Canad. Math. Bull. Vol. **62** (4), 2019 pp. 756–766
http://dx.doi.org/10.4153/S0008439518000644

Simplicial (Co)-homology of $\ell^1(\mathbb{Z}_+)$

Yasser Farhat and Frédéric Gourdeau

Abstract. We consider the unital Banach algebra $\ell^1(\mathbb{Z}_+)$ and prove directly, without using cyclic cohomology, that the simplicial cohomology groups $\mathcal{H}^n(\ell^1(\mathbb{Z}_+), \ell^1(\mathbb{Z}_+)^*)$ vanish for all $n \geqslant 2$. This proceeds via the introduction of an explicit bounded linear operator which produces a contracting homotopy for $n \geqslant 2$. This construction is generalised to unital Banach algebras $\ell^1(\mathcal{S})$, where $\mathcal{S} = \mathcal{G} \cap \mathbb{R}_+$ and $\mathcal{G}$ is a subgroup of $\mathbb{R}_+$.

1 Introduction

In a series of papers, [2, 5, 7, 8], the authors used the Connes–Tzygan exact sequence [9] to help determine the simplicial cohomology groups of various classes of Banach algebras. The proofs generally proceed by an identification of the cyclic cohomology groups that either vanish, are one-dimensional, or are clearly identified, which leads to a determination of the simplicial cohomology groups, using the Connes–Tzygan exact sequence.

Looking for a method enabling one to explain the simplicial cohomology groups directly always seemed desirable. For instance, a direct method may shed more light on the existence of a bi-projective resolution or lead to easier generalizations (with other modules, for instance). It is the purpose of this paper to provide such a direct approach for the Banach algebra $\ell^1(\mathbb{Z}_+)$. In Section 4, this is extended to some other discrete semigroup algebras. This work forms part of the first author's Ph.D. thesis [4], although the presentation and the structures of the proofs have been considerably reworked.

In this paper, $\ell^1(\mathbb{Z}_+)$ is the unital Banach algebra, with convolution product, given by $\{\sum_{n\in\mathbb{Z}_+} c_n\delta_n : c_n \in \mathbb{C}, \sum_{n\in\mathbb{Z}_+} |c_n| < \infty\}$, where δ_n is the Dirac function on $\mathbb{Z}_+$, *i.e.*, $\delta_n(m) = 1$ if $n = m$, and $\delta_n(m) = 0$ if $n \neq m$. This Banach algebra is not weakly amenable; a proof can be obtained by suitably adapting the proof of [1, Theorem 2.3] or by noting that $\ell^1(\mathbb{Z}_+)$ has a non zero bounded point derivation given by $\sum_{n\in\mathbb{Z}_+} c_n\delta_n \to c_1$. However, in [3, Theorem 3.2] and [6], it was shown that $\mathcal{H}^n(\ell^1(\mathbb{Z}_+), \ell^1(\mathbb{Z}_+)^*)$ is zero for $n = 2$ and $n = 3$, respectively. In [5, Theorem 4.9], the authors were finally able to prove that $\mathcal{H}^n(\ell^1(\mathbb{Z}_+), \ell^1(\mathbb{Z}_+)^*)$ is zero for all $n \geq 2$. In this paper, the authors first proved that cyclic cohomology groups of odd degrees vanish, and that those of even degrees are one-dimensional. Using [5, Corollary 4.8], [6, Proposition 2.1], and the Connes–Tzygan exact sequence, they then deduced the vanishing of the

Received by the editors September 5, 2018; revised December 6, 2018.
Published online on Cambridge Core May 3, 2019.
This work was partially supported by the National Sciences and Engineering Research Council of Canada.
AMS subject classification: **46H20**, 43A20, 16E40.
Keywords: simplicial homology, Banach algebra, simplicial cohomology.

simplicial cohomology groups for $n \geq 2$. In this paper we will give an explicit contracting homotopy for the simplicial cochain complex in degrees 2 and above. This avoids use of the Connes–Tzygan exact sequence and cyclic cohomology.

2 Background and Definitions

We now briefly establish our notation and recall some definitions. For a Banach algebra $\mathcal{A}$, we regard $\mathcal{A}^*$, the topological dual space of $\mathcal{A}$, as a Banach $\mathcal{A}$-bimodule in the usual way.

For $n \geq 1$, we denote the Banach space of bounded n-linear operators from

$$\mathcal{A}^n := \underbrace{\mathcal{A} \times \cdots \times \mathcal{A}}_{n}$$

to $\mathcal{A}^*$ by $\mathcal{C}^n(\mathcal{A}, \mathcal{A}^*)$. We define the boundary operator $\Delta^n \colon \mathcal{C}^n(\mathcal{A}, \mathcal{A}^*) \to \mathcal{C}^{n+1}(\mathcal{A}, \mathcal{A}^*)$ as the bounded linear operator given by

$$\begin{aligned}(\Delta^n T)(a_1, \dots, a_{n+1})(a_{n+2}) = T(a_2, \dots, a_{n+1})(a_{n+2}a_1) \\ + \sum_{j=1}^{n} (-1)^j T(a_1, \dots, a_j a_{j+1}, \dots, a_{n+1})(a_{n+2}) \\ + (-1)^{n+1} T(a_1, \dots, a_n)(a_{n+1}a_{n+2}),\end{aligned}$$

where $T \in \mathcal{C}^n(\mathcal{A}, \mathcal{A}^*)$.

By convention, for $n = 0$, we have $\mathcal{C}^n(\mathcal{A}, \mathcal{A}^*) := \mathcal{A}^*$, and $\Delta^0 : \mathcal{C}^0(\mathcal{A}, \mathcal{A}^*) \to \mathcal{C}^1(\mathcal{A}, \mathcal{A}^*)$ is given by $\Delta^0(T)(a_1)(a_2) = T(a_2 a_1) - T(a_1 a_2)$.

