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Vanishing limit for the three-dimensional incompressible Phan-Thien–Tanner system

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This paper focuses on the vanishing limit problem for the three-dimensional incompressible Phan-Thien–Tanner (PTT) system, which is commonly used to describe the dynamic properties of polymeric fluids. Our purpose is to show the relation of the PTT system to the well-known Oldroyd-B system (with or without damping mechanism). The suitable *a priori* estimates and global existence of strong solutions are established for the PTT system with small initial data. Taking advantage of uniform energy and decay estimates for the PTT system with respect to time t and coefficients a and b, then allows us to justify in particular the vanishing limit for all time. More precisely, we prove that the solution (u, τ) of PTT system with $0 \leqslant b \leqslant Ca$ converges globally in time to some limit $(\widetilde{u}, \widetilde{\tau})$ in a suitable Sobolev space when a and b go to zero simultaneously (or, only b goes to zero). We may check that $(\widetilde{u}, \widetilde{\tau})$ is indeed a global solution of the corresponding Oldroyd-B system. In addition, a rate of convergence involving explicit norm will be obtained. As a byproduct, similar results are also true for the local *a priori* estimates in large norm.

Keywords: Navier–Stokes equations; Phan-Thien–Tanner system; global well-posedness; vanishing limit; convergence rate

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1. Introduction

In this paper, we are interested in only homogeneous flow of incompressible isothermal polymer fluids. The Phan-Thien–Tanner (PTT) model originates from the works of Nhan Phan-Thien and Roger I. Tanner in [44, 45]. This model attempts to describe the behaviour of this complex mixture of polymers and fluids, and as such, it presents numerous challenges, simultaneously at the level of their derivation, that of their numerical simulation and mathematical treatment. This flow is governed by the following continuity and momentum balance equations:

$$\begin{cases} \operatorname{div} u = 0, \\ u_t + u \cdot \nabla u = \operatorname{div} T - \nabla p, \end{cases} \tag{1.1}$$

where u, p and T represent the velocity field, the isotropic pressure and the stress tensor, respectively. The stress tensor T, corresponding to the rates of creation and destruction of junctions which depend on the instantaneous elastic energy of the network, or equivalently, the average extension of the network strand, is often divided into the following two parts:

$$T = \mu_1 \tau + 2\mu D(u),$$

where τ is the polymers contribution to the stress tensor and $D(u) := (1/2)(\nabla u + (\nabla u)^T)$. Two constants $\mu_1 > 0$ and $\mu > 0$ are the elastic coefficient and the solvent viscosity coefficient, respectively.

Throughout this paper, we shall focus on the linear PTT model, which is a well-studied phenomenological constitutive model for polymers. One can deduce that

$$\lambda_1 \frac{\mathcal{D}\tau}{\mathcal{D}t} + \frac{\epsilon\lambda_1}{\eta}(\operatorname{tr}\tau)\tau = 2\eta\left(1 - \frac{\lambda_2}{\lambda_1}\right) D(u),$$

and the 'objective derivative' $\mathcal{D}\tau/\mathcal{D}t$ is denoted as follows:

$$\frac{\mathcal{D}\tau}{\mathcal{D}t} := \frac{\partial \tau}{\partial t} + u \cdot \nabla\tau + \tau\Omega(u) - \Omega(u)\tau + \lambda(D(u)\tau + \tau D(u)),$$

where $\Omega(u) := (1/2)(\nabla u - (\nabla u)^T)$, $\lambda_1 > 0$ is the relaxation time of the fluids, λ_2 $(0 \leqslant \lambda_2 < \lambda_1)$ is the retardation time of the fluids, $\eta > 0$ is the polymer viscosity coefficient, ε is a parameter controlling the elongational viscosity coefficient and λ is a constant that is typically in $[-1, 1]$. In particular, we call the system a co-rotational case when $\lambda = 0$.

For the sake of conciseness, we set $a = 1/\lambda_1$, $b = \varepsilon/\eta$ and $\mu_2 = (2\eta/\lambda_1)(1 - \lambda_2/\lambda_1)$. Two constants $a > 0$ and $\mu_2 > 0$ are associated with the Deborah number $De = \mu_2/a$, which indicates the relation between the characteristic flow time and the elastic time (see [4]). The constant $b \geqslant 0$ is related to the rates of creation and destruction for the polymeric network junctions. Using the notations introduced above, we aim to solve the following three-dimensional incompressible PTT system:

$$\begin{cases} u_t + u \cdot \nabla u - \mu\triangle u + \nabla p = \mu_1 \operatorname{div}\tau, & (t,x) \in \mathbb{R}^+ \times \mathbb{R}^3, \\ \tau_t + u \cdot \nabla\tau + (a + b(\operatorname{tr}\tau))\tau + Q(\tau, \nabla u) = \mu_2 D(u), & (t,x) \in \mathbb{R}^+ \times \mathbb{R}^3, \\ \operatorname{div} u = 0, & (t,x) \in \mathbb{R}^+ \times \mathbb{R}^3, \\ u|_{t=0} = u_0(x),\ \tau|_{t=0} = \tau_0(x), & x \in \mathbb{R}^3. \end{cases} \tag{PTT}$$

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Here, $Q(\tau, \nabla u)$ is a given bilinear form

$$Q(\tau, \nabla u) = \tau\Omega(u) - \Omega(u)\tau + \lambda(D(u)\tau + \tau D(u)).$$

