(t) = G_{s_j}(t)$ for $j = 0, 1, \dots, n,\ t \in [0,1]$. We will define it piece by piece on each subinterval $[s_j, s_{j+1}]$ ($j = 0, 1, \dots, n-1$).

Suppose

$$G^*_{s_j} G_{s_{j+1}} = U_j \begin{pmatrix} e^{i\alpha_1(t)} & 0 & \cdots & 0 \\ 0 & e^{i\alpha_2(t)} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & e^{i\alpha_k(t)} \end{pmatrix} U^*_j.$$

Since $\|G^*_{s_j} G_{s_{j+1}} - I\| = \|G_{s_j} - G_{s_{j+1}}\| \leq \delta < 1$, there exists a self-adjoint element $H_j(t)$ in $\mathrm{M}_k(C[0,1])$ with $\|H_j\| < 2\pi$ such that $G^*_{s_j} G_{s_{j+1}}(t) = e^{iH_j(t)}$. (Here $H_j(t) = -i \log[G^*_{s_j}(t) G_{s_{j+1}}(t)]$, which is a smooth function.) Define

$$\widetilde{G}_s(t) = G_{s_j}(t) e^{i \frac{s-s_j}{s_{j+1}-s_j} H_j} \text{ for } s_j \leq s \leq s_{j+1}, \quad t \in [0,1], \quad j = 0, 1, \dots, n-1.$$

Then $\widetilde{G}_s(t)$ ($s_j \leq s \leq s_{j+1}$) is a path in $U(\mathrm{M}_k(C[0,1]))$ from G_{s_j} to $G_{s_{j+1}}$ and $\widetilde{G}_s(t)$ is smooth for $(s,t) \in [s_j, s_{j+1}] \times [0,1]$. Moreover,

$$\begin{aligned}
&\operatorname{length}(\widetilde{G}_s|_{s_j \leq s \leq s_{j+1}}) \\
&= \int_{s_j}^{s_{j+1}} \Big\| \frac{\partial \widetilde{G}_s}{\partial s} \Big\| ds \leq \int_{s_j}^{s_{j+1}} \frac{1}{s_{j+1} - s_j} \|G_{s_j}(t) H_j(t)\| ds \\
&\leq (1+\varepsilon_1) \left\| G_{s_j} U_j \begin{pmatrix} 1 - e^{i\alpha_1(t)} & 0 & \cdots & 0 \\ 0 & 1 - e^{i\alpha_2(t)} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 - e^{i\alpha_k(t)} \end{pmatrix} U^*_j \right\| \\
&= (1+\varepsilon_1) \left\| G_{s_j} \left[I - U_j \begin{pmatrix} e^{i\alpha_1(t)} & 0 & \cdots & 0 \\ 0 & e^{i\alpha_2(t)} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & e^{i\alpha_k(t)} \end{pmatrix} U^*_j \right] \right\| \\
&= (1+\varepsilon_1) \|G_{s_j} [I - G^*_{s_j} G_{s_{j+1}}]\| = (1+\varepsilon_1) \|G_{s_j} - G_{s_{j+1}}\|.
\end{aligned}$$

Therefore, $\widetilde{G}_s$ ($0 \leq s \leq 1$) is a piecewise smooth path in $U(\mathrm{M}_k(C[0,1]))$ and

$$\sum_{j=0}^{n-1} \|G_{s_{j+1}} - G_{s_j}\| \leq \operatorname{length}(\widetilde{G}_s) \leq (1+\varepsilon_1) \sum_{j=0}^{n-1} \|G_{s_{j+1}} - G_{s_j}\|.$$

Thus, by (3.1) we have

$$\operatorname{length}(\widetilde{G}_s)(1+\varepsilon_1)^{-1} - \frac{\varepsilon}{8} \leq \operatorname{length}(\widetilde{F}_s) \leq \operatorname{length}(\widetilde{G}_s) + \varepsilon/2.$$

Finally, pick any smooth monotone function $\xi \colon [0,1] \to [0,1]$ with

$$\xi(0) = 0,\ \xi(1) = 1, \quad \frac{d^n \xi}{ds^n}\Big|_{s=0} = 0, \quad \frac{d^n \xi}{ds^n}\Big|_{s=1} = 0 \ \text{ for all } n \geq 1.$$

Let

$$\widetilde{G}'_s(t) = G_{s_j}(t) e^{i\xi(\frac{s-s_j}{s_{j+1}-s_j}) H_j} \text{ for } s_j \leq s \leq s_{j+1}, \quad t \in [0,1],\ j = 0, 1, \dots, n-1.$$

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Then $\widetilde{G}'_s(t)$ is smooth for all $(s,t) \in [0,1] \times [0,1]$ (since $(\partial \widetilde{G}'_s(t))/(\partial s)|_{s=s_j} = 0$ from both left and right for all $j = 1, 2, \dots, n-1$) and $\text{length}(\widetilde{G}'_s) = \text{length}(\widetilde{G}_s)$. And for each $(s,t) \in [0,1] \times [0,1]$,

$$\begin{aligned}
\|\widetilde{G}'_s(t) - \widetilde{F}_s(t)\| &= \|\widetilde{G}'_s(t) - \widetilde{G}'_{s_j}(t) + \widetilde{G}'_{s_j}(t) - \widetilde{F}_{s_j}(t) + \widetilde{F}_{s_j}(t) - \widetilde{F}_s(t)\| \\
&\le \|\widetilde{G}_{s_{j+1}}(t) - \widetilde{G}_{s_j}(t)\| + \|\widetilde{G}_{s_j}(t) - \widetilde{F}_{s_j}(t)\| + \|\widetilde{F}_{s_j}(t) - \widetilde{F}_s(t)\| \\
&\le \delta + \frac{\delta}{4n} + \frac{\varepsilon}{4} \le \varepsilon/2,
\end{aligned}$$

where s_j satisfies $s_j \le s \le s_{j+1}$.

