THE PRACTICAL USE OF VARIATION PRINCIPLES IN NON-LINEAR MECHANICS

J. M. BLATT and J. N. LYNESS

(received 2 June 1961, revised 11 January 1962)

Abstract

Contrary to the general impression that variation principles are of purely theoretical interest, we show by means of examples that they can lead to considerable practical advantages in the solution of non-linear vibration problems. In this paper, we develop a variation principle for the period of a free oscillation, as a function of the average value of the Lagrangian over one period. Even extremely simple-minded approximations to the true motion lead to excellent values for the period. The stability of such free oscillations against small disturbances of the initial conditions is treated in a previous paper.

1. Introduction

"Variational principles in themselves contain no new physical content, and they rarely simplify the practical solution of a given mechanical problem. Their value lies chiefly as starting points for new formulations of the theoretical structure of classical mechanics" [1]. This opinion of variation principles is generally accepted, we believe wrongly. It is the purpose of this, and the previous [2], paper to point out that variation principles in classical mechanics can be of considerable practical use, and to illustrate their use by some simple examples. For this purpose, we have developed a new variation principle for the period of a periodic motion of a non-linear oscillating system, have adapted a standard variation method to study the stability of such a periodic motion against small perturbations in the initial conditions, and have developed iteration schemes for obtaining better approximations, starting from a given approximation to the motion. For the sake of simplicity, we restrict ourselves here to "free", as opposed to "forced", vibrations; but this restriction is not essential.

If the oscillating system has only one degree of freedom, and is conservative, then the solution of the equations of motion is straightforward, and an explicit solution is known in terms of certain definite integrals.

https://doi.org/10.1017/S144678870002694X Published online by Cambridge University Press

Satisfactory as this might appear at first sight, in practice this method leaves much to be desired. In all but the simplest case of a linear spring, the integrals are at least elliptic integrals, and usually even worse. The solution obtained is awkward and "undurchsichtig". The information of practical interest is of two kinds:

(1) The value of the period of the motion as a function of other parameters, e.g., of the maximum displacement, of the total energy, or some related quantity.

(2) The "shape" of the motion $q = q(t)$, i.e., the admixture of higher harmonics to the sine-wave zero order approximation.

Although the elliptic integrals can be expanded in series form, and the answers to (1) and (2) obtained that way, this is a rather awkward procedure; and all too often the integrals are not even elliptic integrals, and almost nothing is known about their analytic properties. This situation is particularly bad for question (2), since the exact solution gives t as a function of q, rather than q as a function of t. If the integrals are not elliptic, the inversion of $t = t(q)$ so as to find $q = q(t)$ is apt to be quite a problem.

If the oscillating system has several degrees of freedom, the situation is much worse yet; there is no general method of finding periodic solutions of the equations of motion in closed form; and even if such a solution, or a one-parameter family of such solutions (we shall call the latter a "mode") is obtained somehow, there remains the problem of stability against small perturbations of the initial conditions. For example, consider a system of two masses connected to each other and to two walls by non-linear springs, as shown in Fig. 1, and constrained to move in a straight line. This system

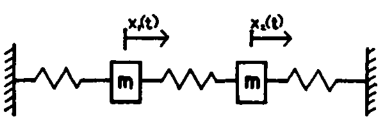

FIGURE I

Fig. 1. A system of springs and masses. The two outer springs are identical and the masses are equal.

was considered by Rosenberg and Atkinson [3], and it was found that there are circumstances under which one mode (e.g., both masses moving in the same direction) is stable, whereas the other mode (masses moving in opposite directions) is unstable against small perturbations of the initial conditions; i.e., if the system is set into motion in a way approximating to the second mode, its motion is not even approximately periodic or similar to the pure second mode.

In the present paper, we describe a variation technique for obtaining practical answers for the periodic motions (the "modes") of the non-linear

https://doi.org/10.1017/S144678870002694X Published online by Cambridge University Press

system. A previous paper [2] is devoted to similar techniques for investigating the stability (against small perturbations of the initial conditions) of the modes so obtained.