For $T \in \mathcal{C}^n(\mathcal{A}, \mathcal{A}^*)$, we say that T is an *n-cocycle* if $\Delta^n T = 0$ and we say that T is an *n-coboundary* if $T = \Delta^{n-1} S$, for some $S \in \mathcal{C}^{n-1}(\mathcal{A}, \mathcal{A}^*)$. Let $\mathcal{Z}^n(\mathcal{A}, \mathcal{A}^*)$ be the subspace of n-cocycles, and $\mathcal{B}^n(\mathcal{A}, \mathcal{A}^*)$ the subspace of n-coboundaries. The n-th *simplicial cohomology group of* $\mathcal{A}$ is the space $\mathcal{H}^n(\mathcal{A}, \mathcal{A}^*) := \frac{\mathcal{Z}^n(\mathcal{A}, \mathcal{A}^*)}{\mathcal{B}^n(\mathcal{A}, \mathcal{A}^*)}$.

Elements of $\mathcal{C}^n(\mathcal{A}, \mathcal{A}^*)$ may be regarded as bounded linear functionals on the space $\mathcal{C}_n(\mathcal{A}, \mathcal{A}) := \mathcal{A}^{\widehat{\otimes} n+1}$, the $(n+1)$-fold completed projective tensor product of $\mathcal{A}$. The bounded linear operator $\Delta^n : \mathcal{C}^n(\mathcal{A}, \mathcal{A}^*) \to \mathcal{C}^{n+1}(\mathcal{A}, \mathcal{A}^*)$ is then the adjoint of the bounded linear operator $d^n : \mathcal{C}_{n+1}(\mathcal{A}, \mathcal{A}) \to \mathcal{C}_n(\mathcal{A}, \mathcal{A})$ defined on elementary tensors $X = a_1 \otimes a_2 \otimes \cdots \otimes a_{n+2} \in \mathcal{C}_{n+1}(\mathcal{A}, \mathcal{A})$ by $\mathbf{d}^n(X) = \sum_{i=0}^{n+1} \mathbf{d}_i^n(X)$, where

$$\begin{aligned} d_0^n(X) &:= a_2 \otimes \cdots \otimes a_{n+1} \otimes a_{n+2}a_1, \\ d_i^n(X) &:= (-1)^i a_1 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_{n+2}, \quad i = 1, \dots, n+1. \end{aligned}$$

The n-th *simplicial homology group of* $\mathcal{A}$ is the space $\mathcal{H}_n(\mathcal{A}, \mathcal{A}) := \frac{\mathcal{Z}_n(\mathcal{A}, \mathcal{A})}{\mathcal{B}_n(\mathcal{A}, \mathcal{A})}$, where $\mathcal{Z}_n(\mathcal{A}, \mathcal{A}) = \operatorname{Ker}(d^{n-1})$ and $\mathcal{B}_n(\mathcal{A}, \mathcal{A}) = \operatorname{Im}(d^n)$.

3 An Explicit Formula for the Contracting Homotopy in $\ell^1(\mathbb{Z}_+)$

In this section $\mathcal{A}$ denotes the Banach algebra $\ell^1(\mathbb{Z}_+)$. We let X denote

$$\delta_{p_1} \otimes \delta_{p_2} \otimes \cdots \otimes \delta_{p_{n+1}} \in \mathcal{C}_n(\mathcal{A}, \mathcal{A}),$$

and call these elementary tensors. As $\mathcal{C}_n(\mathcal{A},\mathcal{A})$ is isometrically isomorphic to $\ell^1((\mathbb{Z}_+ \times \cdots \times \mathbb{Z}_+))$ (with $n+1$ copies of the semigroup $\mathbb{Z}_+$), these elementary tensors form a copy of the standard basis of this ℓ^1, and we can define a bounded linear map by appropriately specifying its values on these elementary tensors; we will do this without further mention. We denote by N_X the value $p_1 + \cdots + p_{n+1}$ that will be called the *degree of* X. It is crucial to note that degree is not changed by the boundary operator, that is, $N_X = N_{d_i^n(X)}$, $i = 0, \dots, n+1$.

It will be useful to distinguish the elementary tensor $\delta_0 \otimes \delta_0 \otimes \cdots \otimes \delta_0 \in \mathcal{C}_n(\mathcal{A},\mathcal{A})$, which will be denoted by X_0^n. Thus X_0^n is the $n+1$-fold tensor product of the identity of $\mathcal{A}$.

For $n \geq 0$ and $1 \leq k \leq n+1$, we define a bounded linear operator $s_k^n : \mathcal{C}_n(\mathcal{A},\mathcal{A}) \to \mathcal{C}_{n+1}(\mathcal{A},\mathcal{A})$ by

$$s_k^n(X) = \begin{cases} \frac{(-1)^k}{N_X} \sum_{j=0}^{p_k-1} \delta_{p_1} \otimes \cdots \otimes \delta_{p_{k-1}} \otimes \delta_{p_k - j} \otimes \delta_j \otimes \delta_{p_{k+1}} \otimes \cdots \otimes \delta_{p_{n+1}} & \text{if } p_k \neq 0, \\ 0 & \text{if } p_k = 0. \end{cases}$$

It is important to note that the degree of each of the terms in the sum is N_X.

Finally, we let $s^n : \mathcal{C}_n(\mathcal{A},\mathcal{A}) \to \mathcal{C}_{n+1}(\mathcal{A},\mathcal{A})$ be the bounded linear operator defined by $s^n(X_0^n) = X_0^{n+1}$, and $s^n(X) = \sum_{k=1}^{n+1} s_k^n(X)$ if $N_X \neq 0$.

