# CERTAIN CONGRUENCE AND QUOTIENT LATTICES RELATED TO COMPLETELY 0-SIMPLE AND PRIMITIVE REGULAR SEMIGROUPS

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G. Lallement [4] has shown that the lattice of congruences, $\Lambda(S)$, on a completely 0-simple semigroup $S$ is semimodular, thus improving G. B. Preston's result [5] that such a lattice satisfies the Jordan–Dedekind chain condition. More recently, J. M. Howie [2] has given a new and more simple proof of Lallement's result using work due to Tamura [9]. The purpose of this note is to extend the semimodularity result to primitive regular semigroups, to establish a theorem relating certain congruence and quotient lattices, and to provide a theorem for congruences on any regular semigroup.

**1. Preliminaries.** If $a$ and $b$ are elements of a lattice $L$, then $a$ is said to *cover* $b$ ($a \succ b$) provided that $a > b$ and $a \geqq c \geqq b$ implies that $a = c$ or $b = c$. The lattice $L$ is called *semimodular* if, whenever $x, y \succ x \wedge y$, then $x \vee y \succ x, y$.

According to Rees [7], every completely 0-simple semigroup is isomorphic to what Clifford and Preston [1] call a Rees $I \times \Lambda$ matrix semigroup $\mathscr{M}^0[G, I, \Lambda, P]$ over a group-with-zero $G^0$ with regular sandwich matrix $P$. The condition of regularity on the $\Lambda \times I$ matrix $P = (p_{\lambda i})$ over $G^0$ is that each row and each column of $P$ contains some nonzero entry.

If $S = \mathscr{M}^0[G, I, \Lambda, P]$ is a completely 0-simple semigroup, then the relation

$$\varepsilon_I = \{(i,j) \in I \times I : p_{\lambda i} = 0 \quad \text{if and only if} \quad p_{\lambda j} = 0\}$$

is an equivalence relation on $I$, as is $\varepsilon_\Lambda$, which is defined on $\Lambda$ in an analogous way. If $e$ is the identity of $G$, the matrix $P$ is called *normal* provided that $i \in J$ implies that there exists $\lambda \in \Lambda$ such that $p_{\lambda i} = p_{\lambda j} = e$ for each $j \in I$ such that $(i,j) \in \varepsilon_I$, and similarly $\lambda \in \Lambda$ implies that there exists $i \in I$ such that $p_{\lambda i} = p_{\mu i} = e$ for each $\mu \in \Lambda$ such that $(\lambda, \mu) \in \varepsilon_\Lambda$. Tamura [9] has shown that any completely 0-simple semigroup is isomorphic to a Rees matrix semigroup $\mathscr{M}^0[G, I, \Lambda, P]$ in which $P$ is normal. Henceforth, it is assumed that $P$ is normal.

A triple $(N, \rho_I, \rho_\Lambda)$ in which $N$ is a normal subgroup of $G$ and $\rho_I$, $\rho_\Lambda$ are equivalence relations on $I$ and $\Lambda$, respectively, such that $\rho_I \subseteq \varepsilon_I$ and $\rho_\Lambda \subseteq \varepsilon_\Lambda$, is called *linked* provided that

(i) $(i,j) \in \rho_I$ and $p_{\lambda i} \neq 0$ imply $p_{\lambda i} p_{\lambda j}^{-1} \in N$,
(ii) $(\lambda, \mu) \in \rho_\Lambda$ and $p_{\lambda i} \neq 0$ imply $p_{\lambda i} p_{\mu i}^{-1} \in N$.