2. A Variation Principle for the Period

In this section we derive an expression for the period and show that it is variationally correct if the time average of the Lagrangian is maintained constant during the variation.

We consider an N degree of freedom system whose configuration is described by the coordinates $q_1, q_2, \ldots, q_N$. We suppose that the kinetic energy K and the potential energy V have the following forms

$$K(q, \dot{q}) = \sum_{i=1}^{N} \sum_{j=1}^{N} \tfrac{1}{2} m_{ij}(q_1, \cdots, q_N)\dot{q}_i \dot{q}_j \tag{2.1}$$

$$V(q) = V(q_1, q_2, \cdots, q_N) \tag{2.2}$$

where $m_{ij} = m_{ji}$. We also suppose that the total energy is conserved, that is

$$K(q, \dot{q}) + V(q) = E \tag{2.3}$$

and that the system is in motion, this motion being periodic with period T. Introducing the new independent variable $\tau = t/T$ and considering $q_i = q_i(\tau)$ as a function of this variable instead of time t, we may write the energy equation (2.3) in the form

$$\frac{1}{T^2} \sum \sum \tfrac{1}{2} m_{ij} \left(\frac{dq_i}{d\tau}\right)\left(\frac{dq_j}{d\tau}\right) = E - V(q). \tag{2.4}$$

We integrate (2.4) with respect to τ from $\tau = 0$ to $\tau = 1$, i.e., over one period. This yields an expression for T^2 in terms of certain integrals, namely

$$T^2 = \frac{J_{1,0}}{E - J_{0,1}}. \tag{2.5}$$

where

$$J_{r,s} = \int_0^1 \left\{\sum \sum \tfrac{1}{2} m_{ij} \frac{dq_i}{d\tau}\frac{dq_j}{d\tau}\right\}^r \{V(q)\}^s d\tau. \tag{2.6}$$

In some motions of physical systems it is known *a priori* that the kinetic energy vanishes at some time during the motion. In this case we may write

$$E = V_{\max}. \tag{2.7a}$$

In general, though, this does not happen, and we require an expression for E in terms of the motion $q(\tau)$, without *a priori* knowledge of the period

T. Such an expression may be obtained by multiplying equation (2.4) by

$$\left\{\sum\sum \tfrac{1}{2} m_{ij} \frac{dq_i}{d\tau}\frac{dq_j}{d\tau}\right\}^r \{V(q)\}^s$$

and then integrating from $\tau = 0$ to $\tau = 1$. This leads to the generalisation of (2.5)

$$\frac{1}{T^2} J_{r+1,s} + J_{r,s+1} = E J_{r,s}.$$

Eliminating $1/T^2$ from two different equations of that type we have for all non-negative integral values of $(r, s) \neq (r', s')$

$$E = \frac{J_{r,s+1}J_{r'+1,s'} - J_{r+1,s}J_{r',s'+1}}{J_{r,s}J_{r'+1,s'} - J_{r+1,s}J_{r',s'}}. \tag{2.7b}$$

Any expression of this type may be used for E. However the integrals can be lengthy and it is usually convenient to use this expression with $r' = s' = r = 0$ and $s = 1$, namely [1]

$$E = \frac{J_{0,2}J_{1,0} - J_{1,1}J_{0,1}}{J_{0,1}J_{1,0} - J_{1,1}}. \tag{2.7c}$$

We now establish a variational property of (2.5). We consider the variation $\delta(T^2)$ of T^2 as a result of replacing $q_i(\tau)$, the exact solution of the equations of motion, by $q_i(\tau) + \delta q_i(\tau)$. Here $\delta q_i(\tau)$ is also periodic in τ, with period 1. It is easily shown that

$$\delta(T^2) = \frac{T^2}{E - J_{0,1}}\left\{\frac{\delta J_{1,0}}{T^2} - \delta(E - J_{0,1})\right\}. \tag{2.8}$$