Thus, by choosing ε_1 appropriately, we have

$$|\,\text{length}(\widetilde{G}'_s) - \text{length}(\widetilde{F}_s)| < \varepsilon/2 \text{ and } \|\widetilde{G}' - \widetilde{F}\| < \varepsilon/2.$$

Since $\widetilde{G}'_s$ can be seen as a smooth map from $[0,1] \times [0,1]$ to $U(\mathrm{M}_k(\mathbb{C}))$, by Lemma 3.3, there exists F such that $\|F - \widetilde{G}'\| < \varepsilon/2$ and $F_s(t)$ has no repeated eigenvalues for all $(s,t) \in [0,1] \times [0,1]$. Moreover,

$$|\,\text{length}(F_s) - \text{length}(\widetilde{G}'_s)| = \Big| \int_0^1 \Big\| \frac{\partial F}{\partial s} \Big\| ds - \int_0^1 \Big\| \frac{\partial \widetilde{G}'}{\partial s} \Big\| ds \Big| < \varepsilon/2.$$

Thus F satisfies properties (i)–(iii), which is what we want.

Moreover, if $\widetilde{F}_1(t)$ has no repeated eigenvalues for all $t \in [0,1]$, then there exists $\eta > 0$ such that $\|u(t) - \widetilde{F}_1(t)\| < \eta$ implies $u(t)$ has no repeated eigenvalues for all $t \in [0,1]$. For $\varepsilon = \eta/2$, by previous arguments of this proof, we can find a path F_s satisfying properties (i)–(iii). Let $s_0 \in [s_{n-1}, 1)$ be such that $\|F_{s_0}(t) - F_1(t)\| \le \eta/4$ (where s_{n-1} is a point of the partition of $[0,1]$ for which we mentioned in the previous arguments of this proof). Then

$$\|F_{s_0}(t) - \widetilde{F}_1(t)\| = \|F_{s_0}(t) - F_1(t) + F_1(t) - \widetilde{F}_1(t)\| \le 3\eta/4.$$

Now let us redefine $F_s(t)$ on the subinterval $[s_0, 1]$ (still using the notation $F_s(t)$) in a similar way as above:

$$F_s(t) = F_{s_0}(t) e^{i \frac{s-s_0}{1-s_0} H_n(t)}, \qquad \text{for } s_0 \le s \le 1,$$

where $H_n(t) = -i \log[F^*_{s_0}(t) \widetilde{F}_1(t)]$. Since this newly defined path F_s lies in the η neighborhood of F_1, $F_s(t)$ has no repeated eigenvalues for all $(s,t) \in [0,1] \times [0,1]$. Thus, this F_s is what we want. ■

Definition 3.6 For a metric space (Y, d), let

$$P^k Y := \{(y_1, y_2, \dots, y_k) : y_i \in Y\} / \sim,$$

where $(y_1, y_2, \dots, y_k) \sim (\widetilde{y}_1, \widetilde{y}_2, \dots, \widetilde{y}_k)$ if there exists $\sigma \in S_k$ such that $y_{\sigma(i)} = \widetilde{y}_i$ for all $1 \le i \le k$. Let $[y_1, y_2, \dots, y_k]$ denote the equivalent class of $(y_1, y_2, \dots, y_k)$ in $P^k Y$. Also define the metric of $P^k Y$ as

$$\text{dist}([y_1, y_2, \dots, y_k], [\widetilde{y}_1, \widetilde{y}_2, \dots, \widetilde{y}_k]) = \min_{\sigma \in S_k} \max_{1 \le i \le k} d(y_i, \widetilde{y}_{\sigma(i)}).$$

The proof of the following lemma is straightforward.

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Lemma 3.7 *Let (Y,d) be a metric space and let*

$$\pi\colon \underbrace{Y\times Y\times\cdots\times Y}_{k} \longrightarrow P^kY$$

be the quotient map. Let

$$X\subset \underbrace{Y\times Y\times\cdots\times Y}_{k}$$

be the set consisting of those elements $(y_1,y_2,\dots,y_k)$ with $y_i\neq y_j$ if $i\neq j$. Then the restriction of π to X is a covering map.

We need the following easy lemma.

