AN INFINITE INTEGRAL INVOLVING A PRODUCT OF TWO MODIFIED BESSEL FUNCTIONS OF THE SECOND KIND

by T. M. MACROBERT

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The formula to be established is

$$\int_0^\infty x^{l-1} K_m(x) K_n(b/x)\, dx$$
$$= \sum_{n,\,-n} 2^{l-2n-3} \Gamma(-n) \Gamma\left(\frac{l+m-n}{2}\right) \Gamma\left(\frac{l-m-n}{2}\right) b^n F\left(;\, 1+n,\, 1-\frac{l+m-n}{2},\, 1-\frac{l-m-n}{2};\, \frac{b^2}{16}\right)$$
$$+ \sum_{m,\,-m} 2^{-l-2m-3} \Gamma(-m) \Gamma\left(\frac{-l-m+n}{2}\right) \Gamma\left(\frac{-l-m-n}{2}\right) b^{l+m}$$
$$\times F\left(;\, 1+m,\, 1+\frac{l+m-n}{2},\, 1+\frac{l+m+n}{2};\, \frac{b^2}{16}\right), \quad \ldots\ldots(1)$$

where l, m, n are any numbers real or complex and $R(b)>0$. A similar result, involving Bessel Functions of the First Kind, was obtained by Hanumanta Rao [*Mess. of Maths.*, XLVII. (1918), pp. 134–137].

The proof is on the same lines as that of Rao. The function $K_n(x)$ satisfies the equation

$$x^2 y'' + xy' - (n^2 + x^2) y = 0. \quad \ldots\ldots(2)$$

It should also be noted that, if n is not integral

$$K_n(x) = \frac{\pi}{2 \sin n\pi} \{I_{-n}(x) - I_n(x)\}, \quad \ldots\ldots(3)$$

and that, if $R(l \pm m) > 0$,

$$\int_0^\infty x^{l-1} K_m(x)\, dx = 2^{l-2} \Gamma\left(\frac{l+m}{2}\right) \Gamma\left(\frac{l-m}{2}\right). \quad \ldots\ldots(4)$$

Denote the integral by I; then

$$\frac{dI}{db} = \int_0^\infty x^{l-2} K_m(x) K_n'(b/x)\, dx = -\frac{1}{b}\Big[x^l K_m(x) K_n(b/x) \Big]_0^\infty$$
$$+ \frac{l}{b} \int_0^\infty x^{l-1} K_m(x) K_n(b/x)\, dx + \frac{1}{b} \int_0^\infty x^l K_m'(x) K_n(b/x)\, dx.$$

Therefore
$$b\frac{dI}{db} - lI = \int_0^\infty x^l K_m'(x) K_n(b/x)\, dx,$$

and
$$b\frac{d^2 I}{db^2} + (1-l)\frac{dI}{db} = \int_0^\infty x^{l-1} K_m'(x) K_n'(b/x)\, dx$$

$$= -\frac{1}{b}\Big[x^{l+1} K_m'(x) K_n(b/x) \Big]_0^\infty + \frac{l+1}{b} \int_0^\infty x^l K_m'(x) K_n(b/x)\, dx + \frac{1}{b} \int_0^\infty x^{l+1} K_m''(x) K_n(b/x)\, dx$$

$$= \frac{l+1}{b}\left\{ b\frac{dI}{db} - lI \right\} + \frac{1}{b} \int_0^\infty x^{l-1} \{(x^2+m^2) K_m(x) - x K_m'(x)\} K_n(b/x)\, dx,$$

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by (2). Hence

$$b^2\frac{d^2I}{db^2}+(1-2l)b\frac{dI}{db}+(l^2-m^2)I=J, \quad \text{.................................(A)}$$

where $J=\int_0^\infty x^{l+1}K_m(x)K_n(b/x)\,dx.$

Again
$$\frac{dJ}{db}=\int_0^\infty x^l K_m(x)K_n'(b/x)\,dx$$

and
$$\frac{d^2J}{db^2}=\int_0^\infty x^{l-1}K_m(x)K_n''(b/x)\,dx$$
$$=\frac{1}{b^2}\int_0^\infty x^{l+1}K_m(x)\left\{\left(\frac{b^2}{x^2}+n^2\right)K_n\left(\frac{b}{x}\right)-\frac{b}{x}K_n'\left(\frac{b}{x}\right)\right\}dx$$
$$=I+\frac{n^2}{b^2}J-\frac{1}{b}\frac{dJ}{db}.$$