It can be seen that the PTT system is derived from the coupling of the constitutive equation with the incompressible Navier–Stokes equations. The coupling in Navier–Stokes equations comes from an extra stress tensor contributed from the polymers. Without taking into account the rates of creation and destruction for the polymeric network junctions (i.e. $a > 0$ and $b = 0$), then the second equation of system (PTT) is replaced by

$$\tau_t + u \cdot \nabla\tau + a\tau + Q(\tau, \nabla u) = \mu_2 D(u).$$

One reduces to the well-known Oldroyd-B system, which models the motion of some simple types of steady flow of certain idealized elastico-viscous liquids (see [**42**]). Note that there is a new nonlinear term $b(\mathrm{tr}\,\tau)\tau$ in the constitutive equation of system (PTT), which is a big difference from the Oldroyd-B system. The limit system as $(a, b) \to (0, 0)$ formally becomes the following three-dimensional incompressible Oldroyd-B system without damping mechanism:

$$\begin{cases} \widetilde{u}_t + \widetilde{u} \cdot \nabla\widetilde{u} - \mu\triangle\widetilde{u} + \nabla\widetilde{p} = \mu_1 \,\mathrm{div}\,\widetilde{\tau}, & (t,x) \in \mathbb{R}^+ \times \mathbb{R}^3, \\ \widetilde{\tau}_t + \widetilde{u} \cdot \nabla\widetilde{\tau} + Q(\widetilde{\tau}, \nabla\widetilde{u}) = \mu_2 D(\widetilde{u}), & (t,x) \in \mathbb{R}^+ \times \mathbb{R}^3, \\ \mathrm{div}\,\widetilde{u} = 0, & (t,x) \in \mathbb{R}^+ \times \mathbb{R}^3, \\ \widetilde{u}|_{t=0} = u_0(x),\ \widetilde{\tau}|_{t=0} = \tau_0(x), & x \in \mathbb{R}^3. \end{cases} \tag{OB}$$

The main objective of this paper is to show the relation of the PTT system to the Oldroyd-B system (with or without damping mechanism). Until now, this question actually remains open. In the sequence, we will give an overview on the study for the PTT and Oldroyd-type models.

To begin, we review some mathematical works dedicated to the PTT model. To the best of our knowledge, it is universally known that the PTT model has been widely studied in numerical analysis (see [**2**, **16**, **43**]). The first mathematical work devoted to the PTT system is by Masmoudi [**41**]. There, the global existence of weak solutions was proven. Recently, Chen *et al.* [**8**] proved the global existence of strong solutions for the periodic PTT system without damping mechanism (i.e. $a = 0$ and $b \geqslant 0$). In a more general context, the global well-posedness of the Cauchy problem for the linear and generalized PTT systems in the whole space was achieved by Chen *et al.* [**6**, **9**] in the critical Besov spaces and [**7**, **10**] in Sobolev spaces. In this paper, we would like to study in a sequel the limit process of the PTT system with $0 \leqslant b \leqslant Ca$ when coefficients a and b go to 0 simultaneously (or, only b goes to 0) and aim to prove in particular that the obtained limit is the solution of the corresponding Oldroyd-B system.

We also review some mathematical works dedicated to the Oldroyd-type models. For all we know, the related Oldroyd-type models have been analysed extensively since 1990. The main breakthrough of the Oldroyd-B system was made by Guillopé and Saut [**17**, **18**]. In the above studies, they proved the local existence of strong solutions and the global existence of one-dimensional shear flows. In the multi-dimensional case, the local well-posedness in Sobolev spaces was first proven by Fernández-Cara *et al.* [**15**]. Later, Chemin and Masmoudi [**5**] not only proved the local well-posedness in the critical Besov spaces, but gave a low bound for the lifespan as well. Then, Feng *et al.* and Lei *et al.* gave some blow up criteria in [**14**, **31**]. In [**39**], the global existence of weak solutions for general initial data in the co-rotational system was studied by Lions and Masmoudi. In regard to strong solutions, the global existence for small smooth perturbations of a stable equilibrium was first presented in [**36**]. Surprisingly, there are a number of studies concerning the global existence of strong solutions of the Cauchy problem with small initial data by using different methods, for example, see [**11**, **32**, **33**, **36**, **52**, **54**] and the references cited therein. In particular, the global existence for the Oldroyd-B system without damping mechanism was discussed in [**54**]. We should mention here that He and Xu [**19**], Lin and Zhang [**37**] and Sun and Zhang [**48**] have obtained similar results involving the initial-boundary problem of the incompressible viscoelastic fluid models. For a class of large initial data, Fang and Zi [**13**] and Jiang and Jiang [**24**] made some contributions to the global existence of strong solutions. The optimal time decay estimates have been established recently in [**20**] for the incompressible Oldroyd-B system and similar results in [**50**] for the compressible case. For more results concerning the related compressible viscoelastic fluids models, one may refer to [**1**, **22**, **23**, **40**, **46**, **53**]. It has been shown that the weak solutions and strong solutions exist globally in time.

In addition, we recall some results related to the limit problems. There has been much progress on the limit problems of various flow models in the last few decades, especially on incompressible limit. The mathematical studies of incompressible limit were stated in [**28**, **38**] for Navier–Stokes equations and in [**21**, **25**] for magnetohydrodynamic equations. As for the Oldroyd-B system, incompressible limit problem was investigated in [**30**, **33**]. Besides, Ju *et al.* investigated the quasineutral limit for the compressible Navier–Stokes–Poisson equations (for both isentropic and non-isentropic cases) in [**26**, **27**]. As in the previous works, the existence of global classical (or weak) solutions for the incompressible case was shown via incompressible limit. More results on the other limit problems of various flow models may be found in [**3**, **12**, **29**, **34**, **35**, **49**, **51**]. Heuristically, we hope that the process of the vanishing limit for the PTT system selects a physical relevant solution of the corresponding Oldroyd-B system.

