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The Mumford relations and the moduli of rank three stable bundles

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Abstract. The cohomology ring of the moduli space $\mathcal{M}(n,d)$ of semistable bundles of coprime rank n and degree d over a Riemann surface M of genus $g \geqslant 2$ has again proven a rich source of interest in recent years. The rank two, odd degree case is now largely understood. In 1991 Kirwan [8] proved two long standing conjectures due to Mumford and to Newstead and Ramanan. Mumford conjectured that a certain set of relations form a complete set; the Newstead-Ramanan conjecture involved the vanishing of the Pontryagin ring. The Newstead–Ramanan conjecture was independently proven by Thaddeus [15] as a corollary to determining the intersection pairings. As yet though, little work has been done on the cohomology ring in higher rank cases. A simple numerical calculation shows that the Mumford relations themselves are not generally complete when $n > 2$. However by generalising the methods of [8] and by introducing new relations, in a sense dual to the original relations conjectured by Mumford, we prove results corresponding to the Mumford and Newstead-Ramanan conjectures in the rank three case. Namely we show (Sect. 4) that the Mumford relations and these 'dual' Mumford relations form a complete set for the rational cohomology ring of $\mathcal{M}(3,d)$ and show (Sect. 5) that the Pontryagin ring vanishes in degree $12g-8$ and above.

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1. Introduction

Let $\mathcal{M}(n,d)$ denote the moduli space of semistable holomorphic vector bundles of coprime rank n and degree d over a Riemann surface M of genus $g \geqslant 2$. Throughout this article we will write

$$\bar{g} = g - 1.$$

Recall that a holomorphic vector bundle E over M is said to be semistable (resp. stable) if every proper subbundle F of E satisfies

$$\mu(F) \leqslant \mu(E) \quad (\text{resp. } \mu(F) < \mu(E)),$$

where $\mu(F) = \text{degree}(F)/\text{rank}(F)$ is the slope of F. Nonsemistable bundles are said to be unstable. When n and d are coprime the stable and semistable bundles coincide.

Let $\mathcal{E}$ be a fixed C^∞ complex vector bundle of rank n and degree d over M. Let $\mathcal{C}$ be the space of all holomorphic structures on $\mathcal{E}$ and let $\mathcal{G}_c$ denote the group of all

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C^∞ complex automorphisms of $\mathcal{E}$. Atiyah and Bott [1] identify the moduli space $\mathcal{M}(n,d)$ with the quotient $\mathcal{C}^{ss}/\mathcal{G}_c$ where $\mathcal{C}^{ss}$ is the open subset of $\mathcal{C}$ consisting of all semistable holomorphic structures on $\mathcal{E}$. In this construction both $\mathcal{C}$ and $\mathcal{G}_c$ are infinite dimensional; there exist other constructions [7] of the moduli space $\mathcal{M}(n,d)$ as genuine geometric invariant theoretic quotients which are in a sense finite dimensional approximations of Atiyah and Bott's construction.

There is a known set of generators [12, 1] for the rational cohomology ring of $\mathcal{M}(n,d)$ as follows. Let V denote a universal bundle over $\mathcal{M}(n,d)\times M$. Atiyah and Bott then define elements

$$\begin{aligned} &a_r \in H^{2r}(\mathcal{M}(n,d);\mathbf{Q}), \qquad b_r^s \in H^{2r-1}(\mathcal{M}(n,d);\mathbf{Q}),\\ &f_r \in H^{2r-2}(\mathcal{M}(n,d);\mathbf{Q}), \end{aligned} \tag{1}$$

where $1 \leqslant r \leqslant n, 1 \leqslant s \leqslant 2g$ by writing

$$c_r(V) = a_r \otimes 1 + \sum_{s=1}^{2g} b_r^s \otimes \alpha_s + f_r \otimes \omega \quad 1 \leqslant r \leqslant n, \tag{2}$$

where ω is the standard generator of $H^2(M;\mathbf{Q})$ and $\alpha_1,\ldots,\alpha_{2g}$ form a fixed canonical cohomology basis for $H^1(M;\mathbf{Q})$. The ring $H^*(\mathcal{M}(n,d);\mathbf{Q})$ is freely generated as a graded algebra over $\mathbf{Q}$ by the elements (1). Notice from the definition that $f_1 = d$. We further introduce the notation

$$\xi_{i,j} = \sum_{s=1}^{g} b_i^s b_j^{s+g}.$$

The universal bundle V is not unique, although its projective class is. We may tensor V by the pullback to $\mathcal{M}(n,d)\times M$ of any holomorphic line bundle K over $\mathcal{M}(n,d)$ to give another bundle with the same universal property. This process changes the generators of $H^*(\mathcal{M}(n,d);\mathbf{Q})$. In particular it changes a_1 by $nc_1(K)$ and $c_1(\pi_! V)$ by $(d-n\bar{g})c_1(K)$ where $\pi\colon \mathcal{M}(n,d)\times M \to \mathcal{M}(n,d)$ is the first projection and $\pi_!$ is the direct image map from K-theory [5, p. 436]. Since n and d are coprime there exist integers u and v such that

$$un + v(d-n\bar{g}) = 1.$$

Thus if we take K to be

$$\det(V|_{\mathcal{M}(n,d)})^u \otimes (\det \pi_! V)^v$$

then $V \otimes \pi^*(K^{-1})$ is a new universal bundle such that

$$ua_1 + vc_1(\pi_! V) = 0. \tag{3}$$

Following Atiyah and Bott [1, p. 582] we replace V by this normalised universal bundle.

The normalised bundle V is universal in the sense that its restriction to $\{[E]\}\times M$ is isomorphic to E for each semistable holomorphic bundle E over M of rank n and degree d and where $[E]$ is the class of E in $\mathcal{M}(n,d)$. Then the stalk of the ith higher direct image sheaf $R^i\pi_*V$ (see [5, Sect. 3.8]) at $[E]$ is

$$H^i(\pi^{-1}([E]), V_{|\pi^{-1}([E])}) = H^i(M, V_{|[E]\times M}) \cong H^i(M,E).$$

Tensoring E with a holomorphic line bundle over M of degree D gives an isomorphism between $\mathcal{M}(n,d)$ and $\mathcal{M}(n,d+nD)$. Since n and d are coprime we may assume without any loss of generality that $2\bar{g}n < d < (2\bar{g}+1)n$ and so we will write

$$d = 2n\bar{g} + \delta \quad (0<\delta<n),$$

from now on. From [11, Lemma 5.2] we know that $H^1(M,E) = 0$ for any semistable holomorphic bundle E of slope greater than $2\bar{g}$. Thus $\pi_!V$ is in fact a vector bundle over $\mathcal{M}(n,d)$ with fibre $H^0(M,E)$ over $[E]\in\mathcal{M}(n,d)$ and, by the Riemann–Roch theorem, of rank $d-n\bar{g} = n\bar{g}+\delta$.

In particular if we express the Chern classes $c_r(\pi_!V)$ in terms of the generators a_r, b_r^s and f_r of $H^*(\mathcal{M}(n,d);\mathbf{Q})$ then knowing the images of the rth Chern classes in $H^*(\mathcal{M}(n,d);\mathbf{Q})$ vanish for $r > n\bar{g}+\delta$ gives us relations in terms of the images of the generators in $H^*(\mathcal{M}(n,d);\mathbf{Q})$. Now from [1, Prop. 9.7] we know that

$$H^*(\mathcal{M}(n,d);\mathbf{Q}) \cong H^*(\mathcal{M}_0(n,d);\mathbf{Q}) \otimes H^*(\mathrm{Jac}(M);\mathbf{Q}), \tag{4}$$

where $\mathrm{Jac}(M)$ is the Jacobian of the Riemann surface M and $\mathcal{M}_0(n,d)$ is the moduli space of rank n bundles with degree d and fixed determinant line bundle. $H^*(\mathrm{Jac}(M);\mathbf{Q})$ is an exterior algebra on $2g$ generators and we can choose the isomorphism (4) so that these generators correspond to $b_1^1,\dots,b_1^{2g}$ and the elements $a_2,\dots,a_n, b_2^1,\dots,b_n^{2g}, f_2,\dots,f_n$ correspond to the generators of $H^*(\mathcal{M}_0(n,d);\mathbf{Q})$. So we can find relations in terms of $a_2,\dots,a_n, b_2^1,\dots,b_n^{2g}$, and $f_2,\dots,f_n$ by equating to zero the coefficients of $\prod_{s\in S} b_1^s$ in the Chern classes $c_r(\pi_!V)$ for $r>n\bar{g}+\delta$ and for every subset $S\subseteq\{1,\dots,2g\}$.

Mumford's conjecture, as proven by Kirwan [8, Sect. 2], was that when the rank n is two then these relations together with the relation (3) from normalising the universal bundle V provide a complete set of relations in $H^*(\mathcal{M}_0(2,d);\mathbf{Q})$. Subsequently a stronger version of Mumford's conjecture has been proven [3] showing the relations coming from the first vanishing Chern class $c_{2g}(\pi_!V)$ generate the relation ideal of $H^*(\mathcal{M}_0(2,d))$ as a $\mathbf{Q}[a_2,f_2]$-module.

Remark 1. In the rank two case the Mumford relations above differ somewhat from the relations ξ_r introduced by Zagier and studied in [2, 6, 14, 17]. In the notation of [17]

$$\Psi_{\{1,\dots,2g\}}\left(\frac{-t-a_1}{2}\right) = \frac{(-1)^{g\bar{g}/2+g}}{2^{2g-1}} t^{\bar{g}} F_0(t^{-1}),$$

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where $\Psi_{\{1,\ldots,2g\}}(x)$ denotes the coefficient of $\prod_{s=1}^{2g} b_1^s$ in $\Psi(x) = \Sigma_{r\geqslant 0}\, c_r(\pi_! V)\, x^{2g-1-r}$ and $F_0(t) = \sum_{r=0}^{\infty} \xi_r t^r$. In the notation of [6] ξ_r appears as $\zeta_r/r!$ and in [14] as $\Phi^{(r)}/r!$.

We will demonstrate later (Remark 3) that the Mumford relations are not complete when the rank n is greater than two. For now we intoduce a new set of relations. Let L be a fixed line bundle over M of degree $4\bar{g}+1$ and let $\phi\colon \mathcal{M}(n,d)\times M \to M$ be the second projection. Then $\pi_!(V^*\otimes \phi^* L)$ is a vector bundle over $\mathcal{M}(n,d)$ of rank $(3\bar{g}+1)n-d = ng-\delta$ with fibre $H^0(M, E^*\otimes L)$ over $[E]$. By equating to zero the coefficients of $\prod_{s\in S} b_1^s$ in the Chern classes $c_r(\pi_!(V^*\otimes\phi^* L))$ for $r > ng-\delta$ and for every subset $S\subseteq\{1,\ldots,2g\}$ we may find relations in terms of the generators $a_2,\ldots,a_n, b_2^1,\ldots,b_n^{2g}$, and $f_2,\ldots,f_n$. We will refer to these new relations as the dual Mumford relations.

Remark 2. The map $E\mapsto E^*\otimes L$ induces an automorphism of $H^*(\mathcal{M}(2,d);\mathbf{Q})$ mapping the Mumford relations to the dual Mumford relations and vice versa. Hence we can deduce that the dual Mumford relations are complete when the rank is two from Kirwan's proof of Mumford's conjecture [8, Sect. 2].

Our first result (to be proved in Section 4) now reads as:

THEOREM 1. *The Mumford and dual Mumford relations together with the relation* (3) *due to the normalisation of the universal bundle V form a complete set of relations for $H^*(\mathcal{M}(3,d);\mathbf{Q})$.*

The Newstead–Ramanan conjecture states [12, Sect. 5a] that the Pontryagin ring of the tangent bundle to $\mathcal{M}(2,d)$ vanishes in degrees $4g$ and higher. The conjecture was proven independently by Thaddeus [15] and Kirwan [8, Sect. 4], and has been proven more recently by King and Newstead [6] and Weitsman [16]. In Section 5 we will use a similar method to Kirwan's but now also involving the dual Mumford relations to prove:

THEOREM 2. *The Pontryagin ring of the moduli space $\mathcal{M}(3,d)$ vanishes in degrees $12g-8$ and above.*

2. Kirwan's approach

The group $\mathcal{G}_c$ is the complexification of the gauge group $\mathcal{G}$ of all smooth automorphisms of $\mathcal{E}$ which are unitary with respect to a fixed Hermitian structure on $\mathcal{E}$ [1, p. 570]. We shall write $\overline{\mathcal{G}}$ for the quotient of $\mathcal{G}$ by its $U(1)$–centre and $\overline{\mathcal{G}}_c$ for the quotient of $\mathcal{G}_c$ by its $\mathbf{C}^*$-centre.

There are natural isomorphisms [1, 9.1]

$$H^*(\mathcal{C}^{ss}/\mathcal{G}_c;\mathbf{Q}) = H^*(\mathcal{C}^{ss}/\overline{\mathcal{G}}_c;\mathbf{Q}) \cong H^*_{\overline{\mathcal{G}}_c}(\mathcal{C}^{ss};\mathbf{Q}) \cong H^*_{\overline{\mathcal{G}}}(\mathcal{C}^{ss};\mathbf{Q})$$

since the $\mathbf{C}^*$-centre of $\mathcal{G}_c$ acts trivially on $\mathcal{C}^{ss}$, $\overline{\mathcal{G}}_c$ acts freely on $\mathcal{C}^{ss}$ and $\overline{\mathcal{G}}_c$ is the complexification of $\overline{\mathcal{G}}$. Atiyah and Bott [1, Th. 7.14] show that the restriction map

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$H^*_{\overline{\mathcal{G}}}(\mathcal{C};\mathbf{Q}) \to H^*_{\overline{\mathcal{G}}}(\mathcal{C}^{ss};\mathbf{Q})$ is surjective. Further $H^*_{\overline{\mathcal{G}}}(\mathcal{C};\mathbf{Q}) \cong H^*(B\overline{\mathcal{G}};\mathbf{Q})$ since $\mathcal{C}$ is an affine space [1, p. 565]. So putting this all together we have

$$H^*(B\overline{\mathcal{G}};\mathbf{Q}) \cong H^*_{\overline{\mathcal{G}}}(\mathcal{C};\mathbf{Q}) \to H^*_{\overline{\mathcal{G}}}(\mathcal{C}^{ss};\mathbf{Q}) \cong H^*(\mathcal{M}(n,d);\mathbf{Q}) \tag{5}$$

is a surjection.

