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CONTINUOUS PREIMAGES OF SPACES WITH FINITE COMPACTIFICATIONS

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1. **Introduction.** A compactification αX of the space X is called an n-point compactification if the remainder $\alpha X - X$ consists of exactly n points. K. D. Magill [5] showed that if Y has an n-point compactification and if $f: X \to f(x) = Y$ is a compact continuous mapping of the space X onto Y, then X also has an n-point compactification. The main purpose here is to study the existence of finite compactifications of a space X that can be mapped continuously onto a space Y for which there is an n-point compactification. Results obtained extend Magill's in that they apply to continuous maps more general than compact ones, and also provide sharper results regarding the number of points in the finite compactifications of X. As a corollary, a characterization of those spaces having an n-point compactification is given in terms of the existence of certain mappings onto compact spaces.

2. **Finite compactification of subspaces.** The following result is used throughout this section.

LEMMA 2.1. *Suppose C is a component of a compact Hausdorff space Y and W is an open neighborhood of C. Then there exists an open neighborhood U of C such that $C \subset U \subset W$ and* $\mathrm{Fr}(U) = \emptyset$.

Proof. Define $P \subset Y$ by

$$P = \{y \in Y \mid \text{there is a separation } Y = R \cup T \text{ with } C \subset R \text{ and } y \in T\}.$$

Clearly P is open, so its complement $Q = Y - P$ is closed.

Suppose Q is not connected. Then we have a separation $Q = A \cup B$. The component $C \subset Q$, so assume $C \subset A$. The sets A and B are compact, hence there are disjoint open neighborhoods W_A and W_B for which $A \subset W_A$ and $B \subset W_B$. Fr $W_A \subset P$, so for each $p \in \mathrm{Fr}\ W_A$, there is a separation $Y = U_p \cup V_p$ such that $C \subset U_p$ and $p \in V_p$.

The collection $\{V_p\}$ covers the compact set Fr W_A, so we may extract a finite subcollection, $\{V_i : i = 1, 2, \ldots, n\}$ which also covers Fr W_A. Each V_i is both open and closed, so it follows that $V = \bigcup \{V_i\}$ is also open and closed.

Now let $\hat{U} = W_A - V$ and note that $\hat{U}$ is open and closed; open because V is closed and closed because V is open and Fr $W_A \subset V$. Thus $Y = \hat{U} \cup (Y - \hat{U})$ is

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a separation with $C \subset \hat{U}$ and $B \subset Y - \hat{U}$, which means that $B \subset P$, a contradiction. So Q is connected, and $Q = C$.

Since $Y - W \subset P$, it is true that for every $x \in Y - W$ there is a separation $Y = R_x \cup T_x$ with $C \subset R_x$, $x \in T_x$. Now $Y - W$ is compact, so if $\{T_i : i = 1, 2, \ldots, m\}$ is a finite subcover of $\{T_x\}$, then $T = \cup \{T_i\}$ is both open and closed. Thus $U = Y - T$ is an open neighborhood of C such that $U \subset W$ and Fr $U = \emptyset$.

We are now ready for the theorem that is the basis of all other results in this paper.

THEOREM 2.1. *Suppose Z is a connected Hausdorff space with an n-point compactification and $X \subset Z$ is open and such that $Z - X$ has at least k compact components. If all components of $Z - X$ are compact, then X has an $(n+k)$-point compactification. Otherwise, X has a $(k+1)$-point compactification.*

Proof. Let αZ be an n-point compactification of Z and let $K = \alpha Z - X$. Define a decomposition $\mathscr{D}$ of αZ by taking the elements of $\mathscr{D}$ to be the individual points of X and the components of K.

We need to show that $\mathscr{D}$ is upper semicontinuous. To do this, let $d \in \mathscr{D}$ and U be an open subset of αZ containing d. We must find an open $V \subset U$ containing d so that if $e \in \mathscr{D}$ and $V \cap e \neq \emptyset$, then $e \subset V$. If $d = \{x\}$ for $x \in X$, this is trivial, so assume d is a component of K.