In this paper, we aim to establish the vanishing limit of the PTT system for incompressible fluids. Let $0 \leqslant a \leqslant 1$ and $0 \leqslant b \leqslant Ca$, under the smallness assumptions on the initial data, our first aim is to prove the global existence of strong solutions for the PTT system with the *a priori* estimates, uniformly with respect to time t and coefficients a and b. Our second purpose is to prove the global existence for the limit system via the vanishing limit. Taking advantage of the uniform energy and decay estimates stated earlier, it is possible to pass to the limit when a and b go to 0 simultaneously (or, only b goes to 0). This result justifies that the limit is indeed a strong solution of the corresponding Oldroyd-B system and has the required regularity. In addition, we investigate the rate of convergence of the PTT system towards the corresponding Oldroyd-B system for solutions having the above regularity. We shall see that the rate strongly depends on time t and coefficients a and b. Moreover, we prove the local *a priori* estimates in large norm. A similar result holds true for some time interval $[0, T]$. As a consequence, this is the first paper devoted to establishing a relation between the PTT system and the Oldroyd-B system with or without damping mechanism.

Notation: We shall introduce some notations. Unless otherwise indicated, we omit $\mathbb{R}^3$ in the rest of the proof. For the sake of simplicity, we restrict our attention here to the case where $\mu = \mu_1 = \mu_2 = 1$. Throughout this paper, C denotes a generic positive constant independent of t, a and b.

We can now describe our main results. We first state the following theorem, the global well-posedness for system (PTT) with small initial data. In passing, we will establish suitable *a priori* estimates, uniformly in both time t and coefficients a and b.

THEOREM 1.1. *Let* $0 \leqslant a \leqslant 1$ *and* $0 \leqslant b \leqslant Ca$. *Assume that the initial data* $(u_0,\, \tau_0)$ *satisfies* $\operatorname{div} u_0 = 0$, $(\tau_0)_{ij} = (\tau_0)_{ji}$ *and* $(|\nabla|^{-1}u_0,\, |\nabla|^{-1}\tau_0) \in H^3(\mathbb{R}^3)$. *There exists a sufficiently small positive constant* δ_0, *such that if*

$$\mathcal{E}(0) := \|(|\nabla|^{-1}u_0, |\nabla|^{-1}\tau_0)\|_{H^3(\mathbb{R}^3)}^2 \leqslant \delta_0,$$

then the problem (PTT) *admits a unique global classical solution* $(u,\, \tau)$, *which satisfies*

$$\begin{aligned}&\sup_{0\leqslant s\leqslant t}\Big(\||\nabla|^{-1}u(s)\|_{H^3(\mathbb{R}^3)}^2 + \||\nabla|^{-1}\tau(s)\|_{H^3(\mathbb{R}^3)}^2\Big)\\ &\quad + \int_0^t \Big(\|u(s)\|_{H^3(\mathbb{R}^3)}^2 + a\||\nabla|^{-1}\tau(s)\|_{H^3(\mathbb{R}^3)}^2\Big)\,\mathrm{d}s \leqslant C_1\mathcal{E}(0), \quad \forall\, t \geqslant 0,\end{aligned}$$

where C_1 *is a positive constant independent of* t, a *and* b. *Moreover, the following time decay estimates hold true*:

$$\begin{aligned}&\|\nabla u(t)\|_{H^1(\mathbb{R}^3)}^2 + \|\mathbb{P}\operatorname{div}\tau(t)\|_{H^1(\mathbb{R}^3)}^2 \leqslant C_1\mathcal{E}(0)(1+t)^{-2}, \quad \forall\, t \geqslant 0,\\ &\|\operatorname{tr}\tau(t)\|_{H^2(\mathbb{R}^3)}^2 \leqslant C_1\mathcal{E}(0)\,\mathrm{e}^{-at}, \quad \forall\, t \geqslant 0.\end{aligned}$$

REMARK 1.2. We should mention that the assumption $0 \leqslant a \leqslant 1$ of theorem 1.1 is reasonable. In this paper, one of the questions to consider is the vanishing limit for system (PTT) when a tends to zero, so here we assume that $0 \leqslant a \leqslant 1$. The results of the case $a = 0$ were proven in some earlier works, see [**7**, **8**, **10**, **54**] for more details.

REMARK 1.3. We should also point out that $0 \leqslant b \leqslant Ca$ is a reasonable technical assumption for our limiting problem. In the case $a = 0$ and $b > 0$, we see that the solution of system (PTT) will blow up in finite time if the initial data $\operatorname{tr}\tau_0(x) < 0$, which have been discussed in [**7**, **8**, **10**]. Indeed, the authors in [**8**] only have to consider the initial data $\operatorname{tr}\tau_0(x) > 0$ in the periodic domain $\mathbb{T}^3$. If we assume that $\tau_0 \in H^s(\mathbb{R}^3)$, then $\lim_{x\to+\infty}\tau_0(x) = 0$ which contracts to $\operatorname{tr}\tau_0(x) > 0$. Moreover, under the assumption that $\operatorname{tr}\tau_0(x) > 0$, the authors in [**7**, **10**] obtained the global existence of strong solutions provided that the initial data are close to a nontrivial particular solution (depending only on t) in the whole space $\mathbb{R}^3$. It makes no sense to consider the global well-posedness for system (PTT) with small initial data $(|\nabla|^{-1}u_0,\, |\nabla|^{-1}\tau_0) \in H^3(\mathbb{R}^3)$ in the case $a = 0$ and $b > 0$.