Lemma 3.8 *Let $F\colon[0,1]\times[0,1]\to P^kS^1$ be a continuous function. Suppose*

$$F(s,t)=[x_1(s,t),x_2(s,t),\dots,x_k(s,t)],$$

and for all $(s,t)\in[0,1]\times[0,1]$, $x_i(s,t)\neq x_j(s,t)$ if $i\neq j$. Then there are continuous functions $f_1,f_2,\dots,f_k\colon[0,1]\times[0,1]\to S^1$ such that

$$F(s,t)=\big[f_1(s,t),f_2(s,t),\dots,f_k(s,t)\big].$$

Proof Let

$$\pi\colon \underbrace{S^1\times S^1\times\cdots\times S^1}_{k}\to P^kS^1$$

denote the quotient map, and let

$$X\subset \underbrace{S^1\times S^1\times\cdots\times S^1}_{k}$$

be the set consisting of those elements $(x_1,x_2,\cdots,x_k)$ with $x_i\neq x_j$ if $i\neq j$. Then by Lemma 3.7, $\pi|_X$ is a covering map from X to $\pi(X)$ (which is a subset of P^kS^1).

Note from the assumption of the lemma, the image of F is contained in $\pi(X)$. Since $[0,1]\times[0,1]$ is simply connected, by the standard lifting theorem for covering spaces, the map $F\colon[0,1]\times[0,1]\to\pi(X)\subset P^kS^1$ can be lifted to a map

$$F_1\colon[0,1]\times[0,1]\to X(\subset \underbrace{S^1\times S^1\times\cdots\times S^1}_{k}).$$

Let $\pi_j\colon S^1\times S^1\times\cdots\times S^1\to S^1$ be the projection onto the j-th coordinate. For $1\le j\le k$, define functions $f_j\colon[0,1]\times[0,1]\to S^1$ by $f_j(s,t)=\pi_j(F_1(s,t))$. Then it is easy to see that the f_j's satisfy the requirements. ■

Remark 3.9 Let F_s be a path in $U(M_k(C[0,1]))$ such that $F_s(t)$ has no repeated eigenvalues for all $(s,t)\in[0,1]\times[0,1]$. Let $\Lambda\colon[0,1]\times[0,1]\to P_kS^1$ be the eigenvalue map of $F_s(t)$; *i.e.,* $\Lambda(s,t)=[x_1(s,t),x_2(s,t),\dots,x_k(s,t)]$, where $\{x_i(s,t)\}_{i=1}^k$ are eigenvalues of the matrix $F_s(t)$. By Lemma 3.8, there are continuous functions $f_1,f_2,\dots,f_k\colon[0,1]\times[0,1]\to S^1$ such that

$$\Lambda(s,t)=\big[f_1(s,t),f_2(s,t),\dots,f_k(s,t)\big].$$

For each fixed $(s,t)\in[0,1]\times[0,1]$, there is a unitary $U_s(t)$ such that

$$F_s(t)=U_s(t)\,\mathrm{diag}\big[f_1(s,t),f_2(s,t),\dots,f_k(s,t)\big]\,U_s(t)^*.$$

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Note that $U_s(t)$ can be chosen to be continuous, but in this paper we do not need this property.

Proposition 3.10 ([1, lines 13–18, p. 71]) *If $U, V \in M_n(\mathbb{C})$ are unitaries with eigenvalues $u_1, u_2, \dots, u_n$ and $v_1, v_2, \dots, v_n$ respectively, then*

$$\min_{\sigma \in S_n} \max_i |u_i - v_{\sigma(i)}| \leq \|U - V\|.$$

The same result for a pair of Hermitian matrices is due to H. Weyl (called Weyl's Inequality see [22]).

Lemma 3.11 *Let F_s be a path in $U(M_n(C[0,1]))$ and $f_s^1(t), f_s^2(t), \dots, f_s^n(t)$ be continuous functions such that*

$$F_s(t) = U_s(t)\,\mathrm{diag}[f_s^1(t), f_s^2(t), \dots, f_s^n(t)]U_s(t)^*,$$

where $U_s(t)$ are unitaries. Suppose for any $(s,t) \in [0,1] \times [0,1]$, $f_s^i(t) \neq f_s^j(t)$ if $i \neq j$, then

$$\mathrm{length}(F_s) \geq \max_{1 \leq j \leq n} \{\mathrm{length}(f_s^j)\}.$$

(In this lemma, we assume that $F_s(t)$ is continuous, but we do not assume $U_s(t)$ is continuous.)

Proof Let

$$\varepsilon = \min\bigl\{ |(f_s^i(t) - f_s^j(t))| : i \neq j,\ 1 \leq i, j \leq n,\ s \in [0,1],\ t \in [0,1] \bigr\}.$$

Since $f_s^j(t)$ is continuous with respect to s for each j, there exists $\delta > 0$ such that for any partition $\mathcal{P} = \{s_1, s_2, \dots, s_\lambda\}$ with $|\mathcal{P}| < \delta$,

$$\|f_{s_i}^j(t) - f_{s_{i-1}}^j(t)\| < \varepsilon/2 \ \text{ for all } 2 \leq i \leq \lambda,\ 1 \leq j \leq n.$$