However, as K is a function of q and $\dot{q}$, we may write $\delta J_{1,0}$ in the following forms

$$\delta J_{1,0} = T\int_0^T \delta K\, dt = \sum_{j=1}^N T\int_0^T \left\{\frac{\partial K}{\partial q_j}\delta q_j + \frac{\partial K}{\partial \dot{q}_j}\delta\dot{q}_j\right\} dt.$$

Using integration by parts and the relation $\delta\dot{q} = (d/dt)\,\delta q$ the final term in the integration may be transformed with the result

$$\delta J_{1,0} = \sum_{j=1}^N T\int_0^T \left(\frac{\partial K}{\partial q_j} - \frac{d}{dt}\frac{\partial K}{\partial \dot{q}_j}\right)\delta q_j\, dt.$$

The boundary terms vanish because of the assumed periodicity of $\delta q_i(\tau)$

[1] The choice $r = 0$, $s = 1$ is preferable to $r = 1$, $s = 0$, because $r = 1$, $s = 0$ involves higher powers of the derivative of the function $q(\tau)$. This can lead to larger errors if $q(\tau)$ is only approximate.

as a function of τ. Now, for the exact motion, Lagrange's equations are valid. That is

$$\frac{\partial K}{\partial q_j} - \frac{d}{dt}\frac{\partial K}{\partial \dot{q}_j} = \frac{\partial V}{\partial q_j}.$$

Using this, it follows that

$$\delta J_{1,0} = T\int_0^T \sum_{j=1}^N \frac{\partial V}{\partial q_j}\,\delta q_j dt = T^2 \delta J_{0,1} \tag{2.9}$$

and substituting this into (2.8) gives

$$\delta(T^2) = -\frac{T^2}{E - J_{0,1}}\,\delta[E - 2J_{0,1}]. \tag{2.10}$$

We define

$$L(q) = E - 2V(q) \tag{2.11a}$$

and

$$\bar{L} = E - 2J_{0,1} = \int_0^1 [E - 2V(q)]d\tau. \tag{2.11b}$$

The function $L(q)$ is numerically equal to the Lagrangian $L = K - V$ provided that $q_i = q_i(t)$ is the correct solution of the equations of motion. Similarly, $\bar{L}$ is numerically equal to the average value of the Lagrangian over one complete period of the motion. It should be noted that, for a one-parameter family of periodic motions of the system, the value of $\bar{L}$ is usually a possible way of specifying this one parameter, in principle just as good as specifying the energy or the amplitude of the motion.[2]

Equation (2.10) can be written in the form

$$\delta\,(T^2) = -\frac{T^2 \delta \bar{L}}{E - \bar{V}}. \tag{2.12}$$

Thus, *the first order variation of expression* (2.5) *for the period vanishes if the variation about the exact motion is such that the average Lagrangian* $\bar{L}$ *given by* (2.11*b*) *is kept constant*; i.e. if $\delta \bar{L} = 0$.

Conversely, suppose we insert into (2.5), instead of the true solution $q_i = q_i(\tau)$, some approximation $q_i = f_i(\tau)$. We then obtain an approximate value of T^2 from (2.5). Let us insert the same approximation $q_i = f_i(\tau)$ into the definition (2.11b); this leads to an approximate value of $\bar{L}$. Suppose that the deviations $f_i(\tau) - q_i(\tau)$ are of order ε; then both T^2 and $\bar{L}$

[2] None of these three distinguish motions of a mode which differ from each other only in phase. However $\bar{L}$ is unsuitable if the mode is in fact linear, as for all the motions in a linear mode $\bar{L} = 0$. From a utilitarian standpoint, it is inconceivable that this variation principle would be used to determine the period of a linear mode.

in general have errors of order ε; however, if there exists a one-parameter family of periodic solutions of the equations of motion, we may choose L as the parameter in question, and we can then compare the exact and approximate values of the *period considered as a function of* L (rather than, say, as a function of the amplitude of the motion). If the comparison is made in that particular way, (2.12) shows that the error in $T(L)$ will be of the order ε^2, i.e., the approximate value obtained for the period will be significantly more accurate than the approximation $q_i = f_i(\tau)$ from which this value was derived.