The definition of s^n is based on [5, §5]. We now wish to analyse precisely how far the map $(s^{n-1}d^{n-1} + d^n s^n)$ is from the identity. It is easily seen that $(s^{n-1}d^{n-1} + d^n s^n)(X_0^n) = X_0^n$. (We note that the value of an empty sum is the zero of the appropriate space.) First, we write $s^{n-1}d^{n-1} + d^n s^n$ as

$$\begin{aligned} &\sum_{k=1}^{n-1}(s_k^{n-1}d_0^{n-1} + d_0^n s_{k+1}^n) + \sum_{1\leq j<k\leq n}(s_k^{n-1}d_j^{n-1} + d_j^n s_{k+1}^n) \\ &\quad + \sum_{1\leq k<j\leq n}(s_k^{n-1}d_j^{n-1} + d_{j+1}^n s_k^n) + (s_n^{n-1}d_0^{n-1} + d_{n+1}^n s_{n+1}^n + d_0^n s_1^n + d_0^n s_{n+1}^n) \\ &\quad + \sum_{i=1}^{n}(s_i^{n-1}d_i^{n-1} + d_i^n s_i^n + d_i^n s_{i+1}^n + d_{i+1}^n s_i^n). \end{aligned}$$

The next lemma is very close to [8, Lemma 3.4]. It shows that the first three sums above all vanish, as terms cancel in pairs.

Lemma 3.1 *With the definitions as above, we have*

$$\begin{aligned} s_k^{n-1}d_0^{n-1} + d_0^n s_{k+1}^n &= 0, \quad 1 \leq k \leq n-1, \\ s_k^{n-1}d_j^{n-1} + d_j^n s_{k+1}^n &= 0, \quad 1 \leq j < k \leq n, \\ s_k^{n-1}d_j^{n-1} + d_{j+1}^n s_k^n &= 0, \quad 1 \leq k < j \leq n. \end{aligned}$$

Proof The results are immediate if $X = X_0^n$. For $X \neq X_0^n$, the proof closely resembles that of [8, Lemma 3.4]. However, the setting is slightly different (because of the need to take into account the normalisation factor N_X) and so the result cannot be invoked directly. We will prove only the first assertion as all proofs are direct. Let $1 \leq k \leq n-1$.

If $p_{k+1} = 0$, then $d_0^n s_{k+1}^n(X) = 0$. Moreover, we have $s_k^{n-1} d_0^{n-1}(X) = s_k^{n-1}(\delta_{p_2} \otimes \cdots \otimes \delta_{p_n} \otimes \delta_{p_{n+1}+p_1})$. By the definition of s_k^{n-1}, we get $s_k^{n-1} d_0^{n-1}(X) = 0$.

If $p_{k+1} \neq 0$, we get

$$s_k^{n-1} d_0^{n-1}(X) = \frac{(-1)^k}{N_{d_0^{n-1}(X)}} \times \sum_{j=0}^{p_{k+1}-1} \delta_{p_2} \otimes \cdots \otimes \delta_{p_k} \otimes \delta_{p_{k+1}-j} \otimes \delta_j \otimes \delta_{p_{k+2}} \otimes \cdots \otimes \delta_{p_{n+1}+p_1}.$$

Recalling that $N_X = N_{d_0^{n-1}(X)}$, we easily see that it is equal to $-d_0^n s_{k+1}^n(X)$. ■

The next two lemmas will enable us to complete our task.

Lemma 3.2 *For $X \neq X_0^n$, $(s_n^{n-1} d_0^{n-1} + d_{n+1}^n s_{n+1}^n + d_0^n s_1^n + d_0^n s_{n+1}^n)(X)$ is given by*

$$\frac{p_{n+1}}{N_X} X - \frac{1}{N_X} \sum_{j=0}^{p_1-1} \delta_j \otimes \delta_{p_2} \otimes \cdots \otimes \delta_{p_n} \otimes \delta_{p_{(n+1)}+p_1-j} + \frac{(-1)^n}{N_X} \sum_{j=0}^{p_1-1} \delta_{p_2} \otimes \cdots \otimes \delta_{p_n} \otimes \delta_{p_{(n+1)}+p_1-j} \otimes \delta_j.$$

Proof We have

$$\begin{aligned}
d_{n+1}^n s_{n+1}^n(X) &= \frac{p_{n+1}}{N_X} X, \\
d_0^n s_1^n(X) &= \frac{-1}{N_X} d_0^n \Big(\sum_{j=0}^{p_1-1} \delta_{p_1-j} \otimes \delta_j \otimes \delta_{p_2} \otimes \cdots \otimes \delta_{p_{n+1}} \Big) \\
&= \frac{-1}{N_X} \sum_{j=0}^{p_1-1} \delta_j \otimes \delta_{p_2} \otimes \cdots \otimes \delta_{p_n} \otimes \delta_{p_{n+1}+p_1-j}, \\
d_0^n s_{n+1}^n(X) &= \frac{(-1)^{n+1}}{N_X} d_0^n \Big(\sum_{j=0}^{p_{n+1}-1} \delta_{p_1} \otimes \cdots \otimes \delta_{p_n} \otimes \delta_{p_{n+1}-j} \otimes \delta_j \Big) \\
&= \frac{(-1)^{n+1}}{N_X} \sum_{j=0}^{p_{n+1}-1} \delta_{p_2} \otimes \cdots \otimes \delta_{p_n} \otimes \delta_{p_{n+1}-j} \otimes \delta_{j+p_1}, \\
s_n^{n-1} d_0^{n-1}(X) &= s_n^{n-1}(\delta_{p_2} \otimes \cdots \otimes \delta_{p_n} \otimes \delta_{p_{n+1}+p_1}) \\
&= \frac{(-1)^n}{N_X} \sum_{j=0}^{p_{n+1}+p_1-1} \delta_{p_2} \otimes \cdots \otimes \delta_{p_n} \otimes \delta_{p_{n+1}+p_1-j} \otimes \delta_j \\
&= \frac{(-1)^n}{N_X} \sum_{j=0}^{p_1-1} \delta_{p_2} \otimes \cdots \otimes \delta_{p_n} \otimes \delta_{p_{n+1}+p_1-j} \otimes \delta_j \\
&\quad + \frac{(-1)^n}{N_X} \sum_{j=0}^{p_{n+1}-1} \delta_{p_2} \otimes \cdots \otimes \delta_{p_n} \otimes \delta_{p_{n+1}-j} \otimes \delta_{j+p_1}.
\end{aligned}$$