Then by Proposition 3.10,

$$\begin{aligned}\mathrm{length}(F_s) &\geq \sum_{i=2}^{\lambda} \|F_{s_i} - F_{s_{i-1}}\| = \sum_{i=2}^{\lambda} \sup_{t \in [0,1]} \|F_{s_i}(t) - F_{s_{i-1}}(t)\| \\ &\geq \sum_{i=2}^{\lambda} \sup_{t \in [0,1]} \Bigl[\min_{\sigma \in S_n} \max_{1 \leq j \leq n} |f_{s_i}^j(t) - f_{s_{i-1}}^{\sigma(j)}(t)|\Bigr].\end{aligned}$$

If $\sigma(j) \neq j$, then

$$|f_{s_i}^j(t) - f_{s_{i-1}}^{\sigma(j)}(t)| \geq \\ |f_{s_i}^j(t) - f_{s_i}^{\sigma(j)}(t)| - |f_{s_i}^{\sigma(j)}(t) - f_{s_{i-1}}^{\sigma(j)}(t)| > \varepsilon - \varepsilon/2 = \varepsilon/2.$$

If $\sigma(j) = j$, then $|f_{s_i}^j(t) - f_{s_{i-1}}^j(t)| < \varepsilon/2$. Therefore,

$$\min_{\sigma \in S_n} \max_{1 \leq j \leq n} |f_{s_i}^j(t) - f_{s_{i-1}}^{\sigma(j)}(t)| = \max_{1 \leq j \leq n} |f_{s_i}^j(t) - f_{s_{i-1}}^j(t)|,$$

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$$\text{(3.2)} \qquad \begin{aligned} \operatorname{length}(F_s) &\geq \sum_{i=2}^{\lambda} \sup_{t\in[0,1]} \Big[\min_{\sigma\in S_n} \max_{1\leq j\leq n} |f_{s_i}^j(t) - f_{s_{i-1}}^{\sigma(j)}(t)| \Big] \\ &\geq \sum_{i=2}^{\lambda} \sup_{t\in[0,1]} \max_{1\leq j\leq n} |f_{s_i}^j(t) - f_{s_{i-1}}^j(t)| \\ &\geq \sum_{i=2}^{\lambda} \max_{1\leq j\leq n} \|f_{s_i}^j - f_{s_{i-1}}^j\| \geq \max_{1\leq j\leq n} \sum_{i=2}^{\lambda} \|f_{s_i}^j - f_{s_{i-1}}^j\|. \end{aligned}$$

Since (3.2) holds for any partition $\mathcal{P}$ with $|\mathcal{P}| < \delta$, we have

$$\operatorname{length}(F_s) \geq \max_{1\leq j\leq n} \operatorname{length}(f_s^j). \qquad \blacksquare$$

Example 3.12 Let $A = \mathrm{M}_{10}(\mathrm{C}[0,1])$. Define

$$u(t) = \begin{pmatrix} e^{-2\pi i t\frac{9}{10}} & 0 & \cdots & 0 \\ 0 & e^{2\pi i t\frac{1}{10}} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & e^{2\pi i t\frac{1}{10}} \end{pmatrix}_{10\times 10}.$$

Then u is a unitary in A with $\det(u) = 1$ and $u \sim_h 1$.

Theorem 3.13 *Let $u \in M_{10}(C[0,1])$ be defined as in 3.12. Then*

$$\operatorname{cel}(u) \geq 2\pi \cdot \frac{9}{10}.$$

Proof Let $\widetilde{F}_s$ be a rectifiable path in $U(\mathrm{M}_{10}(\mathrm{C}[0,1]))$ with $\widetilde{F}_1(t) = u(t)$ and $\widetilde{F}_0 = 1 \in \mathrm{M}_{10}(\mathrm{C}[0,1])$. For any $0 < \varepsilon < 1/4$, by Corollary 3.5 there is a path F_s in $U(\mathrm{M}_{10}(\mathrm{C}[0,1]))$ such that

(a) $\|F - \widetilde{F}\| < \varepsilon/2$,
(b) $F_s(t)$ has no repeated eigenvalues for all $(s,t) \in [0,1]\times[0,1]$, and
(c) $|\operatorname{length}(\widetilde{F}) - \operatorname{length}(F)| < \varepsilon/2$.

By Lemma 3.8 and Remark 3.9, there are continuous maps

$$f^1, f^2, \dots, f^{10}\colon [0,1]\times[0,1] \to S^1$$

and unitaries $U_s(t)$ such that

$$F_s(t) = U_s(t)\operatorname{diag}\big[f_s^1(t), f_s^2(t), \dots, f_s^{10}(t)\big]\,U_s(t)^*.$$

By Lemma 3.11,

$$\operatorname{length}(F_s) \geq \max_{1\leq i\leq 10}\{\operatorname{length}(f_s^i)\}.$$

Since $\|F - \widetilde{F}\| < \varepsilon/2$, $\|f_0^j - 1\| < \varepsilon/2$ for all $1 \leq j \leq 10$.