As shown in [1, Prop. 2.4] the classifying space $B\mathcal{G}$ can be identified with the space $\mathrm{Map}_d(M, BU(n))$ of all smooth maps $f: M \to BU(n)$ such that the pullback to M of the universal vector bundle over $BU(n)$ has degree d. If we pull back this universal bundle using the evaluation map

$$\mathrm{Map}_d(M, BU(n)) \times M \to BU(n)\colon (f,m) \mapsto f(m),$$

then we obtain a rank n vector bundle $\mathcal{V}$ over $B\mathcal{G} \times M$. If we restrict the pullback bundle induced by the maps

$$\mathcal{C}^{ss} \times E\mathcal{G} \times M \to \mathcal{C} \times E\mathcal{G} \times M \to \mathcal{C} \times_{\mathcal{G}} E\mathcal{G} \times M \overset{\simeq}{\to} B\mathcal{G} \times M$$

to $\mathcal{C}^{ss} \times \{e\} \times M$ for some $e \in E\mathcal{G}$ then we obtain a $\mathcal{G}$–equivariant holomorphic bundle on $\mathcal{C}^{ss} \times M$. The $U(1)$–centre of $\mathcal{G}$ acts as scalar multiplication on the fibres, and the associated projective bundle descends to a holomorphic projective bundle over $\mathcal{M}(n,d) \times M$ which is in fact the projective bundle of V [1, pp. 579–580].

By a slight abuse of notation we define elements a_r, b_r^s, f_r in $H^*(B\mathcal{G};\mathbf{Q})$ by writing

$$c_r(\mathcal{V}) = a_r \otimes 1 + \sum_{s=1}^{2g} b_r^s \otimes \alpha_s + f_r \otimes \omega \quad 1 \leqslant r \leqslant n.$$

Atiyah and Bott show [1, Prop. 2.20] that the ring $H^*(B\mathcal{G};\mathbf{Q})$ is freely generated as a graded algebra over $\mathbf{Q}$ by the elements a_r, b_r^s, f_r. The only relations amongst these generators are that the a_r and f_r commute with everything else and that the b_r^s anticommute with each other.

The fibration $BU(1) \to B\mathcal{G} \to B\overline{\mathcal{G}}$ induces an isomorphism [1, p. 577]

$$H^*(B\mathcal{G};\mathbf{Q}) \cong H^*(B\overline{\mathcal{G}};\mathbf{Q}) \otimes H^*(BU(1);\mathbf{Q}).$$

The generators a_r, b_r^s and f_r of $H^*(B\mathcal{G};\mathbf{Q})$ can be pulled back via a section of this fibration to give rational generators of the cohomology ring of $B\overline{\mathcal{G}}$. We may if we wish omit a_1 since its image in $H^*(B\overline{\mathcal{G}};\mathbf{Q})$ can be expressed in terms of the other generators. The only other relations are again the commuting of the a_r and f_r, and the anticommuting of the b_r^s. We may then normalise $\mathcal{V}$ suitably so that these generators for $H^*(B\overline{\mathcal{G}};\mathbf{Q})$ restrict to the generators a_r, b_r^s, f_r for $H^*(\mathcal{M}(n,d);\mathbf{Q})$ under the surjection (5).

The relations amongst these generators for $H^*(\mathcal{M}(n,d);\mathbf{Q})$ are then given by the kernel of the restriction map (5) which in turn is determined by the map

$$\begin{aligned} H^*_{\mathcal{G}}(\mathcal{C};\mathbf{Q}) &\cong H^*_{\overline{\mathcal{G}}}(\mathcal{C};\mathbf{Q}) \otimes H^*(BU(1);\mathbf{Q}) \\ &\to H^*_{\overline{\mathcal{G}}}(\mathcal{C}^{ss};\mathbf{Q}) \otimes H^*(BU(1);\mathbf{Q}) \cong H^*_{\mathcal{G}}(\mathcal{C}^{ss};\mathbf{Q}). \end{aligned}$$

In order to describe this kernel we consider Shatz's stratification of $\mathcal{C}$, the space of holomorphic structures on $\mathcal{E}$ [13]. The stratification $\{\mathcal{C}_\mu : \mu \in \mathcal{M}\}$ is indexed by the partially ordered set $\mathcal{M}$, consisting of all the types of holomorphic bundles of rank n and degree d, as follows.

Any holomorphic bundle E over M of rank n and degree d has a canonical filtration (or flag) [4, p. 221]

$$0 = E_0 \subset E_1 \subset \cdots \subset E_P = E$$

of sub-bundles such that the quotient bundles $Q_p = E_p/E_{p-1}$ are semi-stable and $\mu(Q_p) > \mu(Q_{p+1})$. We will write d_p and n_p respectively for the degree and rank of Q_p. Given such a filtration we define the type of E to be

$$\mu = (\mu(Q_1), \ldots, \mu(Q_P)) \in \mathbf{Q}^n,$$

where the entry $\mu(Q_p)$ is repeated n_p times. When there is no chance of confusion we will also refer collectively to the strata of type $(n_1, \ldots, n_s)$ and we will write Δ for the collection of strata with $n_p = 1$ for each p. The semistable bundles have type $\mu_0 = (d/n, \ldots, d/n)$ and form the unique open stratum. The set $\mathcal{M}$ of all possible types of holomorphic vector bundles over M will provide our indexing set. A partial order on $\mathcal{M}$ is defined as follows. Let $\sigma = (\sigma_1, \ldots, \sigma_n)$ and $\tau = (\tau_1, \ldots, \tau_n)$ be two types; we say that $\sigma \geqslant \tau$ if and only if

$$\sum_{j \leqslant i} \sigma_j \geqslant \sum_{j \leqslant i} \tau_j \quad \text{for } 1 \leqslant i \leqslant n-1.$$

The set $\mathcal{C}_\mu \subseteq \mathcal{C}$, $\mu \in \mathcal{M}$, is defined to be the set of all holomorphic vector bundles of type μ.

The stratification also has the following properties:

(i) The stratification is smooth. That is each stratum $\mathcal{C}_\mu$ is a locally closed $\mathcal{G}_c$-invariant submanifold. Further for any $\mu \in \mathcal{M}$ [1, 7.8]

$$\overline{\mathcal{C}_\mu} \subseteq \bigcup_{\nu \geqslant \mu} \mathcal{C}_\nu. \tag{6}$$

(ii) Each stratum $\mathcal{C}_\mu$ is connected and has finite (complex) codimension d_μ in $\mathcal{C}$. Moreover given any integer N there are only finitely many $\mu \in \mathcal{M}$ such that $d_\mu \leqslant N$. Further d_μ is given by the formula [1, 7.16]

$$d_\mu = \sum_{i>j} (n_i d_j - n_j d_i + n_i n_j \bar{g}) \tag{7}$$

where d_k and n_k are the degree and rank, respectively, of Q_k.

(iii) The gauge group $\mathcal{G}$ acts on $\mathcal{C}$ preserving the stratification which is equivariantly perfect with respect to this action [1, Th. 7.14]. In particular there is an isomorphism of vector spaces

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$$H^k_{\mathcal{G}}(\mathcal{C};\mathbf{Q}) \cong \bigoplus_{\mu\in\mathcal{M}} H^{k-2d_\mu}_{\mathcal{G}}(\mathcal{C}_\mu;\mathbf{Q}) = H^k_{\mathcal{G}}(\mathcal{C}^{ss};\mathbf{Q}) \oplus \bigoplus_{\mu\neq\mu_0} H^{k-2d_\mu}_{\mathcal{G}}(\mathcal{C}_\mu;\mathbf{Q}).$$

The restriction map $H^*_{\mathcal{G}}(\mathcal{C};\mathbf{Q}) \to H^*_{\mathcal{G}}(\mathcal{C}^{ss};\mathbf{Q})$ is the projection onto the summand $H^*_{\mathcal{G}}(\mathcal{C}^{ss};\mathbf{Q})$ and so the kernel is isomorphic as a vector space to

$$\bigoplus_{k\geqslant 0}\bigoplus_{\mu\neq\mu_0} H^{k-2d_\mu}_{\mathcal{G}}(\mathcal{C}_\mu;\mathbf{Q}). \tag{8}$$

Remark 3. We can at this point use a dimension argument to show that the Mumford relations are generally not complete when the rank n is greater than two. From the isomorphism (8) we can see that for the Mumford relations to be complete it is necessary that the least degree of a Mumford relation must be less than or equal to the smallest real codimension of an unstable stratum. The degree of $\sigma^k_{r,S}$ equals $2(n\bar{g}+\delta-nr-k)-|S|$ which is least when $r=-1, k=n-1$, and $S=\{1,\dots,2g\}$. So the smallest degree of a Mumford relation is $2(\delta+(n-1)\bar{g})$. However a simple calculation minimising the codimension formula (7) shows that the least real codimension of an unstable stratum is $2(\delta+(n-1)\bar{g})$ when $\delta<n/2$ and is $2(n-\delta+(n-1)\bar{g})$ when $\delta>n/2$. Hence the Mumford relations are not complete when $n\geqslant 3$ and $\delta>n/2$. A similar argument shows that the dual Mumford relations are not complete when $\delta<n/2$ since the smallest degree of a dual Mumford relation is $2(n-\delta+(n-1)\bar{g})$. Clearly however this simple argument does not tell us anything concerning the union of the Mumford and dual Mumford relations.

To conclude this section we will describe a set of criteria for the completeness of a set of relations in $H^*(\mathcal{M}(n,d);\mathbf{Q})$ and reformulate the Mumford and dual Mumford relations in a way more suited to these criteria. Consider the formal power series

$$c(\pi_!\mathcal{V})(t) = \sum_{r\geqslant 0} c_r(\pi_!\mathcal{V})\cdot t^r \in H^*_{\mathcal{G}}(\mathcal{C};\mathbf{Q})[[t]].$$

The vanishing of the image of $c_r(\pi_!\mathcal{V})$ in $H^*(\mathcal{M}(n,d);\mathbf{Q})$ for $r>n\bar{g}+\delta$ is equivalent to the image of $c(\pi_!\mathcal{V})(t)$ being a polynomial of degree at most $n\bar{g}+\delta$ or equally to the image of

$$\Psi(t) = t^{n\bar{g}+\delta}c(\pi_!\mathcal{V})(t^{-1})$$

being a polynomial of degree at most $n\bar{g}+\delta$ in $H^*(\mathcal{M}(n,d);\mathbf{Q})[t]$. If we write $\Psi(t)$ as the series

$$\Psi(t) = \sum_{r=-\infty}^{\bar{g}} (\sigma^0_r + \sigma^1_r t + \cdots + \sigma^{n-1}_r t^{n-1})(\tilde{\Omega}(t))^r,$$

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where $\tilde{\Omega}(t) = t^n + a_1 t^{n-1} + \cdots + a_n$ then the Mumford relations are equivalent to the vanishing of the images of $\sigma^k_{r,S}(r < 0, 0 \leqslant k \leqslant n-1, S \subseteq \{1, \ldots, 2g\})$ in $H^*(\mathcal{M}_0(n,d); \mathbf{Q})$ when we write

$$\sigma^k_r = \sum_{S \subseteq \{1,\ldots,2g\}} \sigma^k_{r,S} \prod_{s \in S} b^s_1. \tag{9}$$

We will refer to $\sigma^k_{r,S}(r < 0, 0 \leqslant k \leqslant n-1, S \subseteq \{1, \ldots, 2g\})$ as the Mumford relations.

Similarly we know that the restriction of

$$\Psi^*(t) = t^{ng-\delta} c(\pi_!(\mathcal{V}^* \otimes \phi^* L))(-t^{-1})$$

to $H^*(\mathcal{M}(n,d); \mathbf{Q})$ is a polynomial. As before we may put $\Psi^*(t)$ in the form

$$\Psi^*(t) = \sum_{r=-\infty}^{\bar{g}} (\tau^0_r + \tau^1_r t + \cdots + \tau^{n-1}_r t^{n-1})(\tilde{\Omega}(t))^r,$$

where $\tilde{\Omega}(t) = t^n + a_1 t^{n-1} + \cdots + a_n$ and similarly we write

$$\tau^k_r = \sum_{S \subseteq \{1,\ldots,2g\}} \tau^k_{r,S} \prod_{s \in S} b^s_1. \tag{10}$$

We will refer to $\tau^k_{r,S}(r < 0, 0 \leqslant k \leqslant n-1, S \subseteq \{1, \ldots, 2g\})$ as the dual Mumford relations.

The motivation for this is that the restrictions of $\sigma^k_{r,S}$ and $\tau^k_{r,S}$ to the strata $\mathcal{C}_\mu$ are easier to calculate in this form. This is a crucial step in applying the following completeness criteria.

Given $\mu = (\mu_1, \ldots, \mu_n), \nu = (\nu_1, \ldots, \nu_n) \in \mathcal{M}$ then we write $\nu \prec \mu$ if there exists T, $1 \leqslant T \leqslant n$, such that

$$\nu_i = \mu_i \quad \text{for } T < i \leqslant n \quad \text{and} \quad \nu_T > \mu_T.$$

We write $\nu \preceq \mu$ if $\nu \prec \mu$ or $\nu = \mu$. A few easy calculations verify that $\preceq$ is a total order on $\mathcal{M}$ with minimal element μ_0, the semistable type. For an unstable type μ we will write $\mu - 1$ for the type previous to μ with respect to $\preceq$.

PROPOSITION 4 (Completeness criteria). *Let $\mathcal{R}$ be a subset of the kernel of the restriction map*

$$H^*_{\mathcal{G}}(\mathcal{C}; \mathbf{Q}) \to H^*_{\mathcal{G}}(\mathcal{C}^{ss}; \mathbf{Q}).$$

Suppose that for each unstable type μ there is a subset $\mathcal{R}_\mu$ of the ideal generated by $\mathcal{R}$ such that the image of $\mathcal{R}_\mu$ under the restriction map

$$H^*_{\mathcal{G}}(\mathcal{C}; \mathbf{Q}) \to H^*_{\mathcal{G}}(\mathcal{C}_\nu; \mathbf{Q})$$

is zero when $\nu \prec \mu$ *and when* $\nu = \mu$ *contains the ideal of* $H^*_{\mathcal{G}}(\mathcal{C}_\mu; \mathbf{Q})$ *generated by* e_μ*, the equivariant Euler class of* $\mathcal{N}_\mu$*, the normal bundle to the stratum* $\mathcal{C}_\mu$ *in* $\mathcal{C}$*. Then* $\mathcal{R}$ *generates the kernel of the restriction map*

$$H^*_{\mathcal{G}}(\mathcal{C}; \mathbf{Q}) \to H^*_{\mathcal{G}}(\mathcal{C}^{ss}; \mathbf{Q})$$

as an ideal of $H^*_{\mathcal{G}}(\mathcal{C}; \mathbf{Q})$.

Remark 5. The proof of Proposition 4 below follows similar lines to the proof of [8, Prop. 1]. However there are some differences–the order $\preceq$ does not generally coincide with $\leqslant$ – and further the proof of [8, p. 867] as given is true only for the rank two case. For these reasons we include a proof of Proposition 4 below although it clearly owes many of its origins to [8].