As the last sum cancels with $d_0^n s_{n+1}^n(X)$, the result follows. ■

https://doi.org/10.4153/S0008439518000644 Published online by Cambridge University Press

Lemma 3.3 *Let $1 \le i \le n$ and $X \ne X_0^n$. Then*

$$(s_i^{n-1} d_i^{n-1} + d_i^n s_i^n + d_i^n s_{i+1}^n + d_{i+1}^n s_i^n)(X) = \frac{p_i}{N_X} X.$$

Proof For $1 \le i \le n$, we have the following. First

$$d_i^n s_i^n(X) = d_i^n\Big(\frac{(-1)^i}{N_X} \sum_{j=0}^{p_i-1} \delta_{p_1} \otimes \cdots \otimes \delta_{p_i-j} \otimes \delta_j \otimes \cdots \otimes \delta_{p_{n+1}}\Big) = \frac{p_i}{N_X} X.$$

The next two terms cancel out the fourth, as we have

$$\begin{aligned} d_i^n s_{i+1}^n &= d_i^n\Big(\frac{(-1)^{i+1}}{N_X} \sum_{j=0}^{p_{i+1}-1} \delta_{p_1} \otimes \cdots \otimes \delta_{p_{i+1}-j} \otimes \delta_j \otimes \cdots \otimes \delta_{p_{n+1}}\Big) \\ &= \frac{-1}{N_X} \sum_{j=0}^{p_{i+1}-1} \delta_{p_1} \otimes \cdots \otimes \delta_{p_i+p_{i+1}-j} \otimes \delta_j \otimes \cdots \otimes \delta_{p_{n+1}}, \\ d_{i+1}^n s_i^n &= d_{i+1}^n\Big(\frac{(-1)^i}{N_X} \sum_{j=0}^{p_i-1} \delta_{p_1} \otimes \cdots \otimes \delta_{p_i-j} \otimes \delta_j \otimes \cdots \otimes \delta_{p_{n+1}}\Big) \\ &= \frac{-1}{N_X} \sum_{j=0}^{p_i-1} \delta_{p_1} \otimes \cdots \otimes \delta_{p_i-j} \otimes \delta_{j+p_{i+1}} \otimes \cdots \otimes \delta_{p_{n+1}}, \end{aligned}$$

while

$$\begin{aligned} s_i^{n-1} d_i^{n-1} &= s_i^{n-1}\big((-1)^i \delta_{p_1} \otimes \cdots \otimes \delta_{p_i+p_{i+1}} \otimes \cdots \otimes \delta_{p_{n+1}}\big) \\ &= \frac{1}{N_X} \sum_{j=0}^{p_i+p_{i+1}-1} \delta_{p_1} \otimes \cdots \otimes \delta_{p_i+p_{i+1}-j} \otimes \delta_j \otimes \cdots \otimes \delta_{p_{n+1}} \\ &= \frac{1}{N_X} \sum_{j=0}^{p_{i+1}-1} \delta_{p_1} \otimes \cdots \otimes \delta_{p_i+p_{i+1}-j} \otimes \delta_j \otimes \cdots \otimes \delta_{p_{n+1}} \\ &\quad + \frac{1}{N_X} \sum_{j=0}^{p_i-1} \delta_{p_1} \otimes \cdots \otimes \delta_{p_i-j} \otimes \delta_{j+p_{i+1}} \otimes \cdots \otimes \delta_{p_{n+1}}, \end{aligned}$$

The proof is complete. ∎

The three lemmas enable us to deduce directly the following proposition.

Proposition 3.4 *Let $n \ge 1$ and $X = \delta_{p_1} \otimes \delta_{p_2} \otimes \cdots \otimes \delta_{p_{n+1}} \in \mathcal{C}_n(\mathcal{A}, \mathcal{A})$.*

(i) If $X = X_0^n$, then $(s^{n-1} d^{n-1} + d^n s^n)(X) = X$.

(ii) If $X \ne X_0^n$, then

$$\begin{aligned} (s^{n-1} d^{n-1} + d^n s^n)(X) = X &- \frac{1}{N_X} \sum_{j=0}^{p_1-1} \delta_j \otimes \delta_{p_2} \otimes \cdots \otimes \delta_{p_{n+1}+p_1-j} \\ &+ \frac{(-1)^n}{N_X} \sum_{j=0}^{p_1-1} \delta_{p_2} \otimes \cdots \otimes \delta_{p_n} \otimes \delta_{p_{n+1}+p_1-j} \otimes \delta_j. \end{aligned}$$

Proof As stated, the result follows immediately from Lemmas 3.1, 3.2, and 3.3. ∎

https://doi.org/10.4153/S0008439518000644 Published online by Cambridge University Press

If we can modify the map s^n in such a way that the terms

$$\frac{(-1)^n}{N_X}\sum_{j=0}^{p_1-1}\delta_{p_2}\otimes\cdots\otimes\delta_{p_n}\otimes\delta_{p_{n+1}+p_1-j}\otimes\delta_j$$

and

$$\frac{-1}{N_X}\sum_{j=0}^{p_1-1}\delta_j\otimes\delta_{p_2}\otimes\cdots\otimes\delta_{p_{n+1}+p_1-j}$$

are cancelled (for $X\neq X_0^n$), then we will have a contracting homotopy for simplicial homology.

We can, in fact, do precisely this for $n\geq 2$. We note that the definition uses the fact that the algebra is unital.

Definition 3.5 Let $n\geq 1$. We define a bounded linear operator

$$r^n:\mathcal{C}_n(\mathcal{A},\mathcal{A})\longrightarrow\mathcal{C}_{n+1}(\mathcal{A},\mathcal{A})$$

by $r^n(X_0^n):=0$ and, for $X\neq X_0^n$, by

$$r^n(X)=\frac{(-1)^{n+1}}{N_X}\sum_{j=0}^{p_1-1}\delta_j\otimes\delta_{p_2}\otimes\cdots\otimes\delta_{p_n}\otimes\delta_{p_{n+1}+p_1-j}\otimes\delta_0.$$

We now evaluate $r^{n-1}d^{n-1}+d^nr^n(X)$.