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REMARK 1.4. Compared with [7, 10, 54], here our eventual goal is to prove the convergence of the solution (u, τ) to system (PTT) with $0 \leqslant b \leqslant Ca$ when coefficients a and b tend to zero simultaneously (or, only b tends to zero). To prove this, we shall establish the uniform estimates for system (PTT) with respect to t, a and b.

REMARK 1.5. Actually, besides the assumptions of theorem 1.1, we assume that $(u_0, \tau_0) \in L^1(\mathbb{R}^3)$. For any fixed constant coefficients $a \geqslant 0$ and $0 \leqslant b \leqslant Ca$, we deduce not only that there exists a unique global classical solution (u, τ), but that the following time decay estimates hold true:

$$\|\nabla u(t)\|_{H^1(\mathbb{R}^3)} + \|\mathbb{P}\,\mathrm{div}\,\tau(t)\|_{H^1(\mathbb{R}^3)} \leqslant C_1(1+t)^{-(5/4)}, \quad \forall\, t \geqslant 0,$$

where C_1 is a positive constant which does not depend on t, a and b. This conclusion may be proved by using standard energy method, the proof of which is similar to that of this paper. We omit the details, since this is not what we focus on in this paper.

REMARK 1.6. Furthermore, if we also assume that $(u_0, \tau_0) \in L^1(\mathbb{R}^3)$. For any fixed constant coefficients $a > 0$ and $0 \leqslant b \leqslant Ca$, we can infer that the following optimal time decay estimates, for $k = 0, 1, 2$:

$$\|\nabla^k u(t)\|_{L^2(\mathbb{R}^3)} + \|\nabla^k \tau(t)\|_{L^2(\mathbb{R}^3)} \leqslant C_2(1+t)^{-(3/4)-(k/2)}, \quad \forall\, t \geqslant 0,$$

where C_2 is a positive constant independent of t, but may depend on a and b. The proof is similar to that of the work in [20].

We are concerned with the vanishing limit for system (PTT) in the sequel. According to theorem 1.1, we shall prove the convergence of the solution (u, τ) to system (PTT) with $0 \leqslant b \leqslant Ca$ when (a, b) tends to $(0, 0)$. The aim is to show that when (a, b) tends to $(0, 0)$, the global solution (u, τ) of system (PTT) with $0 \leqslant b \leqslant Ca$ converges to some limit solution $(\widetilde{u}, \widetilde{\tau})$, which is a global solution of system (OB).

THEOREM 1.7. *For any $T > 0$, let (u, τ) be a strong solution of system (PTT) in the time interval $[0, T]$ given by theorem 1.1. Then there exists $(\widetilde{u}, \widetilde{\tau})$ such that for $0 \leqslant b \leqslant Ca$, if $(a, b) \to (0, 0)$, it holds true for any $s \in (0, 1/2)$,*

$$|\nabla|^{-1} u \to |\nabla|^{-1}\widetilde{u} \quad \text{strongly in } \mathcal{C}([0, T]; H^{3-s}_{loc}(\mathbb{R}^3)),$$
$$|\nabla|^{-1} \tau \to |\nabla|^{-1}\widetilde{\tau} \quad \text{strongly in } \mathcal{C}([0, T]; H^{3-s}_{loc}(\mathbb{R}^3)).$$

Moreover, $(\widetilde{u}, \widetilde{\tau})$ is a strong solution of system (OB) in the time interval $[0, T]$, and satisfies

$$|\nabla|^{-1}\widetilde{u} \in \mathcal{C}([0, T]; H^3(\mathbb{R}^3)) \cap \mathcal{C}^1([0, T]; H^1(\mathbb{R}^3)), \quad \widetilde{u} \in L^2((0, T); H^3(\mathbb{R}^3)),$$
$$|\nabla|^{-1}\widetilde{\tau} \in \mathcal{C}([0, T]; H^3(\mathbb{R}^3)) \cap \mathcal{C}^1([0, T]; H^2(\mathbb{R}^3)).$$

In addition, there exists a positive constant C_1, independent of t, a and b, such that

$$\||\nabla|^{-1}\widetilde{u}(t)\|^2_{H^3(\mathbb{R}^3)} + \||\nabla|^{-1}\widetilde{\tau}(t)\|^2_{H^3(\mathbb{R}^3)} + \int_0^t \|\widetilde{u}(s)\|^2_{H^3(\mathbb{R}^3)}\,\mathrm{d}s \leqslant C_1\mathcal{E}(0), \quad \forall\, t \geqslant 0.$$

Finally, we investigate the rate of convergence of system (PTT) towards system (OB) for solutions having the above regularity. We shall see that the rate strongly depends on t, a and b.