Proof. Let $\mu \in \mathcal{M}$ and define

$$V_\mu = \bigcup_{\nu \preceq \mu} \mathcal{C}_\nu.$$

We will firstly show that V_μ is an open subset of $\mathcal{C}$ containing $\mathcal{C}_\mu$ as a closed submanifold. Note that if $\nu \leqslant \mu$ then $\nu \preceq \mu$ and thus by property (6) if $\nu \succ \mu$ then $\overline{\mathcal{C}}_\nu \subseteq \mathcal{C} - V_\mu$. The stratification is locally finite and hence V_μ is open. Further note that the closure of $\mathcal{C}_\mu$ in V_μ equals

$$V_\mu \cap \bigcup_{\nu \geqslant \mu} \mathcal{C}_\nu = \mathcal{C}_\mu$$

as required.

Recall now that the composition of the Thom–Gysin map

$$H^{*-2d_\mu}_{\mathcal{G}}(\mathcal{C}_\mu; \mathbf{Q}) \to H^*_{\mathcal{G}}(V_\mu; \mathbf{Q}),$$

with the restriction map

$$H^*_{\mathcal{G}}(V_\mu; \mathbf{Q}) \to H^*_{\mathcal{G}}(\mathcal{C}_\mu; \mathbf{Q})$$

is given by multiplication by the Euler class e_μ which is not a zero-divisor in $H^*_{\mathcal{G}}(\mathcal{C}_\mu; \mathbf{Q})$ [1, p. 569]. It follows from the exactness of the Thom–Gysin sequence

$$\cdots \to H^{*-2d_\mu}_{\mathcal{G}}(\mathcal{C}_\mu; \mathbf{Q}) \to H^*_{\mathcal{G}}(V_\mu; \mathbf{Q}) \to H^*_{\mathcal{G}}(V_{\mu-1}; \mathbf{Q}) \to \cdots$$

that the direct sum of the restriction maps

$$H^*_{\mathcal{G}}(V_\mu; \mathbf{Q}) \to H^*_{\mathcal{G}}(\mathcal{C}_\mu; \mathbf{Q}) \oplus H^*_{\mathcal{G}}(V_{\mu-1}; \mathbf{Q})$$

is injective. Hence inductively the direct sum of restriction maps

$$H^*_{\mathcal{G}}(V_{\mu-1}; \mathbf{Q}) \to \bigoplus_{\nu \prec \mu} H^*_{\mathcal{G}}(\mathcal{C}_\nu; \mathbf{Q})$$

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is injective and in particular the image of any element of $\mathcal{R}_\mu$ under the restriction map

$$H^*_{\mathcal{G}}(\mathcal{C};\mathbf{Q}) \to H^*_{\mathcal{G}}(V_{\mu-1};\mathbf{Q})$$

is zero.

For any given $i \geqslant 0$ there are only finitely many $\nu \in \mathcal{M}$ such that $2d_\nu \leqslant i$ and so for each $i \geqslant 0$ there exists some μ such that

$$H^i_{\mathcal{G}}(\mathcal{C};\mathbf{Q}) = H^i_{\mathcal{G}}(V_\mu;\mathbf{Q}).$$

Hence it is enough to show that for each μ the image in $H^*_{\mathcal{G}}(V_\mu;\mathbf{Q})$ of the ideal generated by $\mathcal{R}$ contains the image in $H^*_{\mathcal{G}}(V_\mu;\mathbf{Q})$ of the kernel of the restriction map

$$H^*_{\mathcal{G}}(\mathcal{C};\mathbf{Q}) \to H^*_{\mathcal{G}}(\mathcal{C}^{ss};\mathbf{Q}). \tag{11}$$

Note that the above is clearly true for $\mu = \mu_0$ as $V_{\mu_0} = \mathcal{C}^{ss}$. We will proceed by induction with respect to $\preceq$.

Assume now that $\mu \neq \mu_0$ and that $\alpha \in H^*_{\mathcal{G}}(\mathcal{C};\mathbf{Q})$ lies in the kernel of (11). Suppose that the image of α in $H^*_{\mathcal{G}}(V_{\mu-1};\mathbf{Q})$ is in the image of the ideal generated by $\mathcal{R}$. We may, without any loss of generality, assume that the image of α in $H^*_{\mathcal{G}}(V_{\mu-1};\mathbf{Q})$ is zero. Thus by the exactness of the Thom–Gysin sequence there exists an element $\beta \in H^{*-2d_\mu}_{\mathcal{G}}(\mathcal{C}_\mu;\mathbf{Q})$ which is mapped to the image of α in $H^*_{\mathcal{G}}(V_\mu;\mathbf{Q})$ by the Thom–Gysin map

$$H^{*-2d_\mu}_{\mathcal{G}}(\mathcal{C}_\mu;\mathbf{Q}) \to H^*_{\mathcal{G}}(V_\mu;\mathbf{Q}).$$

Hence the image of α under the restriction map

$$H^*_{\mathcal{G}}(\mathcal{C};\mathbf{Q}) \to H^*_{\mathcal{G}}(\mathcal{C}_\mu;\mathbf{Q})$$

is βe_μ, and by hypothesis there is an element γ of $\mathcal{R}_\mu$ which maps under the restriction map

$$H^*_{\mathcal{G}}(V_\mu;\mathbf{Q}) \to H^*_{\mathcal{G}}(\mathcal{C}_\mu;\mathbf{Q})$$

to βe_μ. Now the images of γ and α in $H^*_{\mathcal{G}}(V_{\mu-1};\mathbf{Q})$ are both zero and we also know the direct sum of the restriction maps

$$H^*_{\mathcal{G}}(V_\mu;\mathbf{Q}) \to H^*_{\mathcal{G}}(\mathcal{C}_\mu;\mathbf{Q}) \oplus H^*_{\mathcal{G}}(V_{\mu-1};\mathbf{Q})$$

to be injective. Thus the images of γ and α in $H^*_{\mathcal{G}}(V_\mu;\mathbf{Q})$ are the same, completing the proof. $\square$

Remark 6. Kirwan's completeness criteria follow from the above criteria since for each μ

$$V_{\mu-1} \subseteq \mathcal{C} - \bigcup_{\nu \geqslant \mu} \mathcal{C}_\nu.$$

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So if the restriction of a relation to $H^*_{\mathcal{G}}(\mathcal{C}_\nu;\mathbf{Q})$ vanishes for every $\nu \not\succcurlyeq \mu$ then certainly the same relation restricts to zero in $H^*_{\mathcal{G}}(\mathcal{C}_\nu;\mathbf{Q})$ for any $\nu \prec \mu$.

Remark 7. Kirwan's proof of Mumford's conjecture [8, Sect. 2] amounts to showing that for each unstable type $\mu = (d_1, d_2)$ the set

$$\mathcal{R}_\mu = \bigcup \{\sigma^0_{d_2-2g+1,S}, \sigma^1_{d_2-2g+1,S}\},$$

where the union is taken over all subsets $S \subseteq \{1,\dots,2g\}$, satisfies the above criteria. In the rank two case the criteria of proposition 4 are in fact equivalent to Kirwan's completeness criteria since $\preceq$ and $\leqslant$ coincide.

3. Chern class computations

We first describe the restriction maps $H^*_{\mathcal{G}}(\mathcal{C};\mathbf{Q}) \to H^*_{\mathcal{G}}(\mathcal{C}_\mu;\mathbf{Q})$ and our preferred generators for $H^*_{\mathcal{G}}(\mathcal{C}_\mu;\mathbf{Q})$. Let $\mu = (d_1/n_1,\dots,d_P/n_P)$. Let $\mathcal{C}(n_p,d_p)^{ss}$ denote the space of all semistable holomorphic structures on a fixed Hermitian vector bundle of rank n_p and degree d_p and let $\mathcal{G}(n_p,d_p)$ be the gauge group of that bundle. Atiyah and Bott [1, Prop. 7.12] show that the map

$$\prod_{p=1}^{P} \mathcal{C}(n_p,d_p)^{ss} \to \mathcal{C}_\mu,$$

which sends a sequence of semistable bundles $(F_1,\dots,F_P)$ to the direct sum $F_1 \oplus \cdots \oplus F_P$, induces an isomorphism

$$H^*_{\mathcal{G}}(\mathcal{C}_\mu;\mathbf{Q}) \cong \bigotimes_{1\leqslant p\leqslant P} H^*_{\mathcal{G}(n_p,d_p)}(\mathcal{C}(n_p,d_p)^{ss};\mathbf{Q}).$$

Thus we can find generators

$$\begin{aligned}&\textstyle\bigcup_{p=1}^{P} \big(\{a^p_r \mid 1 \leqslant r \leqslant n_p\} \cup \{b^{p,s}_r \mid 1 \leqslant r \leqslant n_p, 1 \leqslant s \leqslant 2g\}\\ &\qquad \cup \{f^p_r | 2 \leqslant r \leqslant n_p\}\big)\end{aligned} \tag{12}$$

corresponding to the generators of $H^*_{\mathcal{G}}(\mathcal{C}^{ss};\mathbf{Q})$ described earlier in (2). As before we also define

$$\xi^{p,q}_{i,j} = \sum_{s=1}^{g} b^{p,s}_i b^{q,s+g}_j.$$

To explicitly describe the restriction map note that $c_r(\mathcal{V})$ restricts to $c_r(\bigoplus_{p=1}^{P} \mathcal{V}_p)$ where $\mathcal{V}_p$ is the universal bundle on $\mathcal{C}(n_p,d_p)$. The restrictions of the generators of $H^*_{\mathcal{G}}(\mathcal{C};\mathbf{Q})$ can be written in terms of the generators of $H^*_{\mathcal{G}}(\mathcal{C}_\mu;\mathbf{Q})$ by taking the appropriate coefficients in the Künneth decomposition.

One problem that we will be faced with in due course is how to calculate the coefficients of $\prod_{s\in S} b^s_1$ once we have restricted to a stratum. Suppose first that the stratum concerned is of type $\mu = (d_1, \ldots, d_n) \in \Delta$ and take $\zeta \in H^*_{\mathcal{G}}(\mathcal{C};\mathbf{Q})$. We can express ζ in terms of the generators

$$\{a_r \mid 1 \leqslant r \leqslant n\} \cup \{b_r^s \mid 1 \leqslant r \leqslant n, 1 \leqslant s \leqslant 2g\} \cup \{f_r | 2 \leqslant r \leqslant n\},$$

but equally we could write ζ in terms of

$$\begin{aligned}&\{a_r \mid 1 \leqslant r \leqslant n\} \cup \{nb_r^s - (n-r+1)a_{r-1}b_1^s \mid 2 \leqslant r \leqslant n, 1 \leqslant s \leqslant 2g\}\\ &\cup\{n^2 f_r - n(n-r+1)(\xi_{r-1,1} + \xi_{1,r-1})\\ &\quad + (n-r+1)(n-r+2)a_{r-2}\xi_{1,1} \mid 2 \leqslant r \leqslant n\},\end{aligned} \tag{13}$$

and $\{b_1^s \mid 1 \leqslant s \leqslant 2g\}$. We shall take the coefficients of $\prod_{s\in S} b_1^s$ when ζ is expressed in this latter form. The reason for this is that the restrictions of the elements (13) in $H^*_{\mathcal{G}}(\mathcal{C}_\mu;\mathbf{Q})$ can then be written in terms of

$$\{a_1^r \mid 1 \leqslant r \leqslant n\} \cup \{b_1^{p,s} - b_1^{n,s} \mid 1 \leqslant p \leqslant n-1, 1 \leqslant s \leqslant 2g\}, \tag{14}$$

(see Remark 8.) We can uniquely write the restriction of ζ in terms of the elements (14) and the restrictions of b_1^s, $(1 \leqslant s \leqslant 2g)$. Hence we may calculate the restrictions of the coefficients of $\prod_{s\in S} b_1^s$ in ζ by taking the coefficients of

$$\prod_{s\in S}(b_1^{1,s} + \cdots + b_1^{n,s})$$

in the restriction of ζ.

We deal with a general type stratum in a similar way. Let $\mu = (d_1/n_1, \ldots, d_P/n_P)$. We define formal symbols $a^{p,k}, b^{p,k,s}$ and $d^{p,k}$ such that the rth Chern class $c_r(\mathcal{V}_p)$ is given by the rth elementary symmetric polynomial in

$$a^{p,k} + \sum_{s=1}^{2g} b^{p,k,s} \otimes \alpha_s + d^{p,k} \otimes \omega \quad (1 \leqslant k \leqslant n_p), \tag{15}$$

when $1 \leqslant r \leqslant n_p$ and $1 \leqslant p \leqslant P$. In terms of $a^{p,k}, b^{p,k,s}$ and $d^{p,k}$ the restriction map to $H^*_{\mathcal{G}}(\mathcal{C}_\mu;\mathbf{Q})$ is formally the same as the restriction map when $\mu \in \Delta$. Again we may uniquely write the restriction of ζ in terms of

$$\begin{aligned}\bigcup_{p=1}^{P}\bigcup_{k=1}^{n_p}\{a^{p,k}, d^{p,k}\} \;\cup\; &\bigcup_{p=1}^{P-1}\bigcup_{k=1}^{n_p}\bigcup_{s=1}^{2g}\{b^{p,k,s} - b^{P,n_P,s}\}\\ \cup\; &\bigcup_{k=1}^{n_P-1}\bigcup_{s=1}^{2g}\{b^{P,k,s} - b^{P,n_P,s}\}\end{aligned} \tag{16}$$

and the restrictions of b_1^s, $(1 \leqslant s \leqslant 2g)$, and we take the coefficients of

$$\prod_{s\in S}(b_1^{1,s} + \cdots + b_1^{P,s})$$

as before.

So in our definitions of the Mumford and dual Mumford relations, (9) and (10), we assume first that σ_r^k and τ_r^k have first been written in terms of the elements (13) before taking the appropriate coefficient.

Remark 8. It is a trivial but tedious calculation to show that the restrictions of the elements (13) in $H^*_{\mathcal{G}}(\mathcal{C}_\mu; \mathbf{Q})$ for $\mu \in \Delta$ can indeed be written in terms of the elements (14). Let a_r^μ denote the restriction of a_r to $H^*_{\mathcal{G}}(\mathcal{C}_\mu; \mathbf{Q})$; this equals the rth elementary symmetric product in $a_1^1, \ldots, a_1^n$. The restrictions of b_r^s and f_r in $H^*_{\mathcal{G}}(\mathcal{C}_\mu; \mathbf{Q})$ equal

$$\sum_{i=1}^{n} b_1^{i,s} \frac{\partial a_r^\mu}{\partial a_1^i}, \qquad \sum_{i=1}^{n} d_i \frac{\partial a_r^\mu}{\partial a_1^i} + \sum_{i=1}^{n}\sum_{j=1}^{n} \xi_{1,1}^{i,j} \frac{\partial^2 a_r^\mu}{\partial a_1^i \partial a_1^j}.$$

The restrictions of the elements (13) can then be seen to equal

$$a_r^\mu, \qquad \sum_{i=1}^{n-1} \left(n \frac{\partial a_r^\mu}{\partial a_1^i} - (n-r+1) a_{r-1}^\mu \right) (b_1^{i,s} - b_1^{n,s})$$

and

$$\begin{aligned} n^2 \sum_{i=1}^{n} d_i \frac{\partial a_r^\mu}{\partial a_1^i} \; + \; & \sum_{i=1}^{n-1}\sum_{j=1}^{n-1}\sum_{s=1}^{g} (b_1^{i,s} - b_1^{n,s})(b_1^{j,s+g} - b_1^{n,s+g}) \\ & \times \; \left(n^2 \frac{\partial^2 a_r^\mu}{\partial a_1^i \partial a_1^j} - n(n-r+1) \left(\frac{\partial a_{r-1}^\mu}{\partial a_1^i} + \frac{\partial a_{r-1}^\mu}{\partial a_1^j} \right) \right. \\ & \left. + \; (n-r+1)(n-r+2) a_{r-1}^\mu \right). \end{aligned}$$

The remains of this section are given over to calculating the Mumford and dual Mumford relations. Our first problem is to obtain their generating functions from their respective Chern characters which we can evaluate using the Grothendieck–Riemann–Roch theorem (GRR).