Tamura [9] has shown that there is a map $\Psi$ from the set of linked triples into the set of nontrivial (not universal) congruences on $S$, defined by $((a, i, \lambda), (b, j, \mu)) \in \Psi(N, \rho_I, \rho_\Lambda)$ if and only if $ab^{-1} \in N$, $(i,j) \in \rho_I$, $(\lambda, \mu) \in \rho_\Lambda$. Moreover, there is a map $\Omega$ from the set of nontrivial congruences on $S$ into the set of linked triples, defined by $\Omega(\rho) = (N, \rho_I, \rho_\Lambda)$, where

(i) $ab^{-1} \in N$ if and only if there exist $i, j \in I$ and $\lambda, \mu \in \Lambda$ such that $((a, i, \lambda), (b, j, \mu)) \in \rho$,
(ii) $(i,j) \in \rho_I$ if and only if there exist $a, b \in G$ and $\lambda, \mu \in \Lambda$ such that $((a, i, \lambda), (b, j, \mu)) \in \rho$,

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(iii) $(\lambda, \mu) \in \rho_\Lambda$ if and only if there exist $a, b \in G$ and $i, j \in I$ such that $((a, i, \lambda), (b, j, \mu)) \in \rho$.

Since $\Psi$ and $\Omega$ are mutually inverse, $\Psi$ is a bijection of the set of linked triples onto the set of nontrivial congruences. The congruence $\Psi(N, \rho_I, \rho_\Lambda)$ is denoted by $[N, \rho_I, \rho_\Lambda]$.

Howie [2] has shown in his note that, if $\rho = [N, \rho_I, \rho_\Lambda]$ and $\sigma = [M, \sigma_I, \sigma_\Lambda]$ are congruences on $S$, then

$$\rho \subseteq \sigma \quad \text{if and only if} \quad M \subseteq N,\ \rho_I \subseteq \sigma_I,\ \rho_\Lambda \subseteq \sigma_\Lambda.$$

If $S$ is any regular semigroup, the relation $\theta = \{(\rho, \sigma) \in \Lambda(S) \times \Lambda(S) : \rho | E = \sigma | E\}$, where $E$ is the set of idempotents of $S$, is such that each $\theta$ class is a complete modular sublattice of $\Lambda(S)$ [8].

**2. A congruence for congruences on a regular semigroup.** If $S$ is an inverse semigroup, then the relation $\theta$ defined above on $\Lambda(S)$ is a complete congruence on $\Lambda(S)$ in the sense that $\theta$ is a congruence, $\Lambda(S)/\theta$ is a complete lattice, and $\theta^\natural : \Lambda(S) \to \Lambda(S)/\theta$ is a complete lattice homomorphism [8]. This section will show that this result is true for regular semigroups.

LEMMA 2.1. *Let $S$ be a regular semigroup and $\sigma, \tau \in \Lambda(S)$ such that $\sigma$ separates idempotents. Then $(\sigma \vee \tau, \tau) \in \theta$.*

*Proof.* First consider the relation $h = \{(a, b) \in S \times S : (a\tau, b\tau) \in \mathscr{H}\}$. Then $h$ is an equivalence relation on $S$ and a routine check will reveal that $\mathscr{H}, \tau \subseteq h$. To see that $h | E \subseteq \tau | E$, let $(e, f) \in h | E$. Then $(e\tau, f\tau) \in \mathscr{H}$ and so $e\tau = f\tau$ since $\mathscr{H}$ separates idempotents in $S/\tau$. Hence $(e, f) \in \tau | E$ and so $\tau | E = h | E$.

Now $\sigma$ separates idempotents and so $\sigma \subseteq \mathscr{H}$ [3, Theorem 2.3]. Consequently

$$\tau \subseteq \sigma \vee \tau \subseteq \mathscr{H} \vee \tau \subseteq h \qquad \text{and so} \qquad \tau | E \subseteq (\sigma \vee \tau) | E \subseteq h | E = \tau | E,$$

which implies that $(\sigma \vee \tau, \tau) \in \theta$.

THEOREM 2.2. *If $S$ is a regular semigroup, the relation $\theta$ is a complete congruence on $\Lambda(S)$.*

*Proof.* To show that $\theta$ is a complete congruence on $\Lambda(S)$, it is sufficient to show that, if $(\rho_i, \sigma_i) \in \theta$ for each $i \in J$, an index set, then both $(\bigvee \rho_i, \bigvee \sigma_i)_{i \in J}$ and $(\bigcap \rho_i, \bigcap \sigma_i)_{i \in J}$ belong to $\theta$. The latter is established by [8, Theorem 5.1].