Note that $|e^{2\pi i t\frac{1}{10}} - e^{-2\pi i t\frac{9}{10}}| \geq \varepsilon$ for each fixed $t \in (\varepsilon, 1-\varepsilon)$. Also note that $\widetilde{F}_1(t) = u(t)$ has one eigenvalue equal to $e^{-2\pi i t\frac{9}{10}}$ and 9 eigenvalues equal to $e^{2\pi i t\frac{1}{10}}$. Therefore, $F_1(t)$ has only one eigenvalue that is in the $\frac{\varepsilon}{2}$-neighborhood of $e^{-2\pi i t\frac{9}{10}}$; all other eigenvalues are in the $\frac{\varepsilon}{2}$-neighborhood of $e^{2\pi i t\frac{1}{10}}$. (Notice that those two

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neighborhoods, namely $\frac{\varepsilon}{2}$-neighborhood of $e^{-2\pi it\frac{9}{10}}$ and $\frac{\varepsilon}{2}$-neighborhood of $e^{2\pi it\frac{1}{10}}$, are disjoint for each $t\in(\varepsilon,1-\varepsilon)$. Thus there exists one and only one j_0 such that

$$|f_1^{j_0}(t)-e^{-2\pi it\frac{9}{10}}|<\varepsilon/2.$$

For other $j\neq j_0$,

$$|f_1^{j}(t)-e^{2\pi it\frac{1}{10}}|<\varepsilon/2.$$

Since all f^j's are continuous, the index j_0 should be the same for all $t\in(\varepsilon,1-\varepsilon)$.

Thus $f_s^{j_0}$ is a path in $U(\mathrm{C}[0,1])$ connecting a point near 1 and a point near $e^{-2\pi it\frac{9}{10}}$. By Lemma 2.5, $\operatorname{length}(f_s^{j_0})\geq\frac{9}{10}\cdot 2\pi-\varepsilon$. Therefore,

$$\operatorname{length}(\widetilde{F}_s)\geq\frac{9}{10}\cdot 2\pi-\varepsilon/2-\varepsilon\geq\frac{9}{10}\cdot 2\pi-2\varepsilon.$$

Since ε is arbitrary, we have $\operatorname{cel}(u)\geq\frac{9}{10}\cdot 2\pi$, which completes the proof. ∎

Example 3.14 We give examples in some simple inductive limit C*-algebras.

Let $\{x_1,x_2,\dots\}$ be a countable distinct dense subset of $[0,1]$ and let $\{k_n\}_{n=2}^{\infty}$ be a sequence of integers satisfying

$$\prod_n\frac{10^{k_n}-1}{10^{k_n}}>\frac{11}{12}.$$

Let

$$A_1=\mathrm{M}_{10}(\mathrm{C}[0,1]),\ A_2=\mathrm{M}_{10^{k_2}}(A_1),\dots,\ A_n=\mathrm{M}_{10^{k_n}}(A_{n-1}),\dots.$$

Let $\varphi_{n,n+1}\colon A_n\to A_{n+1}$ be defined by

$$\varphi_{n,n+1}(f)=\begin{pmatrix} f & 0 & \cdots & 0 & 0\\ 0 & f & \cdots & 0 & 0\\ \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & \cdots & f & 0\\ 0 & 0 & \cdots & 0 & f(x_n)\end{pmatrix}_{10^{k_n}\times 10^{k_n}}$$

and $A=\varinjlim(A_i,\varphi_{i,i+1})$ be the inductive limit C*-algebra. Then A is simple.

Let $u(t)\in A_1$ be defined as in Example 3.12. Then (see Theorem 3.16 and Corollary 3.17)

$$\operatorname{cel}(\varphi_{1,\infty}(u))\geq\frac{9}{10}\cdot 2\pi.$$

Simple inductive limit C*-algebras of such form for a general space X (instead of [0,1]) were studied by Goodearl [5]. Its exponential rank was calculated by Gong and Lin for the case of real rank zero such algebras [4], and by Phillips for the general case [15].

Lemma 3.15 *Let $\theta\colon P^L\mathbb{R}\to(\mathbb{R}^L,d_{\max})$ be defined by*

$$\theta[x_1,x_2,\dots,x_L]=(y_1,y_2,\dots,y_L)$$

if and only if $[x_1,x_2,\dots,x_L]=[y_1,y_2,\dots,y_L]$ and $y_1\leq y_2\leq\cdots\leq y_L$. Then θ is an isometry.

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Proof Let $a = [a_1, a_2, \dots, a_L]$, $b = [b_1, b_2, \dots, b_L]$ be any two elements in $P^L\mathbb{R}$. Without loss of generality, we can assume that $a_1 \le a_2 \le \cdots \le a_L$ and $b_1 \le b_2 \le \cdots \le b_L$. Thus,

$$d_{\max}(\theta(a), \theta(b)) = \max_{1\le i\le L} |a_i - b_i|.$$

If $\operatorname{dist}(a, b) \ne \max_{1\le i\le L} |a_i - b_i|$, then there exists a permutation $\sigma \in S_L$ such that

$$l \triangleq \max_{1\le i\le L} |a_i - b_{\sigma(i)}| < \max_{1\le i\le L} |a_i - b_i|.$$

Since $l < \max_{1\le i\le L} |a_i - b_i|$, there exists k such that $|a_k - b_k| > l$.