Theorem 3.6 *Let $n\geq 2$ and $X\in\mathcal{C}_n(\mathcal{A},\mathcal{A})$, $X\neq X_0^n$. Then*

$$(r^{n-1}d^{n-1}+d^nr^n)(X)=\frac{1}{N_X}\sum_{j=0}^{p_1-1}\delta_j\otimes\delta_{p_2}\otimes\cdots\otimes\delta_{p_{n+1}+p_1-j}+\frac{(-1)^{n+1}}{N_X}\sum_{j=0}^{p_1-1}\delta_{p_2}\otimes\cdots\otimes\delta_{p_n}\otimes\delta_{p_{n+1}+p_1-j}\otimes\delta_j.$$

Proof Let n and X be as in the statement of the theorem. We can write $r^{n-1}d^{n-1}+d^nr^n$ as

$$\sum_{k=2}^{n}(r^{n-1}d_k^{n-1}+d_k^nr^n)+(r^{n-1}d_0^{n-1}+r^{n-1}d_1^{n-1}+d_1^nr^n)+(d_0^nr^n+d_{n+1}^nr^n).$$

It is readily checked that

$$d_0^nr^n(X)=\frac{(-1)^{n+1}}{N_X}\sum_{j=0}^{p_1-1}\delta_{p_2}\otimes\cdots\otimes\delta_{p_n}\otimes\delta_{p_{n+1}+p_1-j}\otimes\delta_j,$$

$$d_{n+1}^nr^n(X)=\frac{1}{N_X}\sum_{j=0}^{p_1-1}\delta_j\otimes\delta_{p_2}\otimes\cdots\otimes\delta_{p_{n+1}+p_1-j}.$$

https://doi.org/10.4153/S0008439518000644 Published online by Cambridge University Press

We claim that for $n \geq 2$, $(r^{n-1}d_0^{n-1} + r^{n-1}d_1^{n-1} + d_1^n r^n)(X) = 0$. This is proved by direct computation. We have

$$\begin{aligned} r^{n-1}d_0^{n-1}(X) &= r^{n-1}(\delta_{p_2} \otimes \cdots \otimes \delta_{p_{n+1}+p_1}) \\ &= \frac{(-1)^n}{N_X} \sum_{j=0}^{p_2-1} \delta_j \otimes \delta_{p_3} \otimes \cdots \otimes \delta_{p_n} \otimes \delta_{p_{n+1}+p_1+p_2-j} \otimes \delta_0. \end{aligned}$$

Next we have

$$\begin{aligned} d_1^n r^n(X) &= d_1^n\Big(\frac{(-1)^{n+1}}{N_X} \sum_{j=0}^{p_1-1} \delta_j \otimes \delta_{p_2} \otimes \cdots \otimes \delta_{p_n} \otimes \delta_{p_{n+1}+p_1-j} \otimes \delta_0\Big) \\ &= \frac{(-1)^n}{N_X} \sum_{j=0}^{p_1-1} \delta_{j+p_2} \otimes \delta_{p_3} \otimes \cdots \otimes \delta_{p_n} \otimes \delta_{p_{n+1}+p_1-j} \otimes \delta_0. \end{aligned}$$

Finally we have

$$\begin{aligned} r^{n-1}d_1^{n-1}(X) &= -r^{n-1}(\delta_{p_1+p_2} \otimes \delta_{p_3} \otimes \cdots \otimes \delta_{p_{n+1}}) \\ &= \frac{(-1)^{n+1}}{N_X} \sum_{j=0}^{p_1+p_2-1} \delta_j \otimes \delta_{p_3} \otimes \cdots \otimes \delta_{p_n} \otimes \delta_{p_{n+1}+p_1+p_2-j} \otimes \delta_0 \\ &= \frac{(-1)^{n+1}}{N_X} \sum_{j=0}^{p_2-1} \delta_j \otimes \delta_{p_3} \otimes \cdots \otimes \delta_{p_n} \otimes \delta_{p_{n+1}+p_1+p_2-j} \otimes \delta_0 \\ &\quad + \frac{(-1)^{n+1}}{N_X} \sum_{j=0}^{p_1-1} \delta_{j+p_2} \otimes \delta_{p_3} \otimes \cdots \otimes \delta_{p_n} \otimes \delta_{p_{n+1}+p_1-j} \otimes \delta_0 \\ &= -(r^{n-1}d_0^{n-1} + d_1^n r^n)(X), \end{aligned}$$

which proves our claim.

Next we claim that $(r^{n-1}d_k^{n-1} + d_k^n r^n)(X) = 0$, for $2 \leq k \leq n$. Consider the case $k \neq n$. We have

$$\begin{aligned} r^{n-1}d_k^{n-1}(X) &= (-1)^k r^{n-1}(\delta_{p_1} \otimes \cdots \otimes \delta_{p_k+p_{k+1}} \otimes \cdots \otimes \delta_{p_{n+1}}) \\ &= \frac{(-1)^{k+n}}{N_X} \sum_{j=0}^{p_1-1} \delta_j \otimes \cdots \otimes \delta_{p_k+p_{k+1}} \otimes \cdots \otimes \delta_{p_{n+1}+p_1-j} \otimes \delta_0, \end{aligned}$$

while

$$\begin{aligned} d_k^n r^n(X) &= \frac{(-1)^{n+1}}{N_X} d_k^n\Big(\sum_{j=0}^{p_1-1} \delta_j \otimes \delta_{p_2} \otimes \cdots \otimes \delta_{p_n} \otimes \delta_{p_{n+1}+p_1-j} \otimes \delta_0\Big) \\ &= \frac{(-1)^{n+1+k}}{N_X} \sum_{j=0}^{p_1-1} \delta_j \otimes \cdots \otimes \delta_{p_k+p_{k+1}} \otimes \cdots \otimes \delta_{p_{n+1}+p_1-j} \otimes \delta_0 \\ &= -r^{n-1}d_k^{n-1}(X). \end{aligned}$$

The case $k = n$ is proved similarly, and the proof is complete. ∎

From Proposition 3.4 and Theorem 3.6 we have the following.

