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Locally finite varieties of groups arising from Cross varieties

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Let $\underline{\underline{V}}$ be a Cross variety and let n be the least integer such that $\underline{\underline{V}}^{(n)}$ is locally finite; then $n \le 2d + 3$ where d is an upper bound for the number of generators of certain critical groups in $\underline{\underline{V}}$.

1. Introduction

If $\underline{\underline{V}}$ is a Cross variety then, by the Oates-Powell Theorem, $\underline{\underline{V}} = \underline{\underline{V}}^{(n)}$ for some n and hence $\underline{\underline{V}}^{(n)}$ is locally finite, but, of course, $\underline{\underline{V}}^{(n)}$ can be locally finite even though $\underline{\underline{V}} \ne \underline{\underline{V}}^{(n)}$; for instance if $\underline{\underline{V}}$ is the variety generated by the dihedral group of order 2^{r+1} then $\underline{\underline{V}}^{(2)}$ is locally finite, although $\underline{\underline{V}} \ne \underline{\underline{V}}^{r-1}$ [4]. Can anything be said in general about the local finiteness of $\underline{\underline{V}}^{(n)}$? Certainly $\underline{\underline{V}}^{(1)}$ is not always locally finite [7], but I conjecture that $\underline{\underline{V}}^{(2)}$ is. Certain evidence to support this exists: R.M. Bryant [1] has shown that the two variable laws of $PSL(2, q)$ imply local finiteness, thus extending the results of [2].

Both proofs of the Oates-Powell Theorem [8], [3] give values of n for which $\underline{\underline{V}}^{(n)}$ is locally finite, though these tend to be somewhat large. In this paper I extend the results of [8], §3, to prove:-

THEOREM A. *Let* $\underline{\underline{V}}$ *be a Cross variety with a chain of subvarieties*

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$$\underline{\underline{E}} = \underline{\underline{V}}_0 \subset \underline{\underline{V}}_1 \subset \ldots \subset \underline{\underline{V}}_r = \underline{\underline{V}} ,$$

each maximal in the succeeding one, and let $\underline{\underline{V}}_i = \mathrm{var}\left(\underline{\underline{V}}_{i-1}, D_i\right)$ *where* D_i *is critical and can be generated by* d *(or fewer) elements; then* $\underline{\underline{V}}^{(2d+3)}$ *is locally finite.*

2. Notation

Notation and terminology follow that of Hanna Neumann [6].

3. Outline of proof

It is clearly sufficient to prove the following theorem:

THEOREM B. *Let* $\underline{\underline{V}}$ *be a Cross variety and* $\underline{\underline{U}}$ *a maximal subvariety of* $\underline{\underline{V}}$ *such that* $\underline{\underline{V}} = \mathrm{var}(\underline{\underline{U}}, D)$ *where* D *is a critical* d*-generator group. If* $\underline{\underline{U}}^{(2d+3)}$ *is locally finite so is* $\underline{\underline{V}}^{(2d+3)}$.

Theorem A follows by induction on r from Theorem B, since it is trivially true for $\underline{\underline{E}}$.

The proof of Theorem B divides into two parts according as σD is abelian or non-abelian.

4. σD abelian

DEFINITION 4.1. Let $\{W_1 = 1, \ldots, W_k = 1\}$ be a basis for the $(2d+3)$-variable laws of $\underline{\underline{U}}$ (by a result of B.H. Neumann [5] such a finite basis exists) and let $W(G)$ be the corresponding word subgroup, so that

$$G/W(G) \in \underline{\underline{U}}^{(2d+3)}$$

and $G/N \notin \underline{\underline{U}}^{(2d+3)}$ if $N < W(G)$. Similarly let $\{w_1 = 1, \ldots, w_l = 1\}$ be a basis for the d-variable laws of $\underline{\underline{U}}$, and $w(G)$ the corresponding word subgroup. Note that, since $\underline{\underline{U}}^{(d)} \supseteq \underline{\underline{U}}^{(2d+3)}$, $w(G) \leq W(G)$.

LEMMA 4.2. *If* $G \in \underline{\underline{V}}$, $w(G) = W(G)$.