If $a_k < b_k$, then $|a_i - b_j| > l$ for any $i \le k$, $j \ge k$. Since the cardinality of the set $\{\sigma(1), \sigma(2), \dots, \sigma(k)\}$ is k, there is at least one element $i_0 \in \{1, 2, \dots, k\}$ with $\sigma(i_0) \ge k$. Then $|a_{i_0} - b_{\sigma(i_0)}| > l$. Therefore, $\max_{1\le i\le k} |a_i - b_{\sigma(i)}| > l$. Similarly, if $a_k > b_k$, one can prove $\max_{k\le i\le L} |a_i - b_{\sigma(i)}| > l$. In either case, it contradicts $l = \max_{1\le i\le L} |a_i - b_{\sigma(i)}|$. Therefore,

$$\operatorname{dist}(a, b) = \max_{1\le i\le L} |a_i - b_i| = d_{\max}(\theta(a), \theta(b)),$$

which means θ is an isometry. ■

Theorem 3.16 *Let A_i, $\varphi_{i,i+1}$, $(i \in \mathbb{N})$ be defined as in Example 3.14, for any $\varepsilon \in (0, \frac{1}{100})$, let $u_\varepsilon \in A_1$ be defined by:*

$$u_\varepsilon(t) = \begin{pmatrix} e^{-2\pi i t(\frac{9}{10}-\varepsilon)} & 0 & \cdots & 0 \\ 0 & e^{2\pi i t(\frac{1}{10}-\varepsilon)} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & e^{2\pi i t(\frac{1}{10}-\varepsilon)} \end{pmatrix}_{10\times 10}.$$

Then

$$\operatorname{cel}(\varphi_{1,n}(u_\varepsilon)) \ge 2\pi\Big(\frac{9}{10} - \varepsilon\Big) - 5\varepsilon \textit{ for all } n \in \mathbb{N}.$$

Proof With easy calculations, we know that

$$\varphi_{1,n}(u_\varepsilon) \in M_L(C[0,1]), \text{ where } L = 10\prod_{i=2}^{n} 10^{k_i}.$$

On the diagonal of $\varphi_{1,n}(u_\varepsilon)$, there are $\prod_{i=2}^{n}(10^{k_i} - 1)$ terms equal to $e^{-2\pi i t(\frac{9}{10}-\varepsilon)}$, $9\cdot\prod_{i=2}^{n}(10^{k_i} - 1)$ terms equal to $e^{2\pi i t(\frac{1}{10}-\varepsilon)}$, and the rest are constants. Let α, β, and γ denote, respectively, the numbers of terms of the forms $e^{-2\pi i t(\frac{9}{10}-\varepsilon)}$, $e^{2\pi i t(\frac{1}{10}-\varepsilon)}$, and constants on the diagonal of $\varphi_{1,n}(u_\varepsilon)$ (i.e. $\alpha = \prod_{i=2}^{n}(10^{k_i} - 1)$, $\beta = 9\cdot\prod_{i=2}^{n}(10^{k_i} - 1)$ and $\gamma = L - \alpha - \beta$). Therefore

$$\frac{\alpha+\beta}{L} = \prod_{i=2}^{n} \frac{10^{k_i} - 1}{10^{k_i}} > \frac{11}{12},$$

which implies $\frac{\alpha}{\gamma} > \frac{11}{10}$.

For each $t \in [0, 1]$, let $E(t)$ be the set consisting of all eigenvalues of $\varphi_{1,n}(u_\varepsilon)(t)$ (counting multiplicities). Define continuous functions $\overline{\overline{y}}_k(t)$ $(1 \le k \le L)$ from $[0, 1]$

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to $[-\frac{9}{10}+\varepsilon, \frac{1}{10}-\varepsilon]$ as follows:

$$\overline{\overline{y}}_k(t)=\begin{cases}-\left(\frac{9}{10}-\varepsilon\right)t, & \text{if } 1\le k\le\alpha,\\ \left(\frac{1}{10}-\varepsilon\right)t, & \text{if } \alpha+1\le k\le\alpha+\beta,\\ -\left(\frac{9}{10}-\varepsilon\right)x_\bullet \text{ or } \left(\frac{1}{10}-\varepsilon\right)x_\bullet, & \text{if } \alpha+\beta+1\le k\le L,\end{cases}$$

for $x_\bullet\in\{x_1,x_2,\dots,x_n\}$ such that for all $t\in[0,1]$,

$$\left\{\exp\{2\pi i\overline{\overline{y}}_k(t)\} : 1\le k\le L\right\}=E(t).$$

Let θ be the map defined in Lemma 3.15 and p_k $(1\le k\le L)$ be projections from $\mathbb{R}^L$ to $\mathbb{R}$ with respect to the k-th coordinate. Define functions $\overline{y}_k(t)\colon[0,1]\to[-\frac{9}{10}+\varepsilon,\frac{1}{10}-\varepsilon]$ for $1\le k\le L$ as follows:

$$\overline{y}_k(t)=p_k\theta\left[\overline{\overline{y}}_1(t),\overline{\overline{y}}_2(t),\dots,\overline{\overline{y}}_L(t)\right].$$

Then $\overline{y}_k(t)$ $(1\le k\le L)$ are continuous functions with $\overline{y}_1(t)\le\overline{y}_2(t)\le\cdots\le\overline{y}_L(t)$ for all $t\in[0,1]$ and (counting multiplicities)

$$\{\overline{y}_1(t),\overline{y}_2(t),\dots,\overline{y}_L(t)\}=\{\overline{\overline{y}}_1(t),\overline{\overline{y}}_2(t),\dots,\overline{\overline{y}}_L(t)\}.$$