As an illustration, consider the following simple example: a mass m is constrained to move in a straight line, and connected to the point $q = 0$ by means of a highly non-linear spring, with potential energy

(2.13) $$V(q) = \beta q^4$$

where β is positive. It is clear that the motion is periodic and that the mass is stationary twice during each period. We may denote by $q_{\max}$ the amplitude of this motion and use (2.7a) for the energy. Thus

(2.13a) $$E = V_{\max} = \beta q_{\max}^4$$

From purely dimensional considerations we obtain the results

(2.14a) $$T^2 = C_1 \frac{m}{\beta (q_{\max})^2}$$

(2.14b) $$L = C_2 \beta (q_{\max})^4$$

where C_1 and C_2 are constants, and it is easily seen that both are positive.

The problem is simple enough to allow a straigthforward exact solution in terms of elliptic integrals. The exact results are

(2.15a) $$C_1 = 4[K(\tfrac{1}{2})]^2 = 13.750$$

(2.15b) $$C_2 = \tfrac{1}{3} = 0.33333.$$

Now suppose we pick the most simple-minded approximation to the motion, namely just a sine-wave shape:

(2.16) $$q(\tau) \cong f(\tau) = q_{\max} \sin (2\pi\tau).$$

Substitution of (2.16) into (2.5) and (2.11b) gives results of the expected form (2.14), but of course with incorrect values of the constants C_1 and C_2, namely

(2.17a) $$\hat{C}_1 = \frac{8\pi^2}{5} = 15.791$$

(2.17b) $$\hat{C}_2 = \tfrac{1}{4} = 0.2500.$$

https://doi.org/10.1017/S144678870002694X Published online by Cambridge University Press

As expected, these values are not very close to the exact values (2.15).

Now, however, let us consider the value L as the parameter specifying the motion, rather than considering q_{max} as the parameter. That is, we eliminate q_{max} between (2.14a) and (2.14b) in order to get

$$T^2 = C_3 \frac{m}{(L)^{1/2}} \tag{2.18a}$$

where

$$C_3 = C_1\sqrt{C_2}. \tag{2.18b}$$

The variation principle leads us to expect that the approximate value of C_3 should lie quite close to the exact value. The two values are

$$C_3 = 7.9388 \ \text{(exact)} \qquad \hat{C}_3 = 7.8957 \ \text{(appr.)} \tag{2.19}$$

Thus, the variational form for the period has led to a result with only $\frac{1}{2}$ percent error! This error must be compared with the errors in the non-variational estimates (2.17), which are 15 percent for C_1 and 25 percent for C_2. The reduction of errors from 25 percent to $\frac{1}{2}$ percent can well be claimed to "simplify the practical solution of a given mechanical problem".

As a further illustration we apply the method of this section to a particular system in which the energy E is not the same as V_{max}, and we use for E the expression (2.7c). We consider a simple pendulum of mass m and length h rotating in a vertical plane. In terms of the variable θ the energy equation of the system is

$$\tfrac{1}{2}mh^2\dot{\theta}^2 + mgh\cos\theta = E. \tag{2.20}$$

The potential energy V is always less than mgh while E can be arbitrarily large. For simplicity we confine our attention to the case when the pendulum is rotating and not oscillating, i.e.

$$E > mgh.$$

We make the substitution

$$z = h\cos\theta$$

so that the energy equation becomes

$$\tfrac{1}{2}\frac{mh^2\dot{z}^2}{h^2 - z^2} + mgz = E. \tag{2.21}$$

The exact solution is straightforward; writing

$$k = \sqrt{\frac{2mgh}{mgh + E}}. \tag{2.22}$$

https://doi.org/10.1017/S144678870002694X Published online by Cambridge University Press

the solution for z, in terms of the Jacobian elliptic functions of modulus k, is

$$\frac{z}{h} = 2sn^2 \left[\frac{t}{k}\sqrt{\frac{g}{h}}\right] - 1. \tag{2.23}$$