We apply Lemma 2.1 to the space K and conclude that there exists an open set $V \subset U$ for which $d \subset V \cap K \subset U \cap K$ and (Fr $V) \cap K = \emptyset$. Thus if $e \in \mathscr{D}$ and $V \cap e \neq \emptyset$, it must follow that $e \subset V \subset U$. This establishes the upper semicontinuity of $\mathscr{D}$.

Let $\hat{X}$ be the decomposition space induced by $\mathscr{D}$ and let h denote the natural map of αZ onto $\hat{X}$. From the upper semicontinuity of $\mathscr{D}$, $\hat{X}$ is a compact Hausdorff space. The continuous map is one to one on X and $X = h^{-1}h(X)$ is an inverse set, so $h \mid X$ is a homeomorphism. As usual, we shall identify $h(X)$ with X and refer to X as a subspace of $\hat{X}$. Note that X is open in $\hat{X}$, so $\hat{X} - X$ is compact.

Suppose A is a connected subset of the remainder $\hat{X} - X$. Then $h^{-1}(A)$ is a connected subset of K since h is a compact monotone map and must lie completely in one component of K. Thus $A = \{p\}$, a single point. Or in other words, $\hat{X} - X$ is totally disconnected.

The space αZ is connected, so $\hat{X}$ is connected. To see that X must be dense in $\hat{X}$, suppose p is a point in the interior of $\hat{X} - X$. The points of $\hat{X} - X$ are the components of this space, so Lemma 2.1 yields the existence of an open neighborhood U of p for which $U \subset \hat{X} - X$, and Fr $U = \emptyset$, contradicting the connectedness of $\hat{X}$. Hence $\hat{X}$ is a compactification of X.

Suppose C is a compact component of $K = Z - X$, and let N be a compact neighborhood of such that $N\cap(\alpha Z-Z)=\emptyset$. Then C is a component of $N\cap K$, and we can once again apply Lemma 2.1 to obtain an open neighborhood V of C so that $V\subset N$ and $(\text{Fr } V)\cap K=\emptyset$. This, of course, means that C is also a component of $\alpha Z-X$, which shows that the number of components of $\alpha Z-X$ is at least as large as the number of compact components of $K=Z-X$.

Next observe that if all components of K are compact, no point of $\alpha Z-Z$ is an accumulation point of a component of K. Thus each $r\in\alpha Z-Z$ is a component of $\alpha Z-X$. We have now shown that if all components of $Z-X$ are compact, there are at least $(n+k)$ components of $\alpha Z-X$. Hence either $\hat{X}-X$ is finite and contains at least $(n+k)$ points or it is infinite and totally disconnected. In either case, we may conclude that there is an $(n+k)$-point compactification of X. (Chandler [3], Theorem 6.32, p. 77; and Lemma 6.13, p. 72.)

Next suppose $Z-X$ has at least one non-compact component C. In this case, we can conclude only that $\alpha Z-X$ has at least $k+1$ components, or $\hat{X}-X$ contains at least $k+1$ points.

3. **Mappings onto spaces with finite compactifications.** Recall that if $f:X\to f(X)=Y$ is a continuous mapping of one locally compact Hausdorff space onto another, then the *singular set* S of f is the collection of all points $p\in Y$ such that in every neighborhood of p there is a compact set with non-compact inverse image ([1], [6]). The set S is closed and a mapping (continuous) is compact (inverse images of compact sets are compact) if and only if $S=\emptyset$ ([1], [6]).

THEOREM 3.1. *Suppose Y has an n-point compactification, X is connected, and $f:X\to f(X)=Y$ is a continuous mapping for which the singular set S has at least k compact components. If all components of S are compact, then S has an $(n+k)$-point compactification. Otherwise, X has a $(k+1)$-point compactification.*

Proof. Let (X_w, f_w) be the Whyburn compactification of the map f. That is, $f_w:X_w\to f(X_w)=Y$, f_w is compact and continuous, X_w is locally compact Hausdorff and contains X as a dense subspace, $f_w\mid X=f$, and $f_w\mid(X_w-X)$ is a homeomorphism onto S. (See [2] and [7]).