LEMMA 9. *Suppose that*

$$\mathrm{ch}(E) = \sum_{i=1}^{M} \alpha_i e^{\delta_i} + \sum_{i=1}^{N} \beta_i e^{\varepsilon_i}, \tag{17}$$

where the β_i, δ_i and the ε_i are formal degree two classes and the α_i are formal degree zero classes. Then as a formal power series

$$c(E)(t) = \sum_{r=0}^{\infty} c_r(E) \cdot t^r = \prod_{i=1}^{M} (1+\delta_i t)^{\alpha_i} \prod_{i=1}^{N} \exp\left\{ \frac{\beta_i t}{1+\epsilon_i t} \right\}. \tag{18}$$

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Proof. The relationship between the Chern character and Chern polynomial is as follows. If $\mathrm{ch}(E) = \Sigma_{i=1}^{K} e^{\gamma_i}$ where γ_i are formal degree two classes then

$$c(E)(t) = \prod_{i=1}^{K}(1+\gamma_i t).$$

If $\mathrm{ch}(E)$ is in the form of (17) then by comparing degrees we find that

$$\sum_{i=1}^{M} \alpha_i (\delta_i)^n + \sum_{i=1}^{N} n\beta_i (\varepsilon_i)^{n-1} = \sum_{i=1}^{K} (\gamma_i)^n$$

for each $n \geqslant 0$. Thus on the level of formal power series $\log c(E)(t)$ equals

$$\sum_{i=1}^{K}\sum_{r=1}^{\infty}(-1)^{r+1}\frac{(\gamma_i t)^r}{r} = \sum_{i=1}^{M}\alpha_i \log(1+\delta_i t) + \sum_{i=1}^{N}\frac{\beta_i t}{1+\varepsilon_i t},$$

and hence the result (18). □

Armed with the above lemma we are now in a position to determine the Chern polynomials $c(\pi_!\mathcal{V})(t)$ and $c(\pi_!(\mathcal{V}^* \otimes \phi^* L))(-t)$. We can, and will, calculate these Chern polynomials in terms of the generators a_r, b_r^s and f_r of $H_{\mathcal{G}}^*(\mathcal{C};\mathbf{Q})$ (see (21) and (22)). However the expressions obtained are somewhat cumbersome and for ease of calculation we will find the formal expressions, (19) and (20), calculated directly from the above lemma of more use.

Proposition 10. *The Chern polynomial* $c(\pi_!\mathcal{V})(t)$ *equals*

$$\Omega(t)^{-\bar{g}}\prod_{k=1}^{n}(1+\delta_k t)^{W_k}\exp\left\{\frac{X_k t}{1+\delta_k t}\right\} \tag{19}$$

and $c(\pi_!(\mathcal{V}^* \otimes \phi^* L))(-t)$ *equals*

$$\Omega(t)^{3\bar{g}+1}\prod_{k=1}^{n}(1+\delta_k t)^{-W_k}\exp\left\{\frac{-X_k t}{1+\delta_k t}\right\}, \tag{20}$$

where $\delta_1, \ldots, \delta_n$ *are formal degree two classes such that their* r*th elementary symmetric polynomial equals* a_r, *and*

$$\Omega(t) = \prod_{k=1}^{n}(1+\delta_k t) = 1 + a_1 t + \cdots + a_n t^n, \qquad \xi_{i,j} = \sum_{s=1}^{g} b_i^s b_j^{s+g},$$

$$W_k = \sum_{i=1}^{n} f_i \frac{\partial \delta_k}{\partial a_i} + \sum_{i=1}^{n}\sum_{j=1}^{n}\xi_{i,j}\frac{\partial^2\delta_k}{\partial a_i \partial a_j}, \qquad X_k = \sum_{i=1}^{n}\sum_{j=1}^{n}\xi_{i,j}\frac{\partial\delta_k}{\partial a_i}\frac{\partial\delta_k}{\partial a_j}.$$

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In terms of the generators a_r, b_r^s and f_r for $H^*_{\mathcal{G}}(\mathcal{C};\mathbf{Q})$ then $c(\pi_!\mathcal{V})(t)$ equals

$$\Omega(t)^{-\bar{g}} \exp\left\{\int_0^t \left(\frac{d}{u} - \sum_{i=1}^n \frac{f_i u^{i-2}}{\Omega(u)} + \sum_{i=1}^n\sum_{j=1}^n \frac{\xi_{i,j}u^{i+j-2}}{\Omega(u)^2}\right)\,\mathrm{d}u\right\}, \tag{21}$$

and $c(\pi_!(\mathcal{V}^*\otimes\phi^*L))(-t)$ equals

$$\Omega(t)^{3\bar{g}+1} \exp\left\{\int_0^t \left(-\frac{d}{u} + \sum_{i=1}^n \frac{f_i u^{i-2}}{\Omega(u)} - \sum_{i=1}^n\sum_{j=1}^n \frac{\xi_{i,j}u^{i+j-2}}{\Omega(u)^2}\right)\,\mathrm{d}u\right\}. \tag{22}$$

Proof. Now $\mathrm{ch}(\mathcal{V}) = e^{\gamma_1}+\cdots+e^{\gamma_n}$ where $\gamma_1,\ldots,\gamma_n$ are formal degree two classes such that their rth elementary symmetric polynomial equals

$$c_r(\mathcal{V}) = a_r\otimes 1 + \sum_{s=1}^{2g} b_r^s\otimes\alpha_s + f_r\otimes\omega \quad (1\leqslant r\leqslant n).$$

For each $k\geqslant 0$ there exist coefficients $\rho^{(k)}_{r_1,\ldots,r_n}$ such that

$$(\gamma_1)^k+\cdots+(\gamma_n)^k = \sum \rho^{(k)}_{r_1,\ldots,r_n}(c_1(\mathcal{V}))^{r_1}\cdots(c_n(\mathcal{V}))^{r_n},$$

where the sum is taken over all nonnegative $r_1,\ldots,r_n$ such that $r_1+2r_2+\cdots+nr_n=k$. Now

$$\left(a_1\otimes 1+\sum_{s=1}^{2g} b_1^s\otimes\alpha_s+f_1\otimes\omega\right)^{r_1}\cdots\left(a_n\otimes 1+\sum_{s=1}^{2g} b_n^s\otimes\alpha_s+f_n\otimes\omega\right)^{r_n}$$

equals

$$\begin{aligned}&(a_1)^{r_1}\cdots(a_n)^{r_n}\otimes 1+\sum_{i=1}^n\sum_{s=1}^{2g} b_i^s\frac{\partial}{\partial a_i}(a_1)^{r_1}\cdots(a_n)^{r_n}\otimes\alpha_s\\ &\quad+\sum_{i=1}^n f_i\frac{\partial}{\partial a_i}(a_1)^{r_1}\cdots(a_n)^{r_n}\otimes\omega\\ &\quad+\sum_{i=1}^n\sum_{j=1}^n\xi_{i,j}\frac{\partial^2}{\partial a_i\partial a_j}(a_1)^{r_1}\cdots(a_n)^{r_n}\otimes\omega.\end{aligned}$$

Since

$$\sum\rho^{(k)}_{r_1,\ldots,r_n}(a_1)^{r_1}\cdots(a_n)^{r_n} = (\delta_1)^k+\cdots+(\delta_n)^k,$$

we find that $\mathrm{ch}(\mathcal{V})$ equals

$$\sum_{k=1}^n e^{\delta_k}\otimes 1+\sum_{i=1}^n\sum_{s=1}^{2g}\sum_{k=1}^n b_i^s\frac{\partial}{\partial a_i}e^{\delta_k}\otimes\alpha_s$$

$$+\sum_{i=1}^{n}\sum_{k=1}^{n} f_i \frac{\partial}{\partial a_i} e^{\delta_k} \otimes \omega$$

$$+\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n} \xi_{i,j} \frac{\partial^2}{\partial a_i \partial a_j} e^{\delta_k} \otimes \omega. \tag{23}$$

From GRR we have $\mathrm{ch}(\pi_! \mathcal{V}) = \pi_*(\mathrm{ch}(\mathcal{V}) \cdot 1 \otimes (1 - \bar{g}\omega))$ and hence $\mathrm{ch}(\pi_! \mathcal{V})$ equals

$$\sum_{i=1}^{n}\sum_{k=1}^{n} f_i \frac{\partial}{\partial a_i} e^{\delta_k} + \sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n} \xi_{i,j} \frac{\partial^2}{\partial a_i \partial a_j} e^{\delta_k} - \bar{g} \sum_{k=1}^{n} e^{\delta_k}$$
$$= \sum_{k=1}^{n} (-\bar{g} + W_k + X_k) e^{\delta_k}.$$

Note that W_k has degree zero and X_k has degree two. Hence by Lemma 9 we see that $c(\pi_! \mathcal{V})(t)$ equals

$$(\Omega(t))^{-\bar{g}} \prod_{k=1}^{n} (1 + \delta_k t)^{W_k} \exp\left\{ \frac{X_k t}{1 + \delta_k t} \right\}$$

to give equation (19).

Now $\frac{\mathrm{d}}{\mathrm{d}t} \log \left(\Omega(t)^{\bar{g}} c(\pi_! \mathcal{V})(t) \right)$ equals

$$\sum_{i=1}^{n}\sum_{k=1}^{n} f_i \frac{\partial \delta_k}{\partial a_i} \frac{\delta_k}{1 + \delta_k t}$$
$$+\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n} \xi_{i,j} \left(\frac{\partial^2 \delta_k}{\partial a_i \partial a_j} \frac{\delta_k}{1 + \delta_k t} + \frac{\partial \delta_k}{\partial a_i} \frac{\partial \delta_k}{\partial a_j} \frac{1}{(1 + \delta_k t)^2} \right)$$
$$= \sum_{i=1}^{n}\sum_{j=1}^{n} \frac{\xi_{i,j}}{t^2} \left(\sum_{k=1}^{n} t \frac{\partial^2 \delta_k}{\partial a_i \partial a_j} \right.$$
$$\left. - \sum_{k=1}^{n} \left(\frac{t}{1 + \delta_k t} \frac{\partial^2 \delta_k}{\partial a_i \partial a_j} - \frac{t^2}{(1 + \delta_k t)^2} \frac{\partial \delta_k}{\partial a_i} \frac{\partial \delta_k}{\partial a_j} \right) \right)$$
$$+ \sum_{i=1}^{n} \frac{f_i}{t} \left(\sum_{k=1}^{n} \frac{\partial \delta_k}{\partial a_i} - \sum_{k=1}^{n} \frac{\partial \delta_k}{\partial a_i} \frac{1}{1 + \delta_k t} \right). \tag{24}$$

Since

$$\sum_{k=1}^{n} \frac{\partial \delta_k}{\partial a_i} = \frac{\partial a_1}{\partial a_i}, \qquad f_1 = d, \quad \text{and} \quad \sum_{k=1}^{n} \frac{\partial^2 \delta_k}{\partial a_i \partial a_j} = \frac{\partial^2 a_1}{\partial a_i \partial a_j} = 0$$

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then (24) reduces to

$$\begin{aligned}
&\frac{d}{t} - \sum_{i=1}^{n}\sum_{k=1}^{n} f_i \frac{\partial}{\partial a_i} \frac{\log(1+\delta_k t)}{t^2} \\
&\quad - \sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n} \xi_{i,j} \frac{\partial^2}{\partial a_i \partial a_j} \frac{\log(1+\delta_k t)}{t^2} \\
&= \frac{d}{t} - \left(\sum_{i=1}^{n} f_i \frac{\partial}{\partial a_i} + \sum_{i=1}^{n}\sum_{j=1}^{n} \xi_{i,j} \frac{\partial^2}{\partial a_i \partial a_j} \right) \frac{\log \Omega(t)}{t^2},
\end{aligned}$$

to give equality (21).

The calculations for the dual case follow in a similar fashion. We have that $\mathrm{ch}(\mathcal{V}^*) = e^{-\gamma_1} + \cdots + e^{-\gamma_n}$ with $\gamma_1, \ldots, \gamma_n$ as before and arguing as in the calculation of (23) we determine that $\mathrm{ch}(\mathcal{V}^*)$ equals

$$\begin{aligned}
&\sum_{k=1}^{n} e^{-\delta_k} \otimes 1 + \sum_{i=1}^{n}\sum_{s=1}^{2g}\sum_{k=1}^{n} b_i^s \frac{\partial}{\partial a_i} e^{-\delta_k} \otimes \alpha_s \\
&\quad + \sum_{i=1}^{n}\sum_{k=1}^{n} f_i \frac{\partial}{\partial a_i} e^{-\delta_k} \otimes \omega + \sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n} \xi_{i,j} \frac{\partial^2}{\partial a_i \partial a_j} e^{-\delta_k} \otimes \omega.
\end{aligned} \tag{25}$$

We know that $\mathrm{ch}(\phi^* L) = \phi^*(e^{(4\bar{g}+1)\omega}) = 1 \otimes (1 + (4\bar{g}+1)\omega)$ and GRR shows that $\mathrm{ch}(\pi_!(\mathcal{V}^* \otimes \phi^* L))$ equals

$$\pi_*(\mathrm{ch}(\mathcal{V}^*) \cdot \mathrm{ch}(\phi^* L) \cdot 1 \otimes (1 - \bar{g}\omega)) = \pi_*(\mathrm{ch}(\mathcal{V}^*) \cdot 1 \otimes (1 + (3\bar{g}+1)\omega))$$

which gives

$$\mathrm{ch}(\pi_!(\mathcal{V}^* \otimes \phi^* L)) = \sum_{k=1}^{n} ((3\bar{g}+1) - W_k + X_k) e^{-\delta_k}. \tag{26}$$

Applying Lemma 9 to expression (26) gives equation (20). Expression (22) is arrived at by calculating $\frac{\mathrm{d}}{\mathrm{d}t} \log\left((\Omega(t))^{-3\bar{g}-1} c(\pi_!(\mathcal{V}^* \otimes \phi^* L))(t)\right)$ and grouping the terms in a similar manner to expression (24). $\square$

Remark 11. Note that δ_k, W_k and X_k are not elements of $H^*_{\mathcal{G}}(\mathcal{C}; \mathbf{Q})$. However the direct sum of the restriction maps

$$H^*_{\mathcal{G}}(\mathcal{C}; \mathbf{Q}) \to \bigoplus_{\mu \in \Delta} H^*_{\mathcal{G}}(\mathcal{C}_\mu; \mathbf{Q})$$

is injective and so we may consider δ_k, W_k and X_k as elements of $\bigoplus_{\mu \in \Delta} H^*_{\mathcal{G}}(\mathcal{C}_\mu; \mathbf{Q})$ corresponding respectively to a_1^k, d_k and $\xi_{1,1}^{k,k}$ in each summand $H^*_{\mathcal{G}}(\mathcal{C}_\mu; \mathbf{Q})$.