The period T, the time average of the Lagrangian $\bar{L}$ and the energy, expressed in terms of the complete elliptic integrals are:

$$T = \frac{2\sqrt{h}}{\sqrt{g}} kK(k) \tag{2.24a}$$

$$\bar{L} = mgh\left\{1 - \frac{2}{k^2} + \frac{4E(k)}{k^2K(k)}\right\} \tag{2.24b}$$

$$E = mgh\left\{\frac{2}{k^2} - 1\right\}. \tag{2.24c}$$

By elimination of the parameter k between (2.24a) and (2.24b) the square of the period may be expressed in terms of $\bar{L}$. This relation is illustrated in fig. 2 by the full line. T^2 may be expressed in terms of a power series expansion in $(\bar{L})^{-1}$. The leading terms are

$$\frac{gT^2}{h^2} = 2\left(\frac{\bar{L}}{mgh}\right)^{-1} - \frac{1}{4}\left(\frac{\bar{L}}{mgh}\right)^{-3} + O\left\{\left(\frac{\bar{L}}{mgh}\right)^{-5}\right\}. \tag{2.25}$$

(2.25) is useful for large energies, i.e., small T^2, large $\bar{L}$ and small k.

So much for the exact solution. Applying the variation calculation, we start with a simple trial function

$$\frac{z}{h} = \cos 2\pi\tau. \tag{2.26}$$

This is clearly the exact solution in the limit of infinite energy. In that limit the potential energy $mgh \cos\theta$ is negligible. Using this function, evaluations of integrals $J_{0,1}$, $J_{0,2}$, $J_{1,0}$ and $J_{1,1}$ leads to a vanishing denominator and finite numerator in the expression for E giving $E = \bar{L} = \infty$ and $T = 0$. These results are in accord with the simple type of trial function (2.26).

We now use a more sophisticated trial function. We bring in a term $\alpha \cos 4\pi$ and a constant term so that z varies between $\pm h$. This leads to

$$\frac{z}{h} = \cos 2\tau - 2\alpha \sin^2 2\tau \qquad |\alpha| \leq \tfrac{1}{4}. \tag{2.27}$$

We expect a high energy approximation for small α. The restriction on the

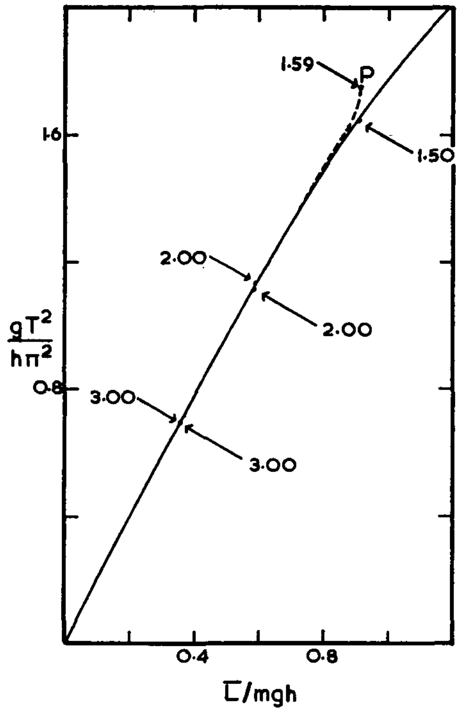

Fig. 2. The square of the period T expressed in terms of the time average of the Lagrangian $\bar{L}$ for the system whose energy equation is (2.21).