For each fixed $t\in[0,1]$, there are at most γ terms (the constants referred to above) in the set $\{\overline{y}_k(t)\}_{k=1}^L$ that could be less than $-(\frac{9}{10}-\varepsilon)t$. Therefore,

$$\overline{y}_k(t)\ge-\left(\frac{9}{10}-\varepsilon\right)t,\quad\text{for } k\ge\gamma+1.$$

Similarly, for each fixed $t\in[0,1]$ there are at most $\gamma+\beta$ terms (the constants or terms of the form $(\frac{1}{10}-\varepsilon)t$) in the set $\{\overline{y}_k(t)\}_{k=1}^L$ which could be greater than $-(\frac{9}{10}-\varepsilon)t$. So

$$\overline{y}_k(t)\le-\left(\frac{9}{10}-\varepsilon\right)t,\quad\text{for } k\le L-\gamma-\beta=\alpha.$$

Since $\alpha>\gamma$,

$$\overline{y}_k(t)=-\left(\frac{9}{10}-\varepsilon\right)t,\quad\text{for } \gamma+1\le k\le\alpha.$$

Let $y_k(t)=\exp\{2\pi i\overline{y}_k(t)\}$ for $1\le k\le L$. Then it is obvious that

$$y_k(t)=e^{-2\pi it(\frac{9}{10}-\varepsilon)},\quad\text{for } \gamma+1\le k\le\alpha,$$

and (counting multiplicities)

$$\{y_k(t):1\le k\le L\}=E(t),\qquad\text{for all } t\in[0,1].$$

Let

$$W(t)=\begin{pmatrix}y_1(t)&0&\cdots&0\\0&y_2(t)&\cdots&0\\\vdots&\vdots&\ddots&\vdots\\0&0&\cdots&y_L(t)\end{pmatrix}.$$

Then for all $t\in[0,1]$, $W(t)$ and $\varphi_{1,n}(u_\varepsilon)(t)$ have exactly the same eigenvalues (counting multiplicities). By [20, Corollary 1.3], $\varphi_{1,n}(u_\varepsilon)$ and W are approximately unitarily conjugate; *i.e.*, there is a sequence Λ_n, $n=1,2,\dots$ of unitaries in $M_L(C[0,1])$ such that $\varphi_{1,n}(u_\varepsilon)=\lim_{n\to\infty}\Lambda_nW\Lambda_n^*$. Therefore, $\mathrm{cel}(\varphi_{1,n}(u_\varepsilon))=\mathrm{cel}(W)$.

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Let ε_j $(1 \leq j \leq L)$ be chosen satisfying $-\varepsilon' < \varepsilon_1 < \varepsilon_2 < \cdots < \varepsilon_\alpha = 0 < \varepsilon_{\alpha+1} < \varepsilon_{\alpha+2} < \cdots < \varepsilon_L < \varepsilon'$, where ε' is a fixed number in $(0, \varepsilon/10)$. Let $\widetilde{y}_k(t) = \exp\{2\pi i(\overline{y}_k(t) + \varepsilon_k)\}$ and

$$\widetilde{W}(t) = \begin{pmatrix} \widetilde{y}_1(t) & 0 & \cdots & 0 \\ 0 & \widetilde{y}_2(t) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \widetilde{y}_L(t) \end{pmatrix} \in U(M_L(C[0,1])).$$

Then $\widetilde{W}(t)$ has no repeated eigenvalues for all $t \in [0,1]$ and

$$\|\widetilde{W}(t) - W(t)\| \leq |e^{2\pi i\varepsilon'} - 1| \leq 2\pi\varepsilon'.$$

Therefore,

$$\mathrm{cel}(\varphi_{1,n}(u_\varepsilon)) = \mathrm{cel}(W) \geq \mathrm{cel}(\widetilde{W}) - \varepsilon.$$

Let $\widetilde{F}_s(t)$ be a rectifiable path in $U(M_L(C[0,1]))$ from 1 to $\widetilde{W}(t)$ with $\widetilde{F}_0(t) = 1$ and $\widetilde{F}_1(t) = \widetilde{W}(t)$. By Corollary 3.5, there is a smooth path F_s in $U(M_L(C[0,1]))$ such that $\|F - \widetilde{F}\| < \varepsilon$, $F_1(t) = \widetilde{F}_1(t) = \widetilde{W}(t)$, $F_s(t)$ has no repeated eigenvalues for all $(s,t) \in [0,1] \times [0,1]$, and $|\operatorname{length}(\widetilde{F}) - \operatorname{length}(F)| \leq \varepsilon$. By Lemma 3.8 and Remark 3.9, there exist continuous maps $f^1, f^2, \ldots, f^L \colon [0,1] \times [0,1] \longrightarrow S^1$ and unitaries $U_s(t)$ such that

$$F_s(t) = U_s(t)\,\mathrm{diag}(f_s^1(t), f_s^2(t), \ldots, f_s^L(t))U_s(t)^*.$$