Corollary 3.7 *For $n \geq 2$, $\mathcal{H}_n(\mathcal{A}, \mathcal{A}) = 0$.*

https://doi.org/10.4153/S0008439518000644 Published online by Cambridge University Press

Proof It follows from Proposition 3.4 and Theorem 3.6 that for $n \geq 2$ and all X, we have $I - \big((s^{n-1} + r^{n-1})d^{n-1} + d^n(s^n + r^n)\big) = 0$. As the linear span of these is dense in $\mathcal{C}_n(\mathcal{A}, \mathcal{A})$, we have the result. ∎

To move from simplicial homology to simplicial cohomology is fairly direct. On one hand, the dual of $(s^{n-1} + r^{n-1})$ is a bounded linear operator from $(\mathcal{C}_n(\mathcal{A}, \mathcal{A}))^*$ to $(\mathcal{C}_{n-1}(\mathcal{A}, \mathcal{A}))^*$. On the other hand, in a canonical way, $(\mathcal{C}_n(\mathcal{A}, \mathcal{A}))^*$ is isometrically isomorphic to $\mathcal{C}^n(\mathcal{A}, \mathcal{A}^*)$. Therefore we can consider the dual of $(s^{n-1} + r^{n-1})$ as a map from $\mathcal{C}^n(\mathcal{A}, \mathcal{A}^*)$ to $\mathcal{C}^{n-1}(\mathcal{A}, \mathcal{A}^*)$. The explicit definition of this map is as follows.

Definition 3.8 For $n \geq 2$, let $\Gamma^{n-1} : \mathcal{C}^n(\mathcal{A}, \mathcal{A}^*) \to \mathcal{C}^{n-1}(\mathcal{A}, \mathcal{A}^*)$ be the bounded linear map defined, for $T \in \mathcal{C}^n(\mathcal{A}, \mathcal{A}^*)$ and $X \neq X_0^n$, by

$$\begin{aligned}\Gamma^{n-1}(T)(\delta_{p_1}, \dots, \delta_{p_{n-1}})(\delta_{p_n}) &= \frac{1}{N_X} \sum_{k=1}^{n} (-1)^k \sum_{j=0}^{p_k - 1} T(\delta_{p_1}, \dots, \delta_{p_k - j}, \delta_j, \dots, \delta_{p_{n-1}})(\delta_{p_n}) \\ &\quad + \frac{(-1)^n}{N_X} \sum_{j=0}^{p_1 - 1} T(\delta_j, \delta_{p_2}, \dots, \delta_{p_n + p_1 - j})(\delta_0),\end{aligned}$$

and by $\Gamma^{n-1}(T)(X_0^n) = 0$.

If we take the dual of $\big[I - \big((s^{n-1} + r^{n-1})d^{n-1} + d^n(s^n + r^n)\big)\big]$ in Corollary 3.7, we obtain the following.

Corollary 3.9 *For $n \geq 2$, $\mathcal{H}^n(\mathcal{A}, \mathcal{A}^*) = 0$.*

Proof It follows, either by duality (as explained above) or by almost identical proofs to those leading to Corollary 3.7, that for $n \geq 2$,

$$(\Delta^{n-1}\Gamma^{n-1} + \Gamma^n \Delta^n)(T) = T,$$

for all $T \in \mathcal{C}^n(\mathcal{A}, \mathcal{A}^*)$. In particular, if $T \in \ker \Delta^n$, then $\Delta^{n-1}(\Gamma^{n-1}(T)) = T$. ∎

It should be noted that we now have an explicit map such that, for $T \in \ker \Delta^n$, we have $S = \Gamma^{n-1}(T)$ such that $\Delta^{n-1}(S) = T$. In the next section, we generalize this result to some other semigroups.

4 Other Ordered Discrete Semigroups

In [5, §6] the authors generalized the results obtained for the semigroup $\mathbb{Z}_+$ to semigroups $\mathcal{S} = \mathcal{G} \cap \mathbb{R}_+$, where $\mathcal{G}$ is a subgroup of $\mathbb{R}$. In this section, we show how to generalize our results to this case.

Let $\mathcal{B} := \ell^1(\mathcal{S})$. We wish to define $\lambda^n : \mathcal{C}^{n+1}(\mathcal{B}, \mathcal{B}^*) \to \mathcal{C}^n(\mathcal{B}, \mathcal{B}^*)$. In the degenerate case, let $\lambda^n T(\delta_0, \delta_0, \dots, \delta_0)(\delta_0) = T(\delta_0, \delta_0, \dots, \delta_0)(\delta_0)$. We note that with this definition,

$$\big[\big(I - (\Delta^{n-1}\lambda^{n-1} + \lambda^n \Delta^n)\big)(T)\big](\delta_0, \delta_0, \dots, \delta_0)(\delta_0) = 0,$$

https://doi.org/10.4153/S0008439518000644 Published online by Cambridge University Press

and we therefore do not need to consider the degenerate case in the remainder of this section.

In the non-degenerate case, let $v = (a_1, a_2, \dots, a_{n+1}) \in \mathcal{S}^{n+1}$ be fixed. We define $\lambda^n T(\delta_{a_1}, \dots, \delta_{a_n})(\delta_{a_{n+1}})$ as follows.

Let $N = a_1 + \cdots + a_{n+1} \neq 0$, and let $\mathcal{S}_N$ be the amenable group $\frac{\mathcal{G}}{\langle N \rangle}$, with an invariant mean m_N. Note that we can identify $\mathcal{S}_N$ with $\{s \in \mathcal{S} : 0 \leq s < N\}$ with addition modulo N, and we will do so from now on, where $\mathcal{S}$ is an intersection between a subgroup of $\mathbb{R}$ and $\mathbb{R}_+$.