The full curve is the exact relation obtained by eliminating k between (2.24a) and (2.24b). The broken curve is the approximation obtained by eliminating α between (2.29a) and (2.29b). The point P corresponds to $\alpha = -0.25$ and the tangent to the curve at P is vertical. The numbers attached to points on the curves are the values of E/mgh derived from (2.24c) and (2.29c) which correspond to the appropriate values of T^2 and $\bar{L}$. It should be noted that adjacent points on the two curves do *not* correspond to the same value of E. This is a consequence of the fact that the expression for T^2 is variationally correct with respect to $\bar{L}$, but the expression for E is *not* variationally correct.

magnitude of α is necessary; if $|\alpha| > \frac{1}{4}$ (2.27) gives a motion in which $|z/h|$ exceeds 1.

The integrals $J_{0,1}$, $J_{0,2}$, $J_{1,0}$ and $J_{1,1}$ are elementary but lengthy and are each functions of α of definite parity. They are

$$\begin{aligned} &J_{0,1}(\alpha) = -mgh\alpha \\ &J_{0,2}(\alpha) = m^2g^2h^2[\tfrac{3}{2}\alpha^2 + \tfrac{1}{2}] \\ (2.28)\quad &J_{1,0}(\alpha) = \frac{m\pi^2h^2}{2\alpha}[16\alpha - (1+4\alpha)^{3/2} + (1-4\alpha)^{3/2}] \qquad |\alpha| \leqq \tfrac{1}{4} \\ &J_{1,1}(\alpha) = \frac{m^2g\pi^2h^2}{2\alpha}[2 + 16\alpha^2 - (1-4\alpha)^{3/2} - (1+4\alpha)^{3/2}] \qquad |\alpha| \leqq \tfrac{1}{4}. \end{aligned}$$

In terms of these expressions, the quantities T, $\bar{L}$ and E are given by

$$T^2(\alpha) = \frac{J_{1,0}(\alpha)}{E(\alpha) - J_{0,1}(\alpha)} \tag{2.29a}$$

$$\bar{L}(\alpha) = E(\alpha) - 2J_{0,1}(\alpha) \tag{2.29b}$$

$$E(\alpha) = \frac{J_{1,0}J_{0,2} - J_{0,1}J_{1,1}}{J_{1,0}J_{0,1} - J_{1,1}} \tag{2.29c}$$

respectively.

By eliminating α between (2.29a) and (2.29b), the square of the period T^2 may be expressed in terms of $\bar{L}$. The expansion for T^2 in terms of $(\bar{L})^{-1}$ differs from the corresponding exact expansion (2.25) only by terms in $(\bar{L}/mgh)^{-5}$ and higher negative powers. In Fig. 2, T^2 is plotted against $\bar{L}$ for values of α between 0 and -0.25. Higher values of α, as mentioned above, correspond to unrealistic motions and in fact lead to divergent integrals. Reference to fig. 2 indicates that the largest error in T as calculated in this way is only one fortieth of the true value of T. Over nearly all the range of α, the proportional error is very much smaller.

3. The Method of Variation of Parameters

It is not necessary to make a definite guess $f(\tau)$ at the solution $q(\tau)$ of the equations of motion, before using the variation principle for the period. Rather, we may introduce some parameters into the "trial function" $f(\tau)$, and use the variation principle itself to determine the best values of the parameters.

This process is best explained by example. Instead of (2.16), let us choose a more complicated form of trial function which includes the next (third) harmonic term; that is let us choose

$$f(\tau) = q_{\max}[(1+\gamma)\sin(2\pi\tau) + \gamma\sin(6\pi\tau)]. \tag{3.1}$$

This contains two parameters, $q_{\max}$ and γ. We need one parameter, let us say $q_{\max}$, in order to keep the average Lagrangian $\bar{L}$ constant during the variation. The other parameter, γ, is free, however, and its actual value is determined by the condition that the period computed from (2.5) is an extremum.