Now, from (A),
$$\frac{dJ}{db}=b^2\frac{d^3I}{db^3}+(3-2l)b\frac{d^2I}{db^2}+(1-2l+l^2-m^2)\frac{dI}{db}$$

and
$$\frac{d^2J}{db^2}=b^2\frac{d^4I}{db^4}+(5-2l)b\frac{d^3I}{db^3}+(4-4l+l^2-m^2)\frac{d^2I}{db^2}.$$

Hence, after some simplification,

$$b^4\frac{d^4I}{db^4}+(6-2l)b^3\frac{d^3I}{db^3}+(7-6l+l^2-m^2-n^2)b^2\frac{d^2I}{db^2}$$
$$+(1-2l+2ln^2+l^2-m^2-n^2)b\frac{dI}{db}+(m^2-l^2)n^2I=b^2I. \quad \text{..........................(B)}$$

Next let $I=\sum_{\nu=0}^{\infty}c_\nu b^{\rho+\nu}$; then, on substituting in (B), the coefficient of c_0b^ρ is

$$\rho(\rho-1)(\rho-2)(\rho-3)+(6-2l)\rho(\rho-1)(\rho-2)+(7-6l+l^2-m^2-n^2)\rho(\rho-1)$$
$$+(1-2l+2ln^2+l^2-m^2-n^2)\rho+(m^2-l^2)n^2$$
$$=\rho^4-2l\rho^3+(l^2-m^2-n^2)\rho^2+2ln^2\rho+(m^2-l^2)n^2$$
$$=(\rho-n)(\rho+n)(\rho-l-m)(\rho-l+m).$$

Thus the indicial equation gives

$$\rho=n,\ -n,\ l+m,\ l-m.$$

If $c_1=0$ then all the c's with odd suffixes vanish, while, for $\nu=0, 1, 2, \ldots$,

$$c_{2\nu+2}(\rho+2\nu+2-n)(\rho+2\nu+2+n)(\rho+2\nu+2-l-m)(\rho+2\nu+2-l+m)=c_{2\nu}.$$

Hence
$$I=Ab^nP+Bb^{-n}Q+Cb^{l+m}R+Db^{l-m}S,$$

where P, Q, R, S are the generalised hypergeometric functions on the right of (1), taken in order.

In determining the values of the coefficients A, B, C, D, it should be noticed that the value of the integral is unaltered if the sign of m or of n is altered. Let it be assumed that $R(n)$ is positive, multiply by b^n and let $b\to 0$; then, if $R(l\pm m+n)>0$, from (3),

$$\frac{\pi}{2\sin n\pi}\frac{2^n}{\Gamma(1-n)}\int_0^\infty x^{l+n-1}K_m(x)\,dx=B,$$

and therefore, from (4),

$$B=2^{l+2n-3}\Gamma(n)\Gamma\left(\frac{l+m+n}{2}\right)\Gamma\left(\frac{l-m+n}{2}\right).$$

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Again, in I replace x by b/x, and it becomes

$$b^l \int_0^\infty x^{-l-1} K_m(b/x) K_n(x)\, dx.$$

From this, assuming that $R(m)$ is positive, on multiplying by b^{m-l} and making $b \to 0$, it is found that, if $R(-l+m \pm n) > 0$,

$$D = 2^{-l+2m-3} \Gamma(m) \Gamma\left(\frac{-l+m+n}{2}\right) \Gamma\left(\frac{-l+m-n}{2}\right).$$

Since I, R and S are symmetrical in n and $-n$, and since $b^n P$ becomes $b^{-n} Q$ when n and $-n$ are interchanged, it follows that, if $R(l \pm m + n) > 0$, A is equal to B with n and $-n$ interchanged, and that C and D are symmetrical in n and $-n$.

Similarly, when $R(-l+m \pm n) > 0$, C is D with m and $-m$ interchanged.

Now, from the continuity of the functions with respect to l, if $R(m)$ and $R(n)$ are taken to be positive, all these values of A, B, C, and D are valid when $R(l) = R(m-n)$. But all the functions in equation (1) are holomorphic in l for all real or complex values of l. Hence, by analytical continuation, formula (1) holds for all values of l.

Note. On putting $l=1$, $m=n$, Hardy's formula

$$\int_0^\infty K_n(x) K_n(b/x)\, dx = \pi K_{2n}(2\surd b),$$

where $R(b) > 0$, is obtained.

UNIVERSITY OF GLASGOW