First, to see that $\theta$ is a congruence on $\Lambda(S)$, suppose that $(\rho, \sigma) \in \theta$ and $\tau \in \Lambda(S)$. Let $\lambda = \bigcap \{\eta \in \Lambda(S) : \eta \in \rho\theta = \sigma\theta\}$. Then $\rho/\lambda$, $\sigma/\lambda$ and $(\tau \vee \lambda)/\lambda$ all belong to $\Lambda(S/\lambda)$ and, furthermore, $\rho/\lambda$ and $\sigma/\lambda$ separate idempotents in the regular semigroup $S/\lambda$. Hence

$$(\rho/\lambda \vee (\tau \vee \lambda)/\lambda,\ \sigma/\lambda \vee (\tau \vee \lambda)/\lambda) = ((\rho \vee \tau)/\lambda,\ (\sigma \vee \tau)/\lambda) \in \theta_{S/\lambda}$$

by Lemma 2.1. Hence $(\rho \vee \tau, \sigma \vee \tau) \in \theta$.

Finally, to see that $\theta$ is complete, assume that $(\rho_i, \sigma_i) \in \theta$ for each $i \in J$ and $(e, f) \in \left(\bigvee_{i \in J} \rho_i\right) | E$. Then there exist $x_1, x_2, \ldots, x_n \in S$ and $i_1, \ldots, i_{n+1} \in J$ such that $(e, x_1) \in \rho_{i_1}$, $(x_1, x_2) \in \rho_{i_2}, \ldots$, $(x_n, f) \in \rho_{i_{n+1}}$. Then

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$$(e,f) \in \left(\bigvee_{j=1}^{n+1} \rho_{i_j}\right) \Big| E = \left(\bigvee_{j=1}^{n+1} \sigma_{i_j}\right) \Big| E \subseteq \left(\bigvee_{i \in J} \sigma_i\right) \Big| E.$$

Symmetrically,

$$\left(\bigvee_{i \in J} \sigma_i\right) \Big| E \subseteq \left(\bigvee_{i \in J} \rho_i\right) \Big| E \quad \text{and so} \quad (\vee \rho_i, \vee \sigma_i)_{i \in J} \in \theta.$$

**3. Completely 0-simple and primitive regular semigroups.**

LEMMA 3.1. *Suppose that $\rho = [N, \rho_I, \rho_\Lambda]$ and $\sigma = [M, \sigma_I, \sigma_\Lambda]$ are nontrivial congruences on a completely 0-simple semigroup $S = \mathcal{M}^0[G, I, \Lambda, P]$. Then $(\rho, \sigma) \in \theta$ if and only if $\rho_I = \sigma_I$ and $\rho_\Lambda = \sigma_\Lambda$.*

*Proof.* Assume that $(\rho, \sigma) \in \theta$ and that $(i, j) \in \rho_I$. Then there exists $\lambda \in \Lambda$ such that $p_{\lambda i} \neq 0$ and hence, since $(N, \rho_I, \rho_\Lambda)$ is a linked triple, $((p_{\lambda i}^{-1}, i, \lambda), (p_{\lambda j}^{-1}, j, \lambda)) \in \rho | E = \sigma | E$ and so $(i, j) \in \sigma_I$. Thus $\rho_I \subseteq \sigma_I$ and similarly $\sigma_I \subseteq \rho_I$. Thus $\rho_I = \sigma_I$ and similarly $\rho_\Lambda = \sigma_\Lambda$.