Straightforward computation gives formulas completely analogous to (2.14) and (2.18), except that C_1, C_2, and C_3 are now all functions of γ, namely

$$C_1(\gamma) = \frac{\pi^2[(1+\gamma)^2 + 9\gamma^2]}{1 - G(\gamma)} \tag{3.2}$$

https://doi.org/10.1017/S144678870002694X Published online by Cambridge University Press

$$C_2(\gamma) = 1 - 2G(\gamma) \tag{3.3}$$

$$C_3(\gamma) = C_1(\gamma)\sqrt{C_2(\gamma)} \tag{3.4}$$

where

$$G(\gamma) = \tfrac{3}{8}(1+\gamma)^4 - \tfrac{1}{2}\gamma(1+\gamma)^3 + \tfrac{3}{2}\gamma^2(1+\gamma)^2 + \tfrac{3}{8}\gamma^4. \tag{3.5}$$

According to (2.18a), the condition for an extremum of the period T, keeping L constant, is simply that C_3 should have an extremum value, i.e.,

$$\frac{dC_3}{d\gamma} = 0. \tag{3.6}$$

This condition determines the "best" value of γ in (3.1). In Fig. 3, we

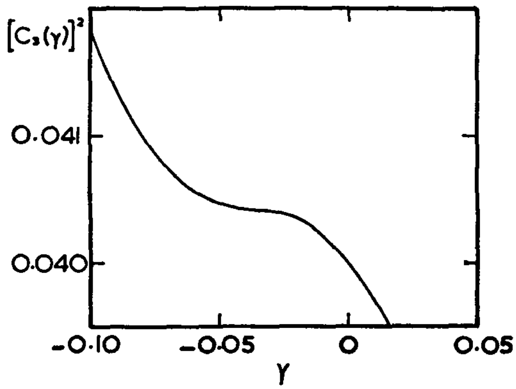

Fig. 3. C_3 as a function of γ using expressions (3.2) to (3.5).

show the function $C_3(\gamma)$ in the neighbourhood of $\gamma = 0$; it is apparent that there is an extremum at

$$\gamma = -0.0350. \tag{3.7}$$

The corresponding value of C_3 is

$$C_3 = 7.9365. \tag{3.8}$$

This must be compared with the exact value given in (2.19), $C_3 = 7.9388$. The error is therefore only 0.03 percent whereas the simpler trial function (2.16) gave $\frac{1}{2}$ percent error in T^2.

The value of the parameter γ is of the right order of magnitude, but not as close as the value of the period. Straightforward (though tedious) Fourier expansion of the exact solution gives

$$\begin{aligned} q = q_{\max}\,\{&0.955008 \sin(2\pi\tau) - 0.043049 \sin(6\pi\tau) + 0.001860 \sin(10\pi\tau) \\ &+ 0.000080 \sin(14\pi\tau) + 0.000003 \sin(18\pi\tau) + \cdots\}. \end{aligned} \tag{3.9}$$

https://doi.org/10.1017/S144678870002694X Published online by Cambridge University Press

Thus the true value of γ is close to -0.043, which differs from the value determined variationally, -0.035. This is to be expected, since only the period itself is variationally correct (i.e., has an error of the order of the square of the error in the trial function); incidental parameters, such as γ, have first-order errors.

It should be noted that the extremum in Fig. 3 is neither a maximum nor a minimum, but a horizontal point of inflection. This shows at once that the variation principle (2.5) does *not* provide either an upper or a lower bound for the true period. This is unfortunate for practical purposes: it would help greatly to know the *sign* of the error.

4. Acknowledgement

The authors are grateful to Dr. L. M. Delves for some illuminating discussions and the pointing out of an error in the original manuscript.

References

[1] H. Goldstein: Classical Mechanics, Addison-Wesley Press, Cambridge, Mass., 1960.
[2] J. N. Lyness and J. M. Blatt: The Practical Use of Variation Principles in the Determination of the Stability of Non Linear Systems, J.A.M.S. Vol. II, Part. 2, 153, 1961
[3] R. M. Rosenberg and C. P. Atkinson: Journal of Appl. Mech., 3, 377, 1959.

Applied Mathematics Department,
The University of New South Wales,
Kensington, N.S.W., Australia.