For a vector $v(x) = (a_1, \dots, a_k - x, x, \dots, a_{n+1}) \in \mathcal{S}_N^{n+2}$ that depends on x (and even if $v = v(x)$ is constant), we let $T[v(x)]$ denote

$$T(\delta_{a_1}, \dots, \delta_{a_k - x}, \delta_x, \dots, \delta_{a_n})(\delta_{a_{n+1}}),$$

where $T \in \mathcal{C}^{n+1}(\mathcal{B}, \mathcal{B}^*)$. We define $T[v(x)]_a^b \in \ell^\infty(\mathcal{S}_N)$ by

$$T[v(x)]_a^b(s) = \begin{cases} T[v(s)] & \text{if } a \leq s < b, \\ 0 & \text{otherwise,} \end{cases}$$

where it is assumed that $0 \leq a \leq N$, $0 \leq b \leq N$.

Finally, for $y \in \mathcal{S}_N$, let χ_y be the characteristic function of $\{s \in \mathcal{S}_N : s < y\}$ and let χ_N be the characteristic function of $\mathcal{S}_N$, seen as elements of $\ell^\infty(\mathcal{S}_N)$.

For $1 \leq k \leq n+1$, we define the bounded linear operator $\lambda_k^n : \mathcal{C}^{n+1}(\mathcal{B}, \mathcal{B}^*) \to \mathcal{C}^n(\mathcal{B}, \mathcal{B}^*)$ by

$$\lambda_k^n T(\delta_{a_1}, \dots, \delta_{a_n})(\delta_{a_{n+1}}) := (-1)^k m_N(T[a_1, a_2, \dots, a_k - x, x, \dots, a_{n+1}]_0^{a_k}).$$

We define λ^n by $\lambda^n := \sum_{k=1}^{n+1} \lambda_k^n$. This construction was given in [5, §6].

With this definition, we can now generalize our results. First, we consider the analogue of Lemmas 3.1, 3.2, 3.3, and Proposition 3.4.

Lemma 4.1 *Let $n \geq 1$ and let the notation and setting be as above. Then*

$$\begin{aligned} \Delta_0^{n-1}\lambda_k^{n-1} + \lambda_{k+1}^n \Delta_0^n &= 0, \quad 1 \leq k \leq n-1, \\ \Delta_j^{n-1}\lambda_k^{n-1} + \lambda_{k+1}^n \Delta_j^n &= 0, \quad 1 \leq j < k \leq n, \\ \Delta_j^{n-1}\lambda_k^{n-1} + \lambda_k^n \Delta_{j+1}^n &= 0, \quad 1 \leq k < j \leq n. \end{aligned}$$

Proof The proof is essentially identical to that of Lemma 3.1, and we only give an outline of the first case. For $T \in \mathcal{C}^n(\mathcal{B}, \mathcal{B}^*)$, and $1 \leq k \leq n-1$, we have

$$\begin{aligned} &\Delta_0^{n-1}(\lambda_k^{n-1} T)(\delta_{a_1}, \dots, \delta_{a_n})(\delta_{a_{n+1}}) \\ &\quad = (\lambda_k^{n-1} T)(\delta_{a_2}, \dots, \delta_{a_n})(\delta_{a_{n+1}+a_1}) \\ &\quad = (-1)^k m_N\big(T[a_2, \dots, a_{k+1} - x, x, \dots, a_n, a_{n+1} + a_1]_0^{a_{k+1}}\big) \end{aligned}$$

and

$$\begin{aligned} &\lambda_{k+1}^n(\Delta_0^n T)(\delta_{a_1}, \dots, \delta_{a_n})(\delta_{a_{n+1}}) \\ &\quad = (-1)^{k+1} m_N\big((\Delta_0^n T)[a_1, \dots, a_{k+1} - x, x, \dots, a_{n+1})]_0^{a_{k+1}}\big). \end{aligned}$$

The conclusion follows directly from the definition of $T[v(x)]$. ∎

https://doi.org/10.4153/S0008439518000644 Published online by Cambridge University Press

Lemma 4.2 *With the notation as above,*

$$\big((\Delta_0^{n-1}\lambda_n^{n-1} + \lambda_{n+1}^n\Delta_{n+1}^n + \lambda_1^n\Delta_0^n + \lambda_{n+1}^n\Delta_0^n)T\big)(\delta_{a_1},\dots,\delta_{a_n})(\delta_{a_{n+1}})$$

is given by

$$m_N(\chi_{a_{n+1}})\cdot T(\delta_{a_1},\dots,\delta_{a_n})(\delta_{a_{n+1}}) \\ - m_N\big(T[x,a_2,\dots,a_n,a_{n+1}+a_1-x]_0^{a_1}\big) \\ + (-1)^n m_N\big(T[a_2,\dots,a_n,a_{n+1}+a_1-x,x]_0^{a_1}\big).$$

Proof Recall that we assume that $N \neq 0$ (so that we are not in the degenerate case). The term $m_N(\chi_{a_{n+1}})\cdot T(\delta_{a_1},\dots,\delta_{a_n})(\delta_{a_{n+1}})$ is obtained from $\lambda_{n+1}^n\delta_{n+1}^n$; the last term results from $\lambda_1^n\delta_0^n$, while the middle one is obtained through the partial cancellation of $\delta_0^{n-1}\lambda_n^{n-1}$ with $\lambda_{n+1}^n\delta_0^n$. Here, we make the simple but crucial use of the fact that the mean is translation invariant. Details are left to the reader, as the proof is entirely analogous to the proof of Lemma 3.2, the invariant mean playing the role of the averaging and the argument being at the dual level. ■

The analogue of Lemma 3.3 is as follows.

Lemma 4.3 *With the notation as above and for $1 \le i \le n$,*

$$\big((\Delta_i^{n-1}\lambda_i^{n-1} + \lambda_i^n\Delta_i^n + \lambda_{i+1}^n\Delta_i^n + \lambda_i^n\Delta_{i+1}^n)T\big)(\delta_{a_1},\dots,\delta_{a_n})(\delta_{a_{n+1}})$$

is given by $m_N(\chi_{a_i})\cdot T(\delta_{a_1},\dots,\delta_{a_n})(\delta_{a_{n+1}})$.