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Remark 12. From (21) we can find an expression for

$$\frac{\Psi'(t)}{\Psi(t)} = \frac{d - n\bar{g}}{t} - \frac{c(\pi_!\mathcal{V})'(t^{-1})}{t^2 c(\pi_!\mathcal{V})(t^{-1})}.$$

In fact we may write $\Psi'(t)/\Psi(t)$ as a rational function with denominator $(\tilde{\Omega}(t))^2$ and a numerator of degree at most $2n-1$. By multiplying by $\Psi(t)$ and comparing coefficients of $t^k(\tilde{\Omega}(t))^r, (r \leqslant \bar{g}, 0 \leqslant k < n)$ we may derive recurrence relations amongst the Mumford relations which determine $\{\sigma_r^k : 0 \leqslant k < n\}$ in terms of $\{\sigma_{r+1}^k, \sigma_{r+2}^k : 0 \leqslant k < n\}$. Similar recurrence relations exist among the dual Mumford relations which determine $\{\tau_r^k : 0 \leqslant k < n\}$ in terms of $\{\tau_{r+1}^k, \tau_{r+2}^k : 0 \leqslant k < n\}$.

The calculation of the restriction of $c(\pi_!\mathcal{V})(t)$ to $H^*_{\mathcal{G}}(\mathcal{C}_\mu; \mathbf{Q})[[t]]$ follows easily from the previous proposition. As in [8, Prop. 2] this restriction can be expressed in terms of elementary functions of the generators of $H^*_{\mathcal{G}}(\mathcal{C}_\mu; \mathbf{Q})$ when $\mu \in \Delta$. However for a general type μ this restriction cannot be expressed so easily and we will find formal expressions similar to (19) of more use.

COROLLARY 13. *Let* $\mu = (d_1/n_1, \ldots, d_P/n_P)$. *The restriction to* $H^*_{\mathcal{G}}(\mathcal{C}_\mu; \mathbf{Q})[[t]]$ *of* $c(\pi_!\mathcal{V})(t)$ *equals the formal power series*

$$\Omega_\mu(t)^{-\bar{g}} \prod_{p=1}^{P} \prod_{k=1}^{n_p} (1 + \delta_k^p t)^{W_k^p} \exp\left\{\frac{X_k^p t}{1 + \delta_k^p t}\right\} \tag{27}$$

and similarly the restriction of $c(\pi_!(\mathcal{V}^* \otimes \phi^* L))(-t)$ *to* $H^*_{\mathcal{G}}(\mathcal{C}_\mu; \mathbf{Q})[[t]]$ *equals*

$$\Omega_\mu(t)^{3\bar{g}+1} \prod_{p=1}^{P} \prod_{k=1}^{n_p} (1 + \delta_k^p t)^{-W_k^p} \exp\left\{\frac{-X_k^p t}{1 + \delta_k^p t}\right\}, \tag{28}$$

where $\delta_1^p, \ldots, \delta_{n_p}^p$ *are formal degree two classes such that their* r*th elementary symmetric polynomial equals* a_r^p, *where* $\Omega_\mu(t) = \prod_{p=1}^{P} \prod_{k=1}^{n_p} (1 + \delta_k^p t)$ *is the restriction of* $\Omega(t)$ *to* $H^*_{\mathcal{G}}(\mathcal{C}_\mu; \mathbf{Q})[t]$, *and where* $\xi_{i,j}^{p,p}, W_k^p$ *and* X_k^p *correspond to the expressions defined in the statement of Proposition* 10.

Proof. Expression (27) is immediate from the previous proposition once we note that the restriction of $\mathrm{ch}(\pi_!\mathcal{V})$ to $H^*_{\mathcal{G}}(\mathcal{C}_\mu; \mathbf{Q})$ equals

$$\sum_{p=1}^{P} \pi_*(\mathrm{ch}(\mathcal{V}_p) \cdot 1 \otimes (1 - \bar{g}\omega))$$

and recall that the Chern polynomial is multiplicative. The dual expression (28) follows in a similar fashion. □

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COROLLARY 14. *Let* $\mu = (d_1, \dots, d_n) \in \Delta$. *Then the restriction of* $c(\pi_! \mathcal{V})(t)$ *to* $H^*_{\mathcal{G}}(\mathcal{C}_\mu; \mathbf{Q})[[t]]$ *equals*

$$\prod_{p=1}^{n} (1 + a_1^p t)^{d_p - \bar{g}} \exp \left\{ \frac{\xi_{1,1}^{p,p} t}{1 + a_1^p t} \right\}.$$

Also the restriction of $c(\pi_!(\mathcal{V}^* \otimes \phi^* L))(-t)$ *to* $H^*_{\mathcal{G}}(\mathcal{C}_\mu; \mathbf{Q})[[t]]$ *equals*

$$\prod_{p=1}^{n} (1 + a_1^p t)^{3\bar{g}+1-d_p} \exp \left\{ \frac{-\xi_{1,1}^{p,p} t}{1 + a_1^p t} \right\}.$$

Proof. Simply note that in this case $\delta_1^p = a_1^p, W_1^p = d_p$ and $X_1^p = \xi_{1,1}^{p,p}$. $\square$

Remark 15. Let $\mu = (d_1/n_1, \dots, d_P/n_P)$. From the calculation (23) and since the Chern character is additive we know that the restriction of $\mathrm{ch}(\mathcal{V})$ to $H^*_{\mathcal{G}}(\mathcal{C}_\mu; \mathbf{Q})$ equals

$$\sum_{p=1}^{P} \sum_{k=1}^{n_p} \exp \left\{ \delta_k^p + \sum_{s=1}^{2g} \left(\sum_{i=1}^{n_p} b_i^{p,s} \frac{\partial \delta_k^p}{\partial a_i^p} \right) \otimes \alpha_s + W_k^p \otimes \omega \right\}.$$

Thus in terms of our earlier notation (15) we have

$$a^{p,k} = \delta_k^p, \qquad b^{p,k,s} = \sum_{i=1}^{n_p} b_i^{p,s} \frac{\partial \delta_k^p}{\partial a_i^p}, \qquad d^{p,k} = W_k^p.$$

We end this section with two further calculations, namely the Chern polynomials of the normal bundle $\mathcal{N}_\mu$ to the stratum $\mathcal{C}_\mu$ in $\mathcal{C}$ (necessary to the completeness criteria) and of the tangent bundle T to the moduli space $\mathcal{M}(n, d)$ (needed for generalising the proof of the Newstead–Ramanan conjecture).

LEMMA 16. *Let* $\mu = (d_1/n_1, \dots, d_P/n_P)$. *Then the Chern polynomial* $c(\mathcal{N}_\mu)(t)$ *of the normal bundle in* $\mathcal{C}$ *to the stratum* $\mathcal{C}_\mu$ *equals*

$$\begin{aligned} &\mathcal{P}_\mu(t)^{\bar{g}} \prod_{I<J} \prod_{k=1}^{n_I} \prod_{l=1}^{n_J} (1 + (\delta_l^J - \delta_k^I)t)^{W_k^I - W_l^J} \\ &\quad \times \exp \left\{ \frac{-\Xi_{k,l}^{I,J} t}{1 + (\delta_l^J - \delta_k^I)t} \right\}, \end{aligned} \tag{29}$$

where

$$\begin{aligned} \Xi_{k,l}^{I,J} &= \sum_{s=1}^{g} \left(\sum_{i=1}^{n_I} b_i^{I,s} \frac{\partial \delta_k^I}{\partial a_i^I} - \sum_{j=1}^{n_J} b_j^{J,s} \frac{\partial \delta_l^J}{\partial a_j^J} \right) \\ &\quad \times \left(\sum_{i=1}^{n_I} b_i^{I,s+g} \frac{\partial \delta_k^I}{\partial a_i^I} - \sum_{j=1}^{n_J} b_j^{J,s+g} \frac{\partial \delta_l^J}{\partial a_j^J} \right), \end{aligned}$$

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and

$$\mathcal{P}_\mu(t) = \prod_{I<J}\prod_{k=1}^{n_I}\prod_{l=1}^{n_J}(1 + (\delta_l^J - \delta_k^I)t).$$

Proof. Kirwan [8, Lemma 2] showed that the normal bundle $\mathcal{N}_\mu$ to $\mathcal{C}_\mu$ in $\mathcal{C}$, equals

$$-\pi_!\left(\bigoplus_{I<J}\mathcal{V}_I^* \otimes \mathcal{V}_J\right).$$

From the proof of the Proposition 10 we can find expressions for $\mathrm{ch}(\mathcal{V}_J)$ and $\mathrm{ch}(\mathcal{V}_I^*)$ corresponding to (23) and (25). The GRR implies that

$$\mathrm{ch}(\mathcal{N}_\mu) = \sum_{I<J}\pi_*(\mathrm{ch}(\mathcal{V}_I^*)\cdot\mathrm{ch}(\mathcal{V}_J)\cdot 1\otimes(\bar{g}\omega - 1)).$$

Substituting in these expressions for $\mathrm{ch}(\mathcal{V}_J)$ and $\mathrm{ch}(\mathcal{V}_I^*)$ we find that $\mathrm{ch}(\mathcal{N}_\mu)$ equals

$$\sum_{I<J}\left\{\sum_{k=1}^{n_I}\sum_{l=1}^{n_J}(\bar{g} + W_k^I - W_l^J - \Xi_{k,l}^{I,J})e^{\delta_l^J - \delta_k^I}\right\}.$$

Applying Lemma 9 produces the required result (29). □

LEMMA 17. *The total Pontryagin class of* $\mathcal{M}(n,d)$ *equals*

$$\prod_{1\leqslant k<l\leqslant n}(1 + (\delta_k - \delta_l)^2)^{2\bar{g}}.$$

In particular the Pontryagin ring of $\mathcal{M}(n,d)$ *is generated by the elementary symmetric polynomials in*

$$\{(\delta_k - \delta_l)^2 \colon 1 \leqslant k < l \leqslant n\}.$$

Proof. Let T denote the tangent bundle of $\mathcal{M}(n,d)$. From [1, p. 582] we know that

$$T + T^* - 2 = \pi_!(\mathrm{End}\, V \otimes (\Omega_M^1 - 1)).$$

Applying GRR we find

$$\mathrm{ch}\, T + \mathrm{ch}\, T^* - 2 = 2\bar{g}\mathrm{ch}(\mathrm{End}\, V|\mathcal{M}(n,d)),$$

which we know to equal

$$2\bar{g}\left(\sum_{k=1}^{n} e^{\delta_k}\right)\left(\sum_{l=1}^{n} e^{-\delta_l}\right)$$

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from expressions (23) and (25).

Now let $p(T)(t) = \sum_{r\geqslant 0} p_r(T)t^r$ denote the Pontryagin polynomial. The relationship between the Pontryagin classes and the Chern classes is given by

$$p(T)(-1) = c(T)(1) \cdot c(T)(-1) \qquad [9, \text{Cor. } 15.5].$$

Hence $p(T)(-1)$ equals

$$\prod_{k\neq l}(1+\delta_k-\delta_l)^{2\bar{g}} = \prod_{k<l}(1-(\delta_k-\delta_l)^2)^{2\bar{g}}.$$

The total Pontryagin class of $\mathcal{M}(n,d)$ then equals $p(T)(1)$ and hence the result. □

4. A complete set of relations

Whilst we observed in Remark 3 that neither the Mumford relations nor the dual Mumford relations are in themselves a complete set of relations when the rank is greater than two, it is still possible to put these relations into the context of the completeness criteria. In terms of these criteria we will show how the Mumford relations contain subsets corresponding to all strata of the form

$$\mu = (d_1/n_1, \dots, d_P/n_P),$$

where $n_P = 1$. Similarly the dual Mumford relations contain subsets corresponding to all those strata with $n_1 = 1$. From this we shall deduce that in the rank three case the Mumford and dual Mumford relations form a complete set.

Before we continue with the main proposition we need a lemma on the vanishing of the Mumford and dual Mumford relations on restriction to a stratum.

LEMMA 18. *Let* $\mu = (d_1/n_1, \dots, d_P/n_P)$. *The image of the Mumford relation* $\sigma^k_{r,S}$ *under the restriction map*

$$H^*_{\mathcal{G}}(\mathcal{C};\mathbf{Q}) \to H^*_{\mathcal{G}}(\mathcal{C}_\mu;\mathbf{Q}),$$

vanishes when $r < d_P/n_P - 2g + 1$. *The image of the dual Mumford relation* $\tau^k_{r,S}$ *under the restriction map vanishes when* $r < 2\bar{g} - d_1/n_1$.