THEOREM 1.8. *Let* (u, τ) *be a strong solution of system* (PTT) *given by theorem* 1.1 *and* $(\widetilde{u}, \widetilde{\tau})$ *be a strong solution of system* (OB) *given by theorem* 1.7*. Then there exists a positive constant* C_1*, independent of* t*,* a *and* b*, such that*

$$\|(u-\widetilde{u}, \tau-\widetilde{\tau})(t)\|_{H^1(\mathbb{R}^3)}^2 \leqslant C_1\left(a^2+b^2\right) t\, \mathrm{e}^{C_1(t+1)}, \quad \forall\ t \geqslant 0.$$

REMARK 1.9. On the one hand, there is no derivative in the additional term $b(\operatorname{tr}\tau)\tau$ of system (PTT), in contrast to the Oldroyd-B system. Then the proof of local well-posedness for system (PTT) is similar to that of the Oldroyd-B system. For any fixed constant coefficients $0 \leqslant a \leqslant 1$ and $0 \leqslant b \leqslant Ca$, let (u, τ) be a local solution of system (PTT) in the time interval $[0, T]$ with the general initial data, there exists $(\widetilde{u}, \widetilde{\tau})$ be a local solution of system (OB) in the time interval $[0, T]$ if $(a, b) \to (0, 0)$. Further, then there exist some $T > 0$ and a positive constant $\widetilde{C}_1$, independent of a and b, but may depend on T, such that for any $0 \leqslant t \leqslant T$, we have

$$\|(u-\widetilde{u}, \tau-\widetilde{\tau})(t)\|_{H^1(\mathbb{R}^3)}^2 \leqslant \widetilde{C}_1\left(a^2+b^2\right).$$

The proof of the above estimate may be obtained by arguing as in theorem 1.8, which will not be explained in detail in this paper. In fact, it turns out to be a simpler problem.

REMARK 1.10. We notice that the PTT system reduces to the well-known Oldroyd-B system when $b = 0$. It goes without saying that similar results may be obtained for $a > 0$ and only set $b \to 0$. It is more of a physical significance with damping term no matter whether it is the PTT system or the Oldroyd-B system. In addition, we may get a better rate of convergence, the proof of which is actually quite similar to that of theorems 1.1, 1.7 and 1.8, because the nonlinear term $b(\operatorname{tr}\tau)\tau$ of the PTT system can be controlled by the damping term $a\tau$.

In the last part of this section, we present the main ideas of the proofs of theorems 1.1, 1.7 and 1.8. From the point of view of the limit system (OB), it will be difficult to establish the *a priori* estimates, uniformly in both time t and coefficients a and b, because system (OB) (i.e. $a = 0$ and $b = 0$) loses damping influence about τ. In order to overcome this difficulty, motivated by the result presented in [**54**], we study the following linearized system:

$$\begin{cases} u_t - \triangle u = \mathbb{P}\operatorname{div}\tau, \\ \tau_t = D(u), \end{cases}$$

where $\mathbb{P} := \mathbb{I} - \triangle^{-1}\nabla\operatorname{div}$ is the Leray projection operator. We decouple the linearized system and find that both u and $\mathbb{P}\operatorname{div}\tau$ satisfy the following damped wave equation:

$$W_{tt} - \triangle W_t - \frac{1}{2}\triangle W = 0.$$

Therefore, we could obtain the time decay estimates of u and $\mathbb{P}\operatorname{div}\tau$, which are useful to deal with the nonlinear terms.

More specifically, the standard energy method be performed in system (PTT) then gives a bound on

$$\||\nabla|^{-1}u\|_{L^\infty_t(H^3)} + \||\nabla|^{-1}\tau\|_{L^\infty_t(H^3)} + \|u\|_{L^2_t(H^3)} + a\||\nabla|^{-1}\tau\|_{L^2_t(H^3)}.$$

Due to the presence of coefficient a in the dissipative term for τ, the standard energy estimates above alone is not enough to prove the global-in-time estimates, uniformly in coefficients a and b. Indeed, the nonlinear terms $u\cdot\nabla u$, $u\cdot\nabla\tau$, $Q(\tau, \nabla u)$ and $b(\mathrm{tr}\,\tau)\tau$ cannot be dealt with here. For this, we use the equation for u in system (PTT) to get the dissipation estimate for $\mathbb{P}\,\mathrm{div}\,\tau$. In fact, due to the loss of the dissipation estimate for τ, we will establish the time decay estimates of u and $\mathbb{P}\,\mathrm{div}\,\tau$ to deal with the nonlinear terms $u\cdot\nabla u$, $u\cdot\nabla\tau$ and $Q(\tau, \nabla u)$. It suffices to prove a bound on

$$\begin{aligned}&\sup_{0\leqslant s\leqslant t}(1+s)^2\left(\|\nabla u(s)\|_{H^1}^2 + \|\mathbb{P}\,\mathrm{div}\,\tau(s)\|_{H^1}^2\right)\\&\quad+\int_0^t(1+s)^2\left(\|\nabla^2 u(s)\|_{H^1}^2 + \|\nabla\mathbb{P}\,\mathrm{div}\,\tau(s)\|_{L^2}^2 + a\|\mathbb{P}\,\mathrm{div}\,\tau(s)\|_{H^1}^2\right)\,\mathrm{d}s.\end{aligned}$$

Note that, however, it does not seem possible to handle the nonlinear term $b(\mathrm{tr}\,\tau)\tau$. This is another difficult task. To achieve this, we may use a method as in [**7**, **8**, **10**]. We shall make full use of the structure of the constitutive equation of system (PTT) to obtain the time decay estimate of the unknown good quantity $\mathrm{tr}\,\tau$. As a consequence, we prove the global-in-time estimates for system (PTT) with small initial data, uniformly in coefficients a and b (see theorem 1.1). Moreover, we prove the convergence of the global solution (u, τ) to system (PTT) with $0\leqslant b\leqslant Ca$ when $(a, b)\to(0, 0)$, and the global solution $(\widetilde{u}, \widetilde{\tau})$ is obtained by passing to the limit. Hence, $(\widetilde{u}, \widetilde{\tau})$ be a global solution of system (OB) (see theorem 1.7). In addition, a rate of convergence involving explicit norms will be obtained by using energy estimate (see theorem 1.8). On the one hand, according to the local well-posedness for the PTT system with general initial data, we also obtain the similar results (see remark 1.9). We mention in passing that in the case $a > 0$ and only set $b\to 0$, similar results are also true (see remark 1.10).