Proof The result follows from the cancellation of $\lambda_{i+1}^n\Delta_i^n + \lambda_i^n\Delta_{i+1}^n$ with $\Delta_i^{n-1}\lambda_i^{n-1}$, once again using translation invariance of the mean, and the remaining term is given by $\lambda_i^n\Delta_i^n$. Details are left to the reader as the proof is entirely analogous to the proof of Lemma 3.3, the invariant mean playing the role of the averaging and the argument being at the dual level. ■

As Proposition 3.4 was only a restatement of the lemmas preceding it, we do not state its generalization here; let us simply note that $\sum_{i=1}^{n+1} m_N(\chi_{a_i}) = 1$, so that we get T by summing terms obtained in Lemmas 4.2 and 4.3. We define the map that generalizes the map introduced in Definition 3.5.

Definition 4.4 Let $n \ge 1$ and let the notation be as above. Let

$$\mu^n : \mathcal{C}^{n+1}(\mathcal{B},\mathcal{B}^*) \longrightarrow \mathcal{C}^n(\mathcal{B},\mathcal{B}^*)$$

be the bounded linear operator defined by

$$\mu^n(T)(\delta_{a_1},\dots,\delta_{a_n})(\delta_{a_{n+1}}) := (-1)^{n+1} m_N\big(T[x,a_2,\dots,a_n,a_{n+1}+a_1-x,0]_0^{a_1}\big).$$

As we indicated at the beginning of the section, $\mu^n(T)$ is defined as being zero in the degenerate case ($N = 0$).

With this new map, we can state the required generalization of Theorem 3.6 and deduce the simplicial cohomology of $\mathcal{B}$ for $n \ge 2$.

Theorem 4.5 *Let $n \geq 2$ and let the notation be as above. Then*

$$(\Delta^{n-1}\mu^{n-1} + \mu^n \Delta^n) T(\delta_{a_1}, \dots, \delta_{a_n})(\delta_{a_{n+1}}) = (-1)^{n+1} m_N\big(T[a_2, \dots, a_n, a_{n+1} + a_1 - x, x]_0^{a_1}\big) + m_N\big(T[x, a_2, \dots, a_n, a_{n+1} + a_1 - x]_0^{a_1}\big).$$

Proof The proof of Theorem 3.6 can be easily adapted to obtain the result. ∎

Corollary 4.6 *With the notation as above, $\mathcal{H}^n(\mathcal{B}, \mathcal{B}^*) = 0$, for $n \geq 2$.*

Proof Let $n \geq 2$. It follows directly from the previous theorem together with the preceding lemmas that $\Delta^{n-1}(\lambda^{n-1} + \mu^{n-1}) + (\lambda^n + \mu^n)\Delta^n = I$. In particular, if $T \in \ker \Delta^n$, then $\Delta^{n-1}\big((\lambda^{n-1} + \mu^{n-1})(T)\big) = T$. ∎

Remark 4.7 The ideas introduced in this paper for $\ell^1(\mathbb{Z}_+)$ and $\ell^1(\mathcal{S})$, where $\mathcal{S}$ is an intersection between a subgroup of $\mathbb{R}$ and $\mathbb{Z}_+$, can be adapted to other settings. Extensions to band semigroup algebras and to the Cuntz semigroup algebra (as defined in [2, 8]) form part of the first author's Ph.D. thesis [4].

References

[1] W. G. Bade, P. C. Curtis Jr., and H. G. Dales, *Amenability and weak amenability for Beurling and Lipschitz algebras*. Proc. Lond. Math. Soc. 55(1987), 359–377. https://doi.org/10.1093/plms/s3-55_2.359.

[2] Y. Choi, F. Gourdeau, and M. C. White, *Simplicial cohomology of band semigroup algebras*. Proc. Roy. Soc. Edinburgh Sect. A 142(2012), no. 4, 715–744. https://doi.org/10.1017/S0308210510000648.

[3] H. G. Dales and J. Duncan, *Second order cohomology in groups of some semigroup algebras*. In: *Banach Algebras '97*. Walter de Gruyter, Berlin, 1998, pp. 101–117.

[4] Y. Farhat, *Homologie et cohomologie de quelques algèbres de Banach*. Ph.D. thesis, Université Laval, Québec, QC, 2013.

[5] F. Gourdeau, B. E. Johnson, and M. C. White, *The cyclic and simplicial cohomology of* $l^1(\mathbb{N})$. Trans. Amer. Math. Soc. 357(2005), no. 12, 5097–5113.

[6] F. Gourdeau and M. C. White, *Vanishing of the third simplicial cohomology group of* $l^1(\mathbb{Z}_+)$. Trans. Amer. Math. Soc. 353(2001), no. 5, 2003–2017.

[7] F. Gourdeau and M. C. White, *The cyclic and simplicial cohomology of the bicyclic semigroup algebra*. Q. J. Math. 62(2011), 607–624. https://doi.org/10.1093/qmath/haq014.

[8] F. Gourdeau and M. C. White, *The cyclic and simplicial cohomology of the Cuntz semigroup algebra*. J. Math. Anal. Appl. 386(2012), no. 2, 921–932. https://doi.org/10.1016/j.jmaa.2011.08.050.

[9] A. Ya. Helemskii, *Banach cyclic (co)homology and the Connes–Tzygan exact sequence*. J. Lond. Math. Soc. 46(1992), 449–462. https://doi.org/10.1112/jlms/s2-46.3.449.

Academic Support Department, Abu Dhabi Polytechnic, P.O. Box 111499, Abu Dhabi, UAE
e-mail: Yasser.Farhat@adpoly.ac.ae

Département de mathématiques et de statistique, Université Laval, 1045, avenue de la Médecine, Québec, QC G1V 0A6
e-mail: Frederic.Gourdeau@mat.ulaval.ca