From Magill's result [6] X_w has an n-point compactification. Also, X_w-X and S are homeomorphic, so the conclusions of the theorem follow directly from Theorem 2.1.

The following example shows that this theorem is the "best" one can hope to get, in that if S has at least one non-compact component, then almost anything is possible.

EXAMPLE. Let $y=[0,1]\times[0,1]-\{(\frac{1}{2},\frac{1}{4}),\ (\frac{1}{2},\frac{3}{4})\}$. It is obvious that Y has a 2-point compactification. Let $A\subset[0,1]\times[0,1]\times[0,1]$ be given by $A=\{(\frac{1}{2},y,1):\frac{1}{4}<y<\frac{3}{4}\}$ and define $X=Y\times[0,1]-A$. It is easy to see that X has a

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one-point compactification; that it does not have an n-point compactification; for $n>1$ follows easily from Theorem (2.6) of Magill [5]. Let $f: X \to f(X) = Y$ be defined by setting $f(x, y, z) = (x, y)$. We see that f is continuous and X does not have an n-point compactification for $n>1$ although Y has a 2-point compactification. It should be clear from this example how one could similarly construct examples of continuous maps of a space X onto a space Y in which X and Y have arbitrary finite compactifications.

Finally we give a characterization of those spaces having an n-point compactification.

THEOREM 3.2. *A connected locally compact Hausdorff space X has an n-point compactification if and only if there is a continuous mapping of X onto a compact Hausdorff space so that the singular set of the mapping consists of exactly n points.*

Proof. Suppose there is a continuous $f: X \to f(X) = Y$ with Y compact and S having exactly n points. Then the existence of an n-point compactification of X follows at once from Theorem 3.1.

Now suppose there is an n-point compactification αX of X. Let $U_1, U_2, \ldots, U_n$ be disjoint compact neighborhoods in αX of the points of $\alpha X - X$. Define the decomposition $\mathcal{D}$ of αX by taking $U_1, \ldots, U_n$ to be members of $\mathcal{D}$ and the individual points of $\alpha X - \bigcup_1 U_i$ as members of $\mathcal{D}$. Clearly $\mathcal{D}$ is upper semicontinuous, so the decomposition space Y induced by $\mathcal{D}$ is compact Hausdorff. Let h denote the natural map of αX onto Y, and define $f: X \to Y$ by $f = h \mid X$. The map f carries X onto Y since each U_i meets X, and the singular set $S = \{h(U_i), i-1, \ldots, n\}$.

REMARK. E. U. Choo [4] has proved a similar result. He essentially establishes the characterization given in the previous theorem, except the continuous mapping is real valued and bounded. (Choo's "accumulation point" of a continuous real valued map is the same as a singular point.)

REFERENCES

1. G. L. Cain, Jr., *Compact and related mappings*, Duke Math. J. **33** (1966), 639–645.
2. G. L. Cain, Jr., *Compactification of mappings*, Proc. Amer. Math. Soc. **23** (1969), 298–303.
3. R. E. Chandler, *Hausdorff Compactifications*, Marcel Dekker, Inc., New York and Basel, 1976.
4. E. U. Choo, *Accumulation points of continuous real-valued functions and compactifications*, Canad. Math. Bull. **20** (1977), 47–52.
5. K. D. Magill, Jr., *N-point compactifications*, Amer. Math. Monthly **72** (1965), 1075–1081.
6. G. T. Whyburn, *Compactification of mappings*, Math. Ann. **166** (1966), 168–174.
7. G. T. Whyburn, *A unified space for mappings*, Trans. Amer. Math. Soc. **74** (1953), 344–350.

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