Since $F_1(t) = \widetilde{W}(t)$ and $\widetilde{y}_\alpha = e^{-2\pi i t(\frac{9}{10} - \varepsilon)}$, we can assume $f_1^\alpha = e^{-2\pi i t(\frac{9}{10} - \varepsilon)}$. Therefore, f_s^α is a path in $U(C[0,1])$ from a point near 1 to $e^{-2\pi i t(\frac{9}{10} - \varepsilon)}$. By Lemma 2.5,

$$\operatorname{length}(f_s^\alpha) \geq 2\pi\Big(\frac{9}{10} - \varepsilon\Big) - \varepsilon.$$

Therefore,

$$\operatorname{length}(\widetilde{F}_s) \geq 2\pi\Big(\frac{9}{10} - \varepsilon\Big) - \varepsilon - \varepsilon = 2\pi\Big(\frac{9}{10} - \varepsilon\Big) - 2\varepsilon,$$

and

$$\mathrm{cel}(\varphi_{1,n}(u_\varepsilon)) \geq 2\pi\Big(\frac{9}{10} - \varepsilon\Big) - 5\varepsilon. \qquad \blacksquare$$

Corollary 3.17 *Let $A = \lim A_i$ and $u \in A_1$ be defined as in 3.14. Then $\varphi_{1,\infty}(u) \in CU(A)$ and*

$$\mathrm{cel}(\varphi_{1,\infty}(u)) \geq 2\pi \cdot \frac{9}{10}.$$

Note that for $A = \mathrm{M}_n(C[0,1])$ $(n \in \mathbb{N})$, $x \in CU(A)$ if and only if $\det(x(t)) = 1$ for each $t \in [0,1]$.

Proof For any $\varepsilon \in (0, \frac{1}{100})$, let $u_\varepsilon \in A_1$ be defined as in Theorem 3.16. Then

$$\mathrm{cel}(\varphi_{1,n}(u_\varepsilon)) \geq 2\pi\Big(\frac{9}{10} - \varepsilon\Big) - 5\varepsilon \text{ for all } n \in \mathbb{N}.$$

Since $\|\varphi_{1,n}(u) - \varphi_{1,n}(u_\varepsilon)\| \leq 2\pi\varepsilon$ for all $n \in \mathbb{N}$,

$$\mathrm{cel}(\varphi_{1,n}(u)) \geq \mathrm{cel}(\varphi_{1,n}(u_\varepsilon)) - 2\pi\varepsilon \geq 2\pi\Big(\frac{9}{10} - 2\varepsilon\Big) - 5\varepsilon.$$

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Since ε is arbitrary, we have $\mathrm{cel}(\varphi_{1,n}(u)) \geq 2\pi \cdot \frac{9}{10}$, for all $n \in \mathbb{N}$. Therefore,

$$\mathrm{cel}(\varphi_{1,\infty}(u)) \geq 2\pi \cdot \frac{9}{10}. \qquad \blacksquare$$

Theorem 3.18 *Let A_i, $\varphi_{i,i+1}$, $(i \in \mathbb{N})$ be defined as in 3.14. For any $\varepsilon > 0$, there exists i such that $\frac{10^{k_i}-1}{10^{k_i}} \geq 1 - \frac{\varepsilon}{2\pi}$. Let $u \in A_i$ be defined by*

$$u(t) = \begin{pmatrix} e^{-2\pi it\frac{10^{k_i}-1}{10^{k_i}}} & 0 & \cdots & 0 \\ 0 & e^{2\pi it\frac{1}{10^{k_i}}} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & e^{2\pi it\frac{1}{10^{k_i}}} \end{pmatrix}_{10^{k_i}\times 10^{k_i}}.$$

Then $\varphi_{i,\infty}(u) \in CU(A)$ and

$$\mathrm{cel}(\varphi_{i,\infty}(u)) \geq 2\pi\frac{10^{k_i}-1}{10^{k_i}} \geq 2\pi\Big(1 - \frac{\varepsilon}{2\pi}\Big) = 2\pi - \varepsilon.$$

Proof By using the same method of proof as for Theorem 3.16 and Corollary 3.17, we can get the result. ■

Theorem 3.19 *Let A be defined as in 3.14. Then*

$$\mathrm{cel}_{CU}(A) \geq 2\pi.$$

Proof This inequality holds by applying Theorem 3.18. ■

Remark 3.20 After reading Lin's article [9], we know that $\mathrm{cel}_{CU}(A) \leq 2\pi$ for A defined in Example 3.14 (see [9, Lemma 4.5]). Therefore, $\mathrm{cel}_{CU}(A) = 2\pi$.

Remark 3.21 Our paper with the results above was first posted on arxiv on Dec. 2012. Later H. Lin posted a paper on Feb. 2013 (see [9]). In his paper, he provides examples with $\mathrm{cel}_{CU}(A) > \pi$ for unital simple AH-algebras A with tracial rank one, whose K_0-group can realize all possible weakly unperforated Riesz group (see [9, 5.11, 5.12]). He also obtained our Theorem 3.13 with different methods. But we construct an example A with $\mathrm{cel}_{CU}(A) \geq 2\pi$, where A is a simple AH-algebra (see 3.14, 3.17, 3.18). Of course, his paper contains many interesting results in other directions.

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