Proof. Recall that the Mumford relations are given by $\sigma^k_{r,S} (r < 0, 0 \leqslant k \leqslant n-1, S \subseteq \{1,\dots,2g\})$ when $\Psi(t) = t^{d-n\bar{g}} c(\pi_!\mathcal{V})(t^{-1})$ is written in the form

$$\sum_{r=-\infty}^{\bar{g}} (\sigma^0_r + \sigma^1_r t + \cdots + \sigma^{n-1}_r t^{n-1})(\tilde{\Omega}(t))^r, \qquad \sigma^k_r = \sum_{S\subseteq\{1,\dots,2g\}} \sigma^k_{r,S} \prod_{s\in S} b^s_1.$$

For $1 \leqslant k \leqslant n$ and any fixed integer R the power t^{-k} appears in

$$\sum_{r=-\infty}^{\bar{g}} (\sigma^0_r + \sigma^1_r t + \cdots + \sigma^{n-1}_r t^{n-1})(\tilde{\Omega}(t))^{r-R-1}$$

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only when $r = R$. Let C_r^i denote the coefficient of t^{-i} in $\Psi(t)(\tilde{\Omega}(t))^{-r-1}$. Then

$$(\sigma_r^0 + \sigma_r^1 t + \cdots + \sigma_r^{n-1} t^{n-1}) = (t^n + a_1 t^{n-1} + \cdots + a_n) \sum_{i=1}^{n} C_r^i t^{-i}$$

modulo negative powers of t and hence

$$\sigma_r^{n-k} = \sum_{i=1}^{k} a_{k-i} C_r^i \quad (r < 0, 1 \leqslant k \leqslant n). \tag{30}$$

Now let K be a fixed line bundle over M of degree D where D is the smallest integer such that

$$\mu(Q_P \otimes K) = \frac{d_P}{n_P} + D > 2\bar{g},$$

where $Q_P = E_P/E_{P-1}$. Since $\mu(Q_p \otimes K) \geqslant \mu(Q_P \otimes K) > 2\bar{g}$ then $\pi_!(\mathcal{V}_p \otimes \phi^* K)$ is a bundle over $\mathcal{C}(n_p, d_p)^{ss}$ of rank $d_p + (D - \bar{g}) n_p$ for each $1 \leqslant p \leqslant P$. In particular

$$\Psi(\pi_!(\mathcal{V}_p \otimes \phi^* K))(t) = t^{d_p + n_p(D - \bar{g})} c(\pi_!(\mathcal{V}_p \otimes \phi^* K))(t^{-1})$$

is a polynomial modulo relations in $H^*_{\mathcal{G}(n_p,d_p)}(\mathcal{C}(n_p, d_p)^{ss}; \mathbf{Q})$. From GRR we have that $\mathrm{ch}(\pi_!(\mathcal{V}_p \otimes \phi^* K))$ equals

$$\mathrm{ch}(\pi_! \mathcal{V}_p) + \pi_*(\mathrm{ch}\, \mathcal{V}_p \cdot 1 \otimes D\omega) = \mathrm{ch}(\pi_! \mathcal{V}_p) + D \sum_{k=1}^{n_p} e^{\delta_k^p}. \tag{31}$$

In terms of Chern polynomials (31) gives

$$c(\pi_!(\mathcal{V}_p \otimes \phi^* K))(t) = (\Omega_p(t))^D c(\pi_! \mathcal{V}_p)(t),$$

where $\Omega_p(t) = \prod_{k=1}^{n_p} (1 + \delta_k^p t)$. Hence

$$\prod_{p=1}^{P} \Psi(\pi_!(\mathcal{V}_p \otimes \phi^* K))(t) = (\tilde{\Omega}_\mu(t))^D \Psi_\mu(t) \tag{32}$$

is a polynomial modulo relations in $H^*_{\mathcal{G}}(\mathcal{C}_\mu; \mathbf{Q})$ where $\Psi_\mu(t)$, and $\tilde{\Omega}_\mu(t)$ are respectively the restrictions to $H^*_{\mathcal{G}}(\mathcal{C}_\mu; \mathbf{Q})$ of $\Psi(t)$ and $\tilde{\Omega}(t)$. Thus the coefficient of t^{-k} in $\Psi_\mu(t) \tilde{\Omega}_\mu(t)^{-r-1}$ is a relation when $r \leqslant -1 - D$. So by (30) the restriction of σ_r^k to $H^*_{\mathcal{G}}(\mathcal{C}_\mu; \mathbf{Q})$ vanishes when $r \leqslant d_P/n_P - 2g$. The dual calculation follows by a similar argument. $\square$

Thus finally we come to

PROPOSITION 19. *Let $\mu = (d_1/n_1, \ldots, d_P/n_P)$ with $n_P = 1$. Then there is a subset $\mathcal{R}_\mu$ of the ideal generated by the Mumford relations such that the image of the ideal generated by $\mathcal{R}_\mu$ under the restriction map*

$$H^*_{\mathcal{G}}(\mathcal{C}; \mathbf{Q}) \to H^*_{\mathcal{G}}(\mathcal{C}_\nu; \mathbf{Q}) \quad \nu = (\tilde{d}_1/\tilde{n}_1, \ldots, \tilde{d}_T/\tilde{n}_T),$$

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is zero when either

$$\text{(i)}\quad \tilde{d}_T/\tilde{n}_T > d_P \quad \text{or} \qquad \text{(ii)}\quad \tilde{n}_T = 1, \tilde{d}_T = d_P, \quad \text{and}\, \nu \not\geqslant \mu,$$

and contains the ideal of $H^*_{\mathcal{G}}(\mathcal{C}_\mu;\mathbf{Q})$ *generated by* e_μ *when* $\nu = \mu$.

Let $\mu = (d_1/n_1, \ldots, d_P/n_P)$ *with* $n_1 = 1$. *Then there is a subset* $\mathcal{R}_\mu$ *of the ideal generated by the dual Mumford relations such that the image of the ideal generated by* $\mathcal{R}_\mu$ *under the restriction map*

$$H^*_{\mathcal{G}}(\mathcal{C};\mathbf{Q}) \to H^*_{\mathcal{G}}(\mathcal{C}_\nu;\mathbf{Q}) \quad \nu = (\tilde{d}_1/\tilde{n}_1, \ldots, \tilde{d}_T/\tilde{n}_T)$$

is zero when either

$$\text{(i)}\, \tilde{d}_1/\tilde{n}_1 < d_1/n_1 \quad \text{or} \qquad \text{(ii)}\quad \tilde{n}_1 = 1, \tilde{d}_1 = d_1 \quad \text{and} \quad \nu \not\geqslant \mu,$$

and contains the ideal of $H^*_{\mathcal{G}}(\mathcal{C}_\mu;\mathbf{Q})$ *generated by* e_μ *when* $\nu = \mu$.

Proof. Let $\Psi(t) = t^{d-n\bar{g}}c(\pi_!\mathcal{V})(t^{-1})$ and let $C^R_K, (R < 0, 1 \leqslant K \leqslant n)$ denote the coefficient of t^{-K} in $\Psi(t)(\tilde{\Omega}(t))^{-R-1}$. Let

$$\mu = (d_1/n_1, \ldots, d_{P-1}/n_{P-1}, d_P)$$

so that $n_P = 1$.

Since the Chern polynomial is multiplicative the restriction in $H^*_{\mathcal{G}}(\mathcal{C}_\mu;\mathbf{Q})$ of C^K_R, which we will write as $C^{K,\mu}_R$, equals the coefficient of t^{-1} in

$$t^{K-1}\prod_{p=1}^{P}\Psi_p(t)(\tilde{\Omega}_p(t))^{-R-1}, \tag{33}$$

where

$$\Psi_p(t) = t^{d_p - n_p\bar{g}}c(\pi_!\mathcal{V}_p)(t^{-1}), \qquad \tilde{\Omega}_p(t) = t^{n_p} + a^p_1 t^{n_p-1} + \cdots + a^p_{n_p},$$

for $1 \leqslant p \leqslant P$. Further from the previous lemma we know that $C^{K,\mu}_R$ vanishes when $R < -D = d_P - 2g + 1$.

We facilitate the proof of Proposition 19 with the following lemma and corollaries

LEMMA 20. *Let* $\theta(t)$ *equal*

$$t^{d-nd_P+(n-1)\bar{g}}\prod_{p=1}^{P-1}\prod_{k=1}^{n_p}(1+(\delta^p_k - a^P_1)/t)^{W^p_k+\bar{g}-d_P}\exp\left\{\frac{\Xi^{p,P}_{k,1}}{t+\delta^p_k - a^P_1}\right\}. \tag{34}$$

Then modulo relations in $H^*_{\mathcal{G}}(\mathcal{C}_\mu;\mathbf{Q})$,

$$C^{K,\mu}_{-D} = (-a^P_1)^{K-1}(\xi^{P,P}_{1,1})^g\Theta,$$

where Θ *is the constant coefficient of* $\theta(t)$.

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Proof. From Corollary 14 we know that

$$\Psi_P(t)(\tilde{\Omega}_P(t))^{D-1} = (t+a_1^P)^{\bar{g}} \exp\left\{\frac{\xi_{1,1}^{P,P}}{t+a_1^P}\right\},$$

where $\xi_{1,1}^{P,P} = \sum_{s=1}^{g} b_1^{P,s} b_1^{P,s+g}$. Also in a Laurent series the coefficient of t^{-1} is invariant under transformations such as $t \mapsto t - a_1^P$. So from (33) $C_{-D}^{K,\mu}$ equals the coefficient of t^{-1} in

$$(t-a_1^P)^{K-1} t^{\bar{g}} \exp(\xi_{1,1}^{P,P}/t) \prod_{p=1}^{P-1} \Psi_p(t-a_1^P)(\tilde{\Omega}_p(t-a_1^P))^{D-1}. \tag{35}$$

From the proof of Lemma 18 (32) we know that

$$\Psi_p(t)(\tilde{\Omega}_p(t))^{D-1} = \Psi(\pi_!(\mathcal{V}_p \otimes \phi^*\mathcal{L}))(t),$$

where $\mathcal{L}$ is a fixed line bundle over M of degree $D-1$. For each $p \neq P$, $Q_p \otimes \mathcal{L}$ is a semistable bundle of slope

$$\frac{d_p}{n_p} - d_P + 2\bar{g} > 2\bar{g}.$$

Hence $\pi_!(\mathcal{V}_p \otimes \phi^*\mathcal{L})$ is a bundle over $\mathcal{C}(n_p, d_p)^{ss}$ and $\Psi_p(t)(\tilde{\Omega}_p(t))^{D-1}$ is a polynomial modulo relations in $H^*_{\mathcal{G}(n_p,d_p)}(\mathcal{C}(n_p,d_p)^{ss}; \mathbf{Q})$. As $(\xi_{1,1}^{P,P})^{g+1} = 0$ it follows from (35) that $C_{-D}^{K,\mu}$ equals the constant coefficient of

$$(\xi_{1,1}^{P,P})^{g}(t-a_1^P)^{K-1} \prod_{p=1}^{P-1} \Psi_p(t-a_1^P)(\tilde{\Omega}_p(t-a_1^P))^{D-1} \tag{36}$$

modulo relations in $H^*_{\mathcal{G}}(\mathcal{C}_\mu; \mathbf{Q})$.

Since $\Sigma_{k=1}^{n_p} W_k^p = d_p$ then we know from Corollary 13 that $\Psi_p(t-a_1^P)$ equals

$$(\tilde{\Omega}_p(t-a_1^P))^{-\bar{g}} t^{d_p} \prod_{k=1}^{n_p} (1+(\delta_k^p - a_1^P)/t)^{W_k^p} \exp\left\{\frac{X_k^p}{t+\delta_k^p - a_1^P}\right\}.$$

Recall from Lemma 16 that

$$\Xi_{k,1}^{p,P} = \sum_{s=1}^{g}\left(\sum_{i=1}^{n_p} b_i^{p,s}\frac{\partial \delta_k^p}{\partial a_i^p} - b_1^{P,s}\right)\left(\sum_{i=1}^{n_p} b_i^{p,s+g}\frac{\partial \delta_k^p}{\partial a_i^p} - b_1^{P,s+g}\right),$$

and we also have that

$$X_k^p = \sum_{s=1}^{g}\left(\sum_{i=1}^{n_p} b_i^{p,s}\frac{\partial \delta_k^p}{\partial a_i^p}\right)\left(\sum_{i=1}^{n_p} b_i^{p,s+g}\frac{\partial \delta_k^p}{\partial a_i^p}\right).$$

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Since

$$(\xi_{1,1}^{P,P})^g = (-1)^{g\bar{g}/2} g! \prod_{s=1}^{2g} b_1^{P,s},$$

then

$$(\xi_{1,1}^{P,P})^g (\Xi_{k,1}^{p,P})^q = (\xi_{1,1}^{P,P})^g (X_k^p)^q \quad (q \geqslant 0).$$

Thus by (36) and the identity $\tilde{\Omega}_p(t - a_1^P) = t^{n_p} \prod_{k=1}^{n_p}(1 + (\delta_k^p - a_1^P)/t)$, we have that $C_{-D}^{K,\mu}$ equals the constant coefficient of

$$(\xi_{1,1}^{P,P})^g (t - a_1^P)^{K-1} \theta(t).$$

Since $(\xi_{1,1}^{P,P})^g \theta(t)$ is a polynomial modulo relations in $H_{\mathcal{G}}^*(\mathcal{C}_\mu; \mathbf{Q})$ then the Lemma follows. □

COROLLARY 21. *Define* $C_{R,S}^K (R < 0, 1 \leqslant K \leqslant n, S \subseteq \{1, \ldots, 2g\})$ *by*

$$C_R^K = \sum_{S \subseteq \{1,\ldots,2g\}} C_{R,S}^K \prod_{s \in S} b_1^s,$$

writing $C_{R,S}^K$ *in terms of the elements* (13) *and also define* $\tilde{a}_r, \tilde{b}_r^s$ *and* $\tilde{f}_r$ *by*

$$c_r(\bigoplus_{p=1}^{P-1} \mathcal{V}_p) = \tilde{a}_r \otimes 1 + \sum_{s=1}^{2g} \tilde{b}_r^s \otimes \alpha_s + \tilde{f}_r \otimes \omega.$$

Then the restriction of $C_{-D,S}^K$ *to* $H_{\mathcal{G}}^*(\mathcal{C}_\mu; \mathbf{Q})$ *equals a nonzero constant multiple of*

$$(a_1^P)^{K-1} \prod_{s \notin S} (\tilde{b}_1^s - (n-1) b_1^{P,s}) \Theta \tag{37}$$

for any subset $S \subseteq \{1, \ldots, 2g\}$.

Proof. We know that $(\xi_{1,1}^{P,P})^g$ equals

$$\begin{aligned}
&(-1)^{g\bar{g}/2} g! \prod_{s=1}^{2g} b_1^{P,s} \\
&\quad = (-1)^{g\bar{g}/2} n^{-2g} g! \prod_{s=1}^{2g} ((\tilde{b}_1^s + b_1^{P,s}) - (\tilde{b}_1^s - (n-1) b_1^{P,s})),
\end{aligned}$$

and also that the restriction of b_1^s in $H_{\mathcal{G}}^*(\mathcal{C}_\mu; \mathbf{Q})$ equals $\tilde{b}_1^s + b_1^{P,s}$. Further

$$\tilde{b}_1^s - (n-1) b_1^{P,s} = \sum_{p=1}^{P-1} \sum_{k=1}^{n_p} \left(\sum_{i=1}^{n_p} b_1^{p,s} \frac{\partial \delta_k^p}{\partial a_i^p} - b_1^{P,s} \right).$$

So the corollary follows once we note from (34) that $\theta(t)$, and hence Θ, can be written in terms of the elements (16). $\square$

COROLLARY 22. *Let* Λ *equal*

$$\bigcup\{\sigma^{n-1}_{-D,S},\ldots,\sigma^{0}_{-D,S}\}, \tag{38}$$

where the union varies over all subsets $S\subseteq\{1,\ldots,2g\}$. *Then all elements of the form*

$$\prod_{k=2}^{n-1}(\tilde f_k)^{m_k}\prod_{k=1}^{n-1}\prod_{s\in S_k}\tilde b_k^s\prod_{k=1}^{n-1}(\tilde a_k)^{r_k}(a_1^P)^r\prod_{s\in S}b_1^{P,s}\Theta \tag{39}$$

lie in the restriction of the ideal generated by Λ, *where* $r,r_1,\ldots,r_{n-1},m_2,\ldots,m_{n-1}$ *are arbitrary nonnegative integers and* $S,S_1,\ldots,S_{n-1}$ *are subsets of* $\{1,\ldots,2g\}$.