The rest of this paper is arranged as follows. In §2, we recall a few basic lemmas. Section 3 is devoted to proving the suitable *a priori* estimates for the PTT system, which will be useful for proving our main results. The proofs of theorems 1.1, 1.7 and 1.8 are presented in § 4.

2. Preliminary

In this section, we recall some very basic facts. The first lemma has been proven in [**8**, **54**]; the details of the proof are omitted.

LEMMA 2.1 [**8**, **54**]. *For any smooth tensor field $[\tau_{ij}]_{3\times 3}$ and three-dimensional vector field u, it always holds that*

$$\mathbb{P}\,\mathrm{div}(u\cdot\nabla\tau) = \mathbb{P}(u\cdot\nabla\mathbb{P}\,\mathrm{div}\,\tau) + \mathbb{P}(\nabla u\cdot\nabla\tau) - \mathbb{P}(\nabla u\cdot\nabla\triangle^{-1}\,\mathrm{div}\,\mathrm{div}\,\tau),$$

$$\mathbb{P}\,\mathrm{div}((\mathrm{tr}\,\tau)\tau) = \mathbb{P}((\mathrm{tr}\,\tau)\mathbb{P}\,\mathrm{div}\,\tau) + \mathbb{P}(\tau\cdot\nabla(\mathrm{tr}\,\tau)) - \mathbb{P}(\nabla(\mathrm{tr}\,\tau)\triangle^{-1}\,\mathrm{div}\,\mathrm{div}\,\tau),$$

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where

$$(\nabla u \cdot \nabla \tau)_i = \sum_j \partial_j u \cdot \nabla \tau_{ij}, \quad (\nabla u \cdot \nabla \triangle^{-1} \operatorname{div}\operatorname{div}\tau)_i = \partial_i u \cdot \nabla \triangle^{-1} \operatorname{div}\operatorname{div}\tau.$$

We introduce the following so-called Aubin–Lions lemma, which will be useful for proving theorem 1.7.

LEMMA 2.2 [**47**]. *Assume that $X \subset Y \subset Z$ are Banach spaces and $X \subset\subset Y$, then the following embeddings are compact:*

$$\{f | f \in L^q((0,T);X), f_t \in L^1((0,T);Z)\} \subset \subset L^q((0,T);Y), \quad \text{if } 1 \leqslant q \leqslant \infty,$$
$$\{f | f \in L^\infty((0,T);X), f_t \in L^r((0,T);Z)\} \subset \subset C([0,T];Y), \quad \text{if } 1 < r \leqslant \infty.$$

3. *Apriori* estimates

In this section, we want to send (a, b) to $(0, 0)$ in system (PTT) with $0 \leqslant a \leqslant 1$ and $0 \leqslant b \leqslant Ca$. For this, we shall need some uniform estimates for smooth solutions with respect to a and b. Before going into further detail, we give some basic energies and time-weighted energies as follows:

$$\begin{aligned}
\mathcal{E}(0) &:= \||\nabla|^{-1}u_0\|_{H^3}^2 + \||\nabla|^{-1}\tau_0\|_{H^3}^2, \\
\mathcal{E}_1(t) &:= \sup_{0\leqslant s\leqslant t} \left(\||\nabla|^{-1}u(s)\|_{H^3}^2 + \||\nabla|^{-1}\tau(s)\|_{H^3}^2\right) \\
&\quad + \int_0^t \left(\|u(s)\|_{H^3}^2 + \||\nabla|^{-1}\mathbb{P}\operatorname{div}\tau(s)\|_{H^2}^2 + a\||\nabla|^{-1}\tau(s)\|_{H^3}^2\right) \mathrm{d}s, \\
\mathcal{E}_2(t) &:= \sup_{0\leqslant s\leqslant t} (1+s)^2 \left(\|\nabla u(s)\|_{H^1}^2 + \|\mathbb{P}\operatorname{div}\tau(s)\|_{H^1}^2\right) \\
&\quad + \int_0^t (1+s)^2 \left(\|\nabla^2 u(s)\|_{H^1}^2 + \|\nabla\mathbb{P}\operatorname{div}\tau(s)\|_{L^2}^2 + a\|\mathbb{P}\operatorname{div}\tau(s)\|_{H^1}^2\right) \mathrm{d}s, \\
\mathcal{E}_3(t) &:= \sup_{0\leqslant s\leqslant t} \left(\mathrm{e}^{as}\|\operatorname{tr}\tau(s)\|_{H^2}^2\right).
\end{aligned}$$

We begin by proving the *a priori* estimates for smooth solutions. The following proposition 3.1 is useful for proving the existence theorem (see theorem 1.1). There, the positive constant C^* that we obtained below does not depend on t, a and b, so that we may pass to the limit in system (PTT) with $0 \leqslant b \leqslant Ca$ when (a, b) goes to $(0, 0)$, which then allows us to get a solution of system (OB) (see theorem 1.7). Moreover, a rate of convergence involving explicit norms will be obtained (see theorem 1.8).