Proof. Suppose there is $G \in \underline{\underline{V}}$ such that $w(G) < W(G)$, then,

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$$(*)\qquad \begin{cases} G/w(G) \in \underline{\underline{V}} \cap \underline{\underline{U}}^{(d)} & \text{and} \\ G/w(G) \notin \underline{\underline{V}} \cap \underline{\underline{U}}^{(2d+3)} . \end{cases}$$

However

$$\underline{\underline{V}} \supseteq \underline{\underline{V}} \cap \underline{\underline{U}}^{(d)} \supseteq \underline{\underline{U}} ,$$

$$\underline{\underline{V}} \supseteq \underline{\underline{V}} \cap \underline{\underline{U}}^{(2d+3)} \supseteq \underline{\underline{U}}$$

and $\underline{\underline{U}}$ is maximal in $\underline{\underline{V}}$.

Moreover $D \notin \underline{\underline{U}}^{(d)}$ (since it is a d-generator group not in $\underline{\underline{U}}$). It follows that

$$\underline{\underline{V}} \cap \underline{\underline{U}}^{(d)} = \underline{\underline{U}} = \underline{\underline{V}} \cap \underline{\underline{U}}^{(2d+3)} ,$$

contradicting (*). Hence $w(G) \nleq W(G)$.

LEMMA 4.3. *If* $G \in \underline{\underline{V}}$ *then* $W(G)$ *is elementary abelian of exponent* p *, where* σD *is a* p*-group.*

Proof. $D/\sigma D \in \underline{\underline{U}}$ (since D is critical and $\underline{\underline{U}}$ is maximal in $\underline{\underline{V}}$) so $W(G) \leq \sigma D$. But $D \notin \underline{\underline{U}}^{(2d+3)}$ so $W(D) \neq 1$. It follows that $W(D) = \sigma D$ is elementary abelian, and D satisfies the laws:

$$\left[W_i, W_j\right] = 1 , \quad W_i^p = 1 \quad (i, j = 1, \ldots, k)$$

where the sets of variables in W_i and W_j are disjoint.

Since $\underline{\underline{U}}$ also satisfies these laws, $\underline{\underline{V}}$ must satisfy them and hence $W(G)$ is elementary abelian for every $G \in \underline{\underline{V}}$.

COROLLARY 4.4. $\underline{\underline{V}}$ *satisfies the* $\left((2d)\text{-}variable\right)$ *laws* $\left[w_i, w_j\right] = 1$, $w_i^p = 1$ $(i, j = 1, \ldots, l)$.

Proof of Theorem B for abelian σD . Let G be a finitely generated group in $\underline{\underline{V}}$. $G/W(G) \in \underline{\underline{U}}^{(2d+3)}$ and so is finite. It follows that $W(G)$ is finitely generated and so is generated by a finite number of words of the form $W_i\left(g_1, \ldots, g_{2d+3}\right)$.

Let $H = gp\left(g_1, \ldots, g_{2d+3}\right)$; then $H \in \underline{\underline{V}}$ and so $W(H) = w(H)$. Thus

$$W_i(g_1, \ldots, g_{2d+3}) \in w(H) \leq w(G) .$$

Hence $W(G) = w(G)$ and $w(G)$ is also finitely generated. But the laws $[w_i, w_j] = 1$ and $w_i^p = 1$ hold in $\underline{\underline{V}}^{(2d+3)}$ (being $(2d)$-variable laws of $\underline{\underline{V}}$). It follows that $w(G)$ is a finitely generated elementary abelian p-group and so is finite. Hence G is finite as required.

5. σ(D) non-abelian

Consideration of Section 3 [8] shows that, for the purposes of proving local finiteness a (the number of variables in a basis for $\underline{\underline{U}}$) can be replaced by $2d + 3$ (the number of variables needed to ensure local finiteness) and b (the size of a generating set for D which includes one for σD) can be replaced by at worst $d + 1$, since it is sufficient to work with a generating set for D which contains one element from σD. Thus, by the results of §3.4 of [8] we have that $\underline{\underline{V}}^{(n)}$ is locally finite, for

$$\max(2d+3, 2(d+1)+1) = 2d + 3 .$$

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