Proof. Let (Λ) denote the ideal of $H^*_{\mathcal{G}}(\mathcal{C};\mathbf{Q})$ generated by Λ. Using induction on (30) we know that the restriction of $C^K_{-D,S}$ lies in the image of (Λ). From (37) and since b_1^s restricts to $\tilde b_1^s+b_1^{P,s}$ it follows that all elements of the form

$$(a_1^P)^{K-1}\prod_{s\in S_1}\tilde b_1^s\prod_{s\in S_2}b_1^{P,s}\Theta$$

for arbitrary $S_1,S_2\subseteq\{1,\ldots,2g\}$ and $1\leqslant K\leqslant n$, lie in the restriction of (Λ). The restriction of a_k in $H^*_{\mathcal{G}}(\mathcal{C}_\mu;\mathbf{Q})$ equals $\tilde a_k+\tilde a_{k-1}a_1^P$. By noting that $(a_1^P)^r$ equals

$$\begin{aligned}&(\tilde a_1+a_1^P)(a_1^P)^{r-1}-(\tilde a_2+\tilde a_1a_1^P)(a_1^P)^{r-2}+\cdots\\&\quad+(-1)^{n-1}(\tilde a_{n-1}a_1^P)(a_1^P)^{r-n}\end{aligned}$$

for $r\geqslant n$, we see that all elements of the form

$$(a_1^P)^r\prod_{s\in S_1}\tilde b_1^s\prod_{s\in S_2}b_1^{P,s}\cdot\Theta\quad(r\geqslant 0)$$

lie in the restriction of (Λ). Finally working inductively on the variables $r_1,\ldots,r_{n-1}$, $S_2,S_3,\ldots,S_{n-1}$ and $m_2,m_3,\ldots,m_{n-1}$ in that order we find that all elements of the form (39) lie in the image of (Λ) since under the restriction map $H^*_{\mathcal{G}}(\mathcal{C};\mathbf{Q})\to H^*_{\mathcal{G}}(\mathcal{C}_\mu;\mathbf{Q})$

$$a_k\mapsto\tilde a_k+\tilde a_{k-1}a_1^P,\qquad b_k^s\mapsto\tilde b_k^s+a_1^P\tilde b_{k-1}^s+\tilde a_{k-1}b_1^{P,s},$$

and

$$f_k\mapsto\tilde f_k+d_P\tilde a_{k-1}+a_1^P\tilde f_{k-1}+\sum_{s=1}^{g}(\tilde b_{k-1}^s b_1^{P,s+g}+b_1^{P,s}\tilde b_{k-1}^{s+g}). \quad\square \tag{40}$$

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We now continue with the proof of Proposition 19. Let $\mathcal{C}' = \mathcal{C}(n-1, d-d_P)$ and let $\mathcal{G}' = \mathcal{G}(n-1, d-d_P)$. Let $\mu' = (d_1/n_1, \ldots, d_{P-1}/n_{P-1})$ and let $e_{\mu'}$ denote the equivariant Euler class of the normal bundle to $\mathcal{C}'_{\mu'}$ in $\mathcal{C}'$. Let

$$U_{\mu'} = \mathcal{C}' - \bigcup_{\nu' > \mu'} \mathcal{C}'_{\nu'}.$$

Then $U_{\mu'}$ is an open subset of $\mathcal{C}'$ which contains $\mathcal{C}'_{\mu'}$ as a closed submanifold. So we have the maps

$$\begin{tikzcd}
 & H^*_{\mathcal{G}'}(\mathcal{C}'; \mathbf{Q}) \arrow[d] & \\
H^{*-2d_{\mu'}}_{\mathcal{G}'}(\mathcal{C}'_{\mu'}; \mathbf{Q}) \arrow[r] \arrow[dr, "\text{multiplication by } e_{\mu'}"'] & H^*_{\mathcal{G}'}(U_{\mu'}; \mathbf{Q}) \arrow[r] \arrow[d] & H^*_{\mathcal{G}'}(U_{\mu'} - \mathcal{C}'_{\mu'}; \mathbf{Q}) \\
 & H^*_{\mathcal{G}'}(\mathcal{C}'_{\mu'}; \mathbf{Q}). &
\end{tikzcd}$$

Let $a'_r, b^{s\prime}_r$ and f'_r denote the generators of $H^*_{\mathcal{G}'}(\mathcal{C}'; \mathbf{Q})$. Also take $\nu' \ngeq \mu'$ and let $\hat{a}_r, \hat{b}^s_r, \hat{f}_r$ denote the restrictions of $a'_r, b^{s\prime}_r, f'_r$ in $H^*_{\mathcal{G}'}(\mathcal{C}'_{\nu'}; \mathbf{Q})$. Since the stratification is equivariantly perfect then the restriction map

$$H^*_{\mathcal{G}'}(\mathcal{C}'; \mathbf{Q}) \to H^*_{\mathcal{G}'}(U_{\mu'}; \mathbf{Q})$$

is surjective [8, p. 859]. From the exactness of the Thom–Gysin sequence we have that for every element of the form $\alpha e_{\mu'}$ in $H^*_{\mathcal{G}'}(\mathcal{C}'_{\mu'}; \mathbf{Q}) e_{\mu'}$ there is some $\beta(a'_r, b^{s\prime}_r, f'_r)$ in $H^*_{\mathcal{G}'}(\mathcal{C}'; \mathbf{Q})$ such that

$$\beta(\tilde{a}_r, \tilde{b}^s_r, \tilde{f}_r) = \alpha e_{\mu'} \quad \text{and} \quad \beta(\hat{a}_r, \hat{b}^s_r, \hat{f}_r) = 0.$$

Since every element of the form (39) lies in the restriction of (Λ) to $H^*_{\mathcal{G}}(\mathcal{C}_\mu; \mathbf{Q})$ then every element of the form

$$\beta(\tilde{a}_r, \tilde{b}^s_r, \tilde{f}_r)(a^P_1)^r \prod_{s \in S} b^{P,s}_1 \Theta \quad (r \geqslant 0, S \subseteq \{1, \ldots, 2g\}) \tag{41}$$

similarly lies in the restriction of (Λ). Now let $\nu = (\nu', d_P)$ with $\nu' \ngeq \mu'$. Note that the restriction map

$$H^*_{\mathcal{G}}(\mathcal{C}; \mathbf{Q}) \to H^*_{\mathcal{G}}(\mathcal{C}_\nu; \mathbf{Q})$$

is formally the same as (40) but with $\hat{a}_r, \hat{b}^s_r, \hat{f}_r$ replacing $\tilde{a}_r, \tilde{b}^s_r, \tilde{f}_r$. Thus there are elements of (Λ) which restrict to (41) under (40) and have restriction

$$\beta(\hat{a}_r, \hat{b}^s_r, \hat{f}_r)(a^P_1)^r \prod_{s \in S} b^{P,s}_1 \hat{\Theta} = 0$$

in $H^*_{\mathcal{G}}(\mathcal{C}_\nu;\mathbf{Q})$.

Define $\mathcal{R}_\mu$ to be all those elements of (Λ) which restrict to an element of the form

$$\alpha e_{\mu'}(a_1^P)^r \prod_{s\in S} b_1^{P,s}\Theta \quad (r \geqslant 0, S \subseteq \{1,\ldots,2g\}, \alpha \in H^*_{\mathcal{G}'}(\mathcal{C}'_{\mu'};\mathbf{Q})),$$

in $H^*_{\mathcal{G}}(\mathcal{C}_\mu;\mathbf{Q})$ and which restrict to zero in $H^*_{\mathcal{G}}(\mathcal{C}_\nu;\mathbf{Q})$ for any $\nu = (\nu', d_P)$ with $\nu' \not\geqslant \mu'$.

From the definition of Θ (34) we know that $e_{\mu'}\Theta$ is the constant coefficient of

$$(-1)^{d_{\mu'}} t^{d_{\mu'}} c(\mathcal{N}_{\mu'})(-t^{-1})\theta(t), \tag{42}$$

where $\mathcal{N}_{\mu'}$ is the normal bundle to $\mathcal{C}'_{\mu'}$ in $\mathcal{C}'$ and $d_{\mu'}$ is the codimension of $\mathcal{C}'_{\mu'}$ in $\mathcal{C}'$. From Lemma 16 and the fact that

$$d_{\mu'} + d - nd_P + (n-1)\bar{g} = d_\mu,$$

we know (42) equals

$$(-1)^{d_{\mu'}} t^{d_\mu} c(\mathcal{N}_\mu)(-t^{-1}),$$

which has constant coefficient $(-1)^{d_{\mu'}+d_\mu} e_\mu$. Hence the ideal

$$H^*_{\mathcal{G}}(\mathcal{C}_\mu;\mathbf{Q})e_\mu$$

lies in the restriction of $\mathcal{R}_\mu$ to $H^*_{\mathcal{G}}(\mathcal{C}_\mu;\mathbf{Q})$.

Finally from Lemma 18 and the definition of Λ (38) we know that the image of $\mathcal{R}_\mu$ under the restriction map

$$H^*_{\mathcal{G}}(\mathcal{C};\mathbf{Q}) \to H^*_{\mathcal{G}}(\mathcal{C}_\nu;\mathbf{Q}) \quad \nu = (\tilde{d}_1/\tilde{n}_1, \ldots, \tilde{d}_T/\tilde{n}_T),$$

vanishes when $\tilde{d}_T/\tilde{n}_T > d_P/n_P$ proving the first half of Proposition 19.

The proof of the dual case follows in a similar fashion. □

In the general rank case there are strata of types not covered in the previous proposition. Moreover the strata on which the restrictions of the relations have been demonstrated to vanish do not generally coincide with the strata mentioned in the hypotheses of the completeness criteria. However in the rank two and rank three cases all unstable strata are covered by the above proposition. In the rank two case Proposition 19 shows that the Mumford relations and the dual Mumford relations both form complete sets, simply duplicating Kirwan's work [8] and Remark 2. In the rank three case we have the following:

THEOREM 1. *The Mumford and dual Mumford relations together with the relation* (3) *due to the normalisation of the universal bundle V form a complete set of relations for $H^*(\mathcal{M}(3,d);\mathbf{Q})$.*

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Proof. The unstable strata are now of types (2,1), (1,1,1) and (1,2). From the previous proposition we may meet the completeness criteria for the (2,1) and (1,1,1) strata using the Mumford relations. In these cases those strata where the restriction of $\mathcal{R}_\mu$ have been shown to vanish are those strata $\mathcal{C}_\nu$ such that $\nu \prec \mu$. The criteria for the (1,2) types may be met using the dual Mumford relations. In this case those strata where the restriction of $\mathcal{R}_\mu$ vanishes (according to Proposition 19) are those strata $\mathcal{C}_\nu$ such that $\nu \not\succeq \mu$ which certainly includes those strata such that $\nu \prec \mu$. □

Remark 23. As remarked earlier it was shown in [3, Th. 4] that the Mumford relations $\sigma^1_{-1,S}$ for $S \subseteq \{1, \ldots, 2g\}$ generate the relation ideal of $H^*(\mathcal{M}_0(2,1); \mathbf{Q})$ as a $\mathbf{Q}[a_2, f_2]$-module. Evidence for this theorem appears in the Poincaré polynomial of the relation ideal which equals [1, p. 593]

$$\frac{t^{2g}(1+t)^{2g}}{(1-t^2)(1-t^4)}.$$

Similarly in the rank three case the Poincaré polynomial of the ideal of relations among our generators for $H^*(\mathcal{M}_0(3,1); \mathbf{Q})$ equals

$$\frac{(1+t^2)^2 t^{4g-2}(1+t)^{2g}(1+t^3)^{2g} - (1+t^2+t^4)t^{6g-2}(1+t)^{4g}}{(1-t^2)(1-t^4)^2(1-t^6)}.$$

The first Mumford relation $\sigma^2_{-1,\{1,\ldots,2g\}}$ has degree $4g-2$ and the first dual Mumford relation $\tau^2_{-1,\{1,\ldots,2g\}}$ has degree $4g$. This strongly suggests that the relations

$$\{\sigma^i_{-1,S}, \tau^i_{-1,S} \colon i = 1, 2, S \subseteq \{1, \ldots, 2g\}\},$$

generate the relation ideal of $H^*(\mathcal{M}_0(3,1); \mathbf{Q})$ as a

$$\mathbf{Q}[a_2, a_3, f_2, f_3] \otimes \Lambda^*\{b_2^1, \ldots, b_2^{2g}\}$$

module.

5. On the vanishing of the Pontryagin ring

We now move on to discuss the Pontryagin ring of the moduli space in the rank three case. For each $S \subseteq \{1, \ldots, 2g\}$ we define $\Psi_S(t)$ and $\Psi^*_S(t)$ by writing

$$\Psi(t) = \sum_{S \subseteq \{1,\ldots,2g\}} \Psi_S(t) \prod_{s \in S} b_1^s, \qquad \Psi^*(t) = \sum_{S \subseteq \{1,\ldots,2g\}} \Psi^*_S(t) \prod_{s \in S} b_1^s.$$

Kirwan proved the Newstead–Ramanan conjecture [8, Sect. 4] by considering relations derived from the expression

$$\Psi_{\{1,\ldots,2g\}}(t)\Psi_{\{1,\ldots,2g\}}(-t-a_1).$$

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Arguing along similar lines but now considering the expression

$$\Phi(t) = \Psi_{\{1,\dots,2g\}}(t)\Psi^*_{\{1,\dots,2g\}}(t),$$

we will show that in the rank three case the Pontryagin ring vanishes in degree $12g-8$ and above – Theorem 2 below.