PROPOSITION 3.1. *Suppose that (u, τ) is a solution of system* (PTT) *in the time interval $[0, T]$ and satisfies the assumptions of theorem* 1.1. *There exists a positive* *constant* δ_0*, such that if*

$$\mathcal{E}_1(t)+\mathcal{E}_2(t)+\mathcal{E}_3(t)\leqslant 4C^*\delta_0,\quad \forall\ t\in[0,T], \tag{3.1}$$

then

$$\mathcal{E}_1(t)+\mathcal{E}_2(t)+\mathcal{E}_3(t)< 4C^*\delta_0,\quad \forall\ t\in[0,T],$$

where C^* *is a positive constant independent of* t*,* a *and* b*.*

3.1. Estimate of $\mathcal{E}_1(t)$

First of all, we shall prove the following standard energy estimates.

LEMMA 3.2. *Under the assumptions of proposition 3.1, it holds that*

$$\mathcal{E}_{11}(t)\leqslant \mathcal{E}(0)+C\left(\mathcal{E}_1^{3/2}(t)+\mathcal{E}_2^{3/2}(t)\right), \tag{3.2}$$

where C *is a positive constant independent of* t*,* a *and* b*, and*

$$\begin{aligned}\mathcal{E}_{11}(t):=&\sup_{0\leqslant s\leqslant t}\left(\||\nabla|^{-1}u(s)\|_{H^3}^2+\||\nabla|^{-1}\tau(s)\|_{H^3}^2\right)\\&+2\int_0^t\left(\|u(s)\|_{H^3}^2+a\||\nabla|^{-1}\tau(s)\|_{H^3}^2\right)\mathrm{d}s.\end{aligned} \tag{3.3}$$

Proof. Applying the operator $|\nabla|^{-1}\nabla^k$ ($k=0,\,1,\,2,\,3$) to system (PTT), and taking the L^2 inner product with $|\nabla|^{-1}\nabla^k u$ and $|\nabla|^{-1}\nabla^k\tau$, respectively, we find that

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\||\nabla|^{-1}u\|_{H^3}^2+\||\nabla|^{-1}\tau\|_{H^3}^2\right)+\|u\|_{H^3}^2+a\||\nabla|^{-1}\tau\|_{H^3}^2\\&=\sum_{k=0}^3\int\left(\nabla^k|\nabla|^{-1}(\operatorname{div}\tau)\cdot\nabla^k|\nabla|^{-1}u+\nabla^k|\nabla|^{-1}(D(u))\cdot\nabla^k|\nabla|^{-1}\tau\right)\mathrm{d}x\\&\quad-b\sum_{k=0}^3\int\nabla^k|\nabla|^{-1}\left((\operatorname{tr}\tau)\tau\right)\cdot\nabla^k|\nabla|^{-1}\tau\mathrm{d}x\\&\quad-\sum_{k=0}^3\int\nabla^k|\nabla|^{-1}\left(u\cdot\nabla u\right)\cdot\nabla^k|\nabla|^{-1}u\mathrm{d}x\\&\quad-\sum_{k=0}^3\int\nabla^k|\nabla|^{-1}\left(u\cdot\nabla\tau+Q(\tau,\nabla u)\right)\cdot\nabla^k|\nabla|^{-1}\tau\,\mathrm{d}x\\&:=I_1+I_2+I_3+I_4.\end{aligned} \tag{3.4}$$

According to $\operatorname{div}u=0$ and $\tau_{ij}=\tau_{ji}$, and integrating by parts in the first term, we get $I_1=0$. Using the facts that $\operatorname{div}u=0$, $\tau_{ij}=\tau_{ji}$ and $0\leqslant b\leqslant Ca$, and integrating

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by parts, we deduce that

$$\int_0^t |I_2|\,\mathrm{d}s \leqslant Cb\int_0^t \||\nabla|^{-1}\tau(s)\|_{H^3}^2\|\tau(s)\|_{H^2}\,\mathrm{d}s$$
$$\leqslant Cb\sup_{0\leqslant s\leqslant t}\||\nabla|^{-1}\tau(s)\|_{H^3}\int_0^t \||\nabla|^{-1}\tau(s)\|_{H^3}^2\,\mathrm{d}s \leqslant C\mathcal{E}_1^{3/2}(t),$$
$$\int_0^t |I_3|\,\mathrm{d}s \leqslant C\sup_{0\leqslant s\leqslant t}\||\nabla|^{-1}u(s)\|_{H^3}\int_0^t \|u(s)\|_{H^3}^2\,\mathrm{d}s \leqslant C\mathcal{E}_1^{3/2}(t).$$

To deal with the last term I_4, we use Hölder's inequality to get

$$\int_0^t |I_4|\,\mathrm{d}s \leqslant C\int_0^t \||\nabla|^{-1}\tau(s)\|_{H^3}^2\left(\|\nabla^2 u(s)\|_{H^1} + \|\nabla u(s)\|_{L^2}^{1/2}\|\nabla^2 u(s)\|_{L^2}^{1/2}\right)\mathrm{d}s$$
$$\leqslant \sup_{0\leqslant s\leqslant t}\||\nabla|^{-1}\tau(s)\|_{H^3}^2\left\{\left(\int_0^t (1+s)^{-2}\,\mathrm{d}s\right)^{1/2}\right.$$
$$\times\left(\int_0^t (1+s)^2\|\nabla^2 u(s)\|_{H^1}^2\,\mathrm{d}s\right)^{1/2}$$
$$+\left(\int_0^t (1+s)^{-(3/2)}\,\mathrm{d}s\right)^{1/2}\left(\int_0^t (1+s)\|\nabla u(s)\|_{L^2}^2\,\mathrm{d}s\right)^{1/4}$$
$$\left.\times\left(\int_0^t (1+s)^2\|\nabla^2 u(s)\|_{L^2}^2\,\mathrm{d}s\right)^{1/4}\right\}$$
$$\leqslant C\left(\mathcal{E}_1^{3/2}(t) + \mathcal{E}_2^{3/2}(t)\right),$$