Conversely, if $\rho_I = \sigma_I$ and $\rho_\Lambda = \sigma_\Lambda$, suppose that $((p_{\lambda i}^{-1}, i, \lambda), (p_{\mu j}^{-1}, j, \mu)) \in \rho | E$. Then $(i, j) \in \rho_I = \sigma_I$ and $(\lambda, \mu) \in \rho_\Lambda = \sigma_\Lambda$. Hence $p_{\lambda i} p_{\lambda j}^{-1} \in M$ and $p_{\lambda j} p_{\mu j}^{-1} \in M$, since $(M, \sigma_I, \sigma_\Lambda)$ is linked. Thus $p_{\lambda i} p_{\mu j}^{-1} \in M$ and so $((p_{\lambda i}^{-1}, i, \lambda), (p_{\mu j}^{-1}, j, \mu)) \in \sigma | E$. Thus $\rho | E \subseteq \sigma | E$. Similarly, $\sigma | E \subseteq \rho | E$ and so $(\rho, \sigma) \in \theta$.

If $S$ is any regular semigroup, $S$ is called *$\theta$-reduced* provided that each $\theta$ class is a singleton. A congruence $\rho \in \Lambda(S)$ is called a *$\theta$-reduced congruence* provided that $S/\rho$ is $\theta_{S/\rho}$-reduced.

THEOREM 3.2. *Let $S = S^0$ be a completely 0-simple semigroup. Then $\mu$ (where $\mu$ is the maximum idempotent separating congruence on $S$) is $\theta$-reduced. Hence the natural map of $\Lambda(S/\mu)$ onto $\Lambda(S)/\theta$ is a complete lattice isomorphism.*

*Proof.* Since the identity congruence on $S$ is $[\{1\}, i_I, i_\Lambda]$, where $i_I$ and $i_\Lambda$ are the identity equivalence relations on $I$ and $\Lambda$, then $\mu = [G, i_I, i_\Lambda]$ by Lemma 3.1. Thus the basic group of $S/\mu$ is $\{1\}$. Consequently no two distinct congruences on $S/\mu$ are $\theta_{S/\mu}$-equivalent and so $S/\mu$ is $\theta_{S/\mu}$-reduced. Since $\theta$ is a congruence on $\Lambda(S)$, it follows that $\mu \subseteq \bigvee \{\sigma \in \Lambda(S) : \sigma \in \rho\theta\}$ for each $\rho \in \Lambda(S)$. Let $(\rho, \sigma) \in \theta$ with $\rho = [N, \tau_I, \tau_\Lambda]$ and $\sigma = [M, \tau_I, \tau_\Lambda]$. If, in addition, $\mu \subseteq \rho, \sigma$, then $N = M = G$ and so $\rho = \sigma$. Thus $\rho/\mu \to \rho\theta$ is a complete lattice isomorphism of $\Lambda(S/\mu)$ onto $\Lambda(S)/\theta$.

THEOREM 3.3. *Let $S = S^0$ be a primitive regular semigroup. Then $\Lambda(S)$ is semimodular. Further, $\mu$ is $\theta$-reduced and so $\Lambda(S/\mu)$ and $\Lambda(S)/\theta$ are completely lattice isomorphic.*

*Proof.* By [6, Theorem 1], $S$ is a 0-direct union of completely 0-simple subsemigroups $\{S_i : i \in I\}$. That is, $S$ is the 0-disjoint union of $\{S_i : i \in I\}$ and $S_i S_j = \{0\}$ if $i \neq j$. From this it follows that $\rho \to \prod_{i \in I} (\rho | S_i)$ is an order preserving bijection of $\Lambda(S)$ onto $\prod_{i \in I} \Lambda(S_i)$. Thus $\Lambda(S)$ is semimodular since semimodularity is preserved by direct products. Further, $(\rho, \sigma) \in \theta_S$ if and only if $(\rho | S_i, \sigma | S_i) \in \theta_{S_i}$ for each $i \in I$. Hence

$$\mu = \{(x, y) \in S \times S : (x, y) \in \mu_i \text{ for some } i \in I\},$$

where $\mu_i$ is the maximum idempotent separating congruence on $S_i$, and so $\mu$ is $\theta$-reduced.

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