LEMMA 24. *Let $\mu = (d_1, d_2, \dots, d_n) \in \Delta$. The restriction of $\Phi(t)$ to $H^*_{\mathcal{G}}(\mathcal{C}_\mu; \mathbf{Q})$ equals*

$$(-1)^g \frac{A(t)^{2g}}{n^{4g}\tilde{\Omega}_\mu(t)},$$

where

$$\tilde{\Omega}_\mu(t) = \prod_{p=1}^n (t + a_1^p), \qquad A(t) = \sum_{p=1}^n \prod_{q\neq p} (t + a_1^q).$$

Proof. From Corollary 14 we know that the restriction of $\Psi(t)$ to $H^*_{\mathcal{G}}(\mathcal{C}_\mu; \mathbf{Q})$ equals

$$\prod_{p=1}^n (t + a_1^p)^{d_p - \bar{g}} \exp\left\{\frac{\xi_p}{t + a_1^p}\right\},$$

where $\xi_p = \xi_{1,1}^{p,p} = \Sigma_{s=1}^g b_1^{p,s} b_1^{p,s+g}$. Let $v_s = b_1^{1,s} + \dots + b_1^{n,s}$ denote the restriction of b_1^s to $H^*_{\mathcal{G}}(\mathcal{C}_\mu; \mathbf{Q})$ and let $w_{i,j}^s = b_1^{i,s} - b_1^{j,s}$ (see (14)). Then $nb_1^{i,s} = v_s + \Sigma_{j=1}^n w_{i,j}^s$ and hence

$$\begin{aligned} n^2\xi_i \; &= \; \sum_{s=1}^g v_s v_{s+g} + \sum_{s=1}^g \left(v_s \sum_{j=1}^n w_{i,j}^{s+g} + \sum_{j=1}^n w_{i,j}^s v_{s+g} \right) \\ &\quad + \sum_{s=1}^g \sum_{j=1}^n \sum_{k=1}^n w_{i,j}^s w_{i,k}^{s+g}. \end{aligned}$$

Note that

$$\sum_{p=1}^n \frac{\xi_p}{t + a_1^p} = \frac{1}{\tilde{\Omega}_\mu(t)} \sum_{i=1}^n \sum_{q\neq i} \xi_i (t + a_1^q). \tag{43}$$

Thus (43) equals

$$\frac{1}{n^2\tilde{\Omega}_\mu(t)} \left\{ A(t) \sum_{s=1}^g v_s v_{s+g} + \sum_{s=1}^g (B_s(t) v_{s+g} + v_s B_{s+g}(t)) + \Gamma(t) \right\},$$

where

$$A(t) = \sum_{i=1}^n \prod_{q\neq i} (t + a_1^q), \quad B_s(t) = \sum_{i=1}^n \sum_{j=1}^n w_{i,j}^s \prod_{q\neq i} (t + a_1^q),$$

$$\Gamma(t) = \sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n}\sum_{s=1}^{g} w_{i,j}^{s} w_{i,k}^{s+g} \prod_{q\neq i}(t + a_1^q).$$

The exponential of (43) equals

$$\exp\left\{\frac{\Gamma(t)}{n^2\tilde{\Omega}_\mu(t)}\right\}\prod_{s=1}^{g}\left[1 + \frac{B_s(t)v_{s+g} + v_s B_{s+g}}{n^2\tilde{\Omega}_\mu(t)} + \left(\frac{A(t)}{n^2\tilde{\Omega}_\mu(t)} - \frac{B_s B_{s+g}}{n^4\tilde{\Omega}_\mu(t)^2}\right) v_s v_{s+g}\right].$$

The coefficient of $\prod_{s=1}^{2g} v_s$ in the above then equals

$$(-1)^{g\bar{g}/2}\exp\left\{\frac{\Gamma(t)}{n^2\tilde{\Omega}_\mu(t)}\right\}\prod_{s=1}^{g}\left(\frac{A(t)}{n^2\tilde{\Omega}_\mu(t)} - \frac{B_s B_{s+g}}{n^4\tilde{\Omega}_\mu(t)^2}\right),$$

or equivalently

$$(-1)^{g\bar{g}/2}\exp\left\{\frac{\Gamma(t)}{n^2\tilde{\Omega}_\mu(t)}\right\}\left(\frac{A(t)}{n^2\tilde{\Omega}_\mu(t)}\right)^g \exp\left\{\frac{-\xi(t)}{n^2 A(t)\tilde{\Omega}_\mu(t)}\right\},$$

where $\xi(t) = \Sigma_{s=1}^{g} B_s(t)B_{s+g}(t)$. Thus the restriction of $\Psi_{\{1,\ldots,2g\}}(t)$ to $H^*_{\mathcal{G}}(\mathcal{C}_\mu;\mathbf{Q})$ equals

$$(-1)^{g\bar{g}/2}\left(\prod_{p=1}^{n}(t + a_1^p)^{d_p-\bar{g}}\right)\exp\left\{\frac{\Gamma(t)}{n^2\tilde{\Omega}_\mu(t)}\right\}\left(\frac{A(t)}{n^2\tilde{\Omega}_\mu(t)}\right)^g \exp\left\{\frac{-\xi(t)}{n^2 A(t)\tilde{\Omega}_\mu(t)}\right\},$$

and similarly the restriction of $\Psi^*_{\{1,\ldots,2g\}}(t)$ to $H^*_{\mathcal{G}}(\mathcal{C}_\mu;\mathbf{Q})$ equals

$$(-1)^{g\bar{g}/2}\left(\prod_{p=1}^{n}(t + a_1^p)^{3\bar{g}+1-d_p}\right)\exp\left\{\frac{-\Gamma(t)}{n^2\tilde{\Omega}_\mu(t)}\right\}\left(\frac{-A(t)}{n^2\tilde{\Omega}_\mu(t)}\right)^g \exp\left\{\frac{\xi(t)}{n^2 A(t)\tilde{\Omega}_\mu(t)}\right\}.$$

The result then follows. □

Now if we write $\Phi(t)$ in the form

$$\sum_{r=-\infty}^{2g-1}(\rho_r^0 + \rho_r^1 t + \cdots + \rho_r^{n-1}t^{n-1})(\tilde{\Omega}(t))^r,$$

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where $\tilde{\Omega}(t) = t^n + a_1 t^{n-1} + \cdots + a_n$ then we know that the elements $\rho_r^k, (r < 0, 0 \leqslant k \leqslant n-1)$ lie in the kernel of the restriction map

$$H_{\mathcal{G}}^*(\mathcal{C}; \mathbf{Q}) \to H_{\mathcal{G}}^*(\mathcal{C}^{ss}; \mathbf{Q}).$$

From Lemma 24 we know that the restriction of $\Phi(t)$ to $H_{\mathcal{G}}^*(\mathcal{C}_\mu; \mathbf{Q})$ equals

$$(-1)^g \frac{A(t)^{2g}}{n^{4g} \tilde{\Omega}_\mu(t)},$$

for any $\mu \in \Delta$. Let $\rho_r^{k,\mu}$ denote the restriction of ρ_r^k in $H_{\mathcal{G}}^*(\mathcal{C}_\mu; \mathbf{Q})$. Thus we have that

$$\frac{(-1)^g}{n^{4g}} A(t)^{2g} = \sum_{k=0}^{n-1} \rho_{-1}^{k,\mu} t^k \quad \bmod \tilde{\Omega}_\mu(t).$$

Hence by substituting $t = -a_1^i$ for each i we obtain

$$\frac{(-1)^g}{n^{4g}} \left(\prod_{p=1, p \neq i}^{n} (a_1^i - a_1^p) \right)^{2g} = \sum_{k=0}^{n-1} \rho_{-1}^{k,\mu} (-a_1^i)^k.$$

Since the direct sum of restriction maps

$$H_{\mathcal{G}}^*(\mathcal{C}; \mathbf{Q}) \to \bigoplus_{\mu \in \Delta} H_{\mathcal{G}}^*(\mathcal{C}_\mu; \mathbf{Q})$$

is injective [8, Prop. 3] we have that

$$\frac{(-1)^g}{n^{4g}} \left(\prod_{p=1, p \neq i}^{n} (\delta_i - \delta_p) \right)^{2g} = \sum_{k=0}^{n-1} \rho_{-1}^{k} (-\delta_i)^k. \tag{44}$$

Solving the equations (44) we obtain

$$\rho_{-1}^k = \frac{(-1)^{g+n}}{n^{4g}} \sum_{i=1}^{n} S_i^k \left(\prod_{p=1, p \neq i}^{n} (\delta_i - \delta_p) \right)^{2g-1}, \tag{45}$$

where S_i^k equals the kth elementary symmetric polynomial in $\{\delta_p : p \neq i\}$.

We will show later, in Proposition 27, that the relations ρ_{-1}^k above are insufficient to prove any vanishing of the Pontryagin ring in ranks greater than three. For now consider the rank three case. We write

$$\alpha = \delta_1 - \delta_2, \qquad \beta = \delta_2 - \delta_3, \qquad \gamma = \delta_3 - \delta_1.$$

We know from Lemma 17 that the Pontryagin ring is generated by the elementary symmetric polynomials in α^2, β^2 and γ^2. The relations $\rho_{-1}^0, \rho_{-1}^1, \rho_{-1}^2$ read as

$$(\alpha\beta)^{2g-1} + (\beta\gamma)^{2g-1} + (\gamma\alpha)^{2g-1} = 0, \tag{46}$$

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$$(\delta_1+\delta_3)(\alpha\beta)^{2g-1}+(\delta_2+\delta_1)(\beta\gamma)^{2g-1}+(\delta_3+\delta_2)(\gamma\alpha)^{2g-1}=0, \qquad (47)$$

$$(\delta_1\delta_3)(\alpha\beta)^{2g-1}+(\delta_2\delta_1)(\beta\gamma)^{2g-1}+(\delta_3\delta_2)(\gamma\alpha)^{2g-1}=0. \qquad (48)$$

The equations (46), $a_1\times$ (46) $-$ (47), and (48) $+a_1\times$ (47) $-a_2\times$ (46) then show

$$(\delta_2)^k(\alpha\beta)^{2g-1}+(\delta_3)^k(\beta\gamma)^{2g-1}+(\delta_1)^k(\gamma\alpha)^{2g-1}=0, \qquad (49)$$

for $k=0,1,2$. Note that

$$(\delta_i)^{r+3}=a_1(\delta_i)^{r+2}-a_2(\delta_i)^{r+1}+a_3(\delta_i)^r,$$

and hence equation (49) holds for all nonnegative k. Further note that

$$\gamma^2=(a_1)^2-4a_2+2a_1\delta_2-3(\delta_2)^2, \qquad (50)$$

and so combining equation (49) with equation (50) and two similar equations for α^2 and β^2 we see that

$$\gamma^{2l}(\delta_2)^k(\alpha\beta)^{2g-1}+\alpha^{2l}(\delta_3)^k(\beta\gamma)^{2g-1}+\beta^{2l}(\delta_1)^k(\gamma\alpha)^{2g-1}=0,$$

for any nonnegative k,l. Let r,s,t be three nonnegative integers with an even sum. Note

$$2\alpha=(a_1-3\delta_2)-\gamma, \qquad 2\beta=(3\delta_2-a_1)-\gamma,$$

and hence $(\alpha^r\beta^s+\alpha^s\beta^r)\gamma^t$, when written in terms of a_1,δ_2 and γ is an even function in γ.

Now any element of the Pontryagin ring can be written as a sum of elements of the form

$$\begin{aligned} F(u,v,w) \;=\; & \alpha^u\beta^v\gamma^w+\alpha^v\beta^w\gamma^u+\alpha^w\beta^u\gamma^v+\alpha^u\beta^w\gamma^v \\ & +\alpha^v\beta^u\gamma^w+\alpha^w\beta^v\gamma^u, \end{aligned}$$

where $u+v+w$ is even. From the argument above we know that

$$F(2g-1+r,2g-1+s,t)=0, \qquad (51)$$

for $r,s,t\geqslant 0$ and $r+s+t$ even. If $u\geqslant 1$ then we have

$$F(u,v,w)=-F(u-1,v,w+1)-F(u-1,v+1,w), \qquad (52)$$

since $\alpha+\beta+\gamma=0$.

Suppose now that $u\geqslant v\geqslant w$. We claim $F(u,v,w)=0$ if $u+v+w\geqslant 6g-4$. Note that

$$\max\{u,v,w\}>\max\{u-1,v+1,w+1\},$$

unless $u-v$ equals zero or one. In either case we find that $u\geqslant v\geqslant 2g-1$ and hence $F(u,v,w)=0$ by (51). Hence by repeated applications of identity (52) we see that $F(u,v,w)=0$ when $u+v+w\geqslant 6g-4$ and so we have:

THEOREM 2. *The Pontryagin ring of the moduli space* $\mathcal{M}(3,d)$ *vanishes in degrees* $12g-8$ *and above.*

Remark 25. Theorem 2 falls short of Neeman's conjecture [10] which states that the Pontryagin ring of $\mathcal{M}(n,d)$ should vanish in degrees above $2gn^2-4g(n-1)+2$. When $n=3$ this gives $10g+2$.

Remark 26. In the rank two case the relations (45) show that

$$((a_1)^2-4a_2)^g=0$$

and that the Pontryagin ring of $\mathcal{M}(2,d)$ vanishes in degrees greater than or equal to $4g$, duplicating Kirwan's proof of the Newstead–Ramanan conjecture.

To conclude we show now that the relations ρ_{-1}^k are inadequate to show any vanishing of the Pontryagin ring when $n\geqslant 4$. From equation (45) we see that the ideal of the Pontryagin ring is contained in the ideal generated by the formal expressions

$$\left(\prod_{p=1,p\neq i}^{n}(\delta_i-\delta_p)\right)^{2g-1}. \tag{53}$$

Let I denote the ideal generated by the relations (53) and consider this as an ideal of $\mathbf{C}[\delta_1,\ldots,\delta_n]$. By Hilbert's Nullstellensatz the radical $\sqrt{I}$ of I consists of those elements of the Pontryagin ring which vanish on the intersection of the subspaces given by

$$\prod_{p\neq i}(\delta_i-\delta_p)=0,\quad i=1,\ldots,n. \tag{54}$$

We shall consider the even and odd cases for n separately.

(i) n is even – write $n=2m$. The intersection of the subspaces (54) consists of $(2m)!/(2^m m!)$ distinct m-dimensional subspaces of $\mathbf{C}^n$. One of these subspaces is given by the equations

$$\delta_{2k-1}=\delta_{2k},\quad k=1,\ldots,m. \tag{55}$$

We know from Lemma 17 that the total Pontryagin class $p(T)$ of $\mathcal{M}(n,d)$ equals

$$\prod_{1\leqslant k<l\leqslant n}(1+(\delta_k-\delta_l)^2)^{2\bar{g}},$$

and in the subspace (55) $p(T)$ then equals

$$\prod_{1\leqslant <l\leqslant m}(1+(\delta_{2k-1}-\delta_{2l-1})^2)^{8\bar{g}}.$$

In particular we see that none of the Pontryagin classes of $\mathcal{M}(n,d)$ vanish on the subspace (55).

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(ii) n is odd – write $n = 2m+1$. The intersection of the subspaces (54) consists of $(2k+1)!/(3\cdot 2^k(k-1)!)$ distinct k-dimensional subspaces of $\mathbf{C}^n$. One of these subspaces is given by the equations

$$\delta_1 = \delta_2 = \delta_3, \qquad \delta_{2k} = \delta_{2k+1}, \quad k = 2, \ldots, m. \tag{56}$$

In the subspace (56) the total Pontryagin class of $\mathcal{M}(n,d)$ equals

$$\left(\prod_{2\leqslant k\leqslant m} (1+(\delta_1-\delta_{2k})^2)^{12\bar{g}}\right)\left(\prod_{2\leqslant k<l\leqslant m} (1+(\delta_{2k}-\delta_{2l})^2)^{8\bar{g}}\right).$$

In particular we see that none of the Pontryagin classes of $\mathcal{M}(n,d)$ vanish on the subspace (56).

Thus we see that none of the Pontryagin classes $p_r(T)$ are nilpotent modulo the formal relations (53). Hence:

PROPOSITION 27. *For $n \geqslant 4$ the Pontryagin classes $p_r(T) \in H^{4r}(\mathcal{M}(n,d);\mathbf{Q})$ are not nilpotent modulo ρ^k_{-1} for $0 \leqslant k \leqslant n-1$. In particular these relations are inadequate to prove any non-trivial vanishing of the Pontryagin ring.*

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