which in the last inequality we have used for the fact that

$$\int_0^t (1+s)\|\nabla u(s)\|_{L^2}^2\,\mathrm{d}s \leqslant C\left(\int_0^t \|u(s)\|_{L^2}^2\,\mathrm{d}s\right)^{1/2}\left(\int_0^t (1+s)^2\|\nabla^2 u(s)\|_{L^2}^2\,\mathrm{d}s\right)^{1/2}.$$

Integrating (3.4) with respect to t, and plugging the above inequalities into the resulting inequality, we get the desired inequality (3.2). □

Next, we aim to recover the dissipation estimate for $\mathbb{P}\operatorname{div}\tau$.

Lemma 3.3. *Under the assumptions of proposition 3.1, it holds that*

$$\mathcal{E}_{12}(t) \leqslant \mathcal{E}(0) + C\mathcal{E}_{11}(t) + C\left(\mathcal{E}_1^{3/2}(t) + \mathcal{E}_2^{3/2}(t) + \mathcal{E}_3^{3/2}(t)\right), \tag{3.5}$$

where C is a positive constant independent of t, a and b, and

$$\mathcal{E}_{12}(t) := -\sup_{0\leqslant s\leqslant t}\sum_{k=0}^{2}\int |\nabla|^{-1}\nabla^k u\cdot|\nabla|^{-1}\nabla^k\mathbb{P}\operatorname{div}\tau(s)\,\mathrm{d}x$$
$$+\frac{1}{2}\int_0^t \||\nabla|^{-1}\mathbb{P}\operatorname{div}\tau(s)\|_{H^2}^2\,\mathrm{d}s. \tag{3.6}$$

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Proof. Applying the operator $|\nabla|^{-1}\nabla^k\mathbb{P}$ ($k = 0, 1, 2$) to the first equation of system (PTT), we obtain the equation that

$$|\nabla|^{-1}\nabla^k u_t + |\nabla|^{-1}\nabla^k\mathbb{P}(u\cdot\nabla u) - |\nabla|^{-1}\nabla^k\triangle u = |\nabla|^{-1}\nabla^k\mathbb{P}\,\mathrm{div}\,\tau.$$

Taking the L^2 inner product with $|\nabla|^{-1}\nabla^k\mathbb{P}\,\mathrm{div}\,\tau$, and summing up k from 0 to 2, we find that

$$\begin{aligned}\||\nabla|^{-1}\mathbb{P}\,\mathrm{div}\,\tau\|_{H^2}^2 &= \sum_{k=0}^{2}\int |\nabla|^{-1}\nabla^k u_t\cdot|\nabla|^{-1}\nabla^k\mathbb{P}\,\mathrm{div}\,\tau\,\mathrm{d}x\\ &\quad+\sum_{k=0}^{2}\int |\nabla|^{-1}\nabla^k\mathbb{P}\,(u\cdot\nabla u)\cdot|\nabla|^{-1}\nabla^k\mathbb{P}\,\mathrm{div}\,\tau\,\mathrm{d}x\\ &\quad-\sum_{k=0}^{2}\int |\nabla|^{-1}\nabla^k\triangle u\cdot|\nabla|^{-1}\nabla^k\mathbb{P}\,\mathrm{div}\,\tau\,\mathrm{d}x\\ &:= J_1+J_2+J_3. \end{aligned}\tag{3.7}$$

In order to deal with J_1, we turn the time derivative of u to τ. Hence, we can transform the time derivative to the spatial derivative, i.e.

$$\begin{aligned}J_1 &= \frac{\mathrm{d}}{\mathrm{d}t}\sum_{k=0}^{2}\int |\nabla|^{-1}\nabla^k u\cdot|\nabla|^{-1}\nabla^k\mathbb{P}\,\mathrm{div}\,\tau\,\mathrm{d}x\\ &\quad-\sum_{k=0}^{2}\int |\nabla|^{-1}\nabla^k u\cdot|\nabla|^{-1}\nabla^k\mathbb{P}\,\mathrm{div}\,\tau_t\,\mathrm{d}x\\ &:= J_{11}+J_{12}.\end{aligned}$$

Applying the operator $|\nabla|^{-1}\nabla^k\mathbb{P}\,\mathrm{div}$ ($k = 0, 1, 2$) to the second equation of system (PTT), we see that

$$\begin{aligned}|\nabla|^{-1}\nabla^k\mathbb{P}\,\mathrm{div}\,\tau_t &= \frac{1}{2}|\nabla|^{-1}\nabla^k\triangle u - a|\nabla|^{-1}\nabla^k\mathbb{P}\,\mathrm{div}\,\tau\\ &\quad-|\nabla|^{-1}\nabla^k\mathbb{P}\,\mathrm{div}\,(u\cdot\nabla\tau+Q(\tau,\nabla u)+b(\mathrm{tr}\,\tau)\tau)\,.\end{aligned}$$

Now, using the facts that $0\leqslant a\leqslant 1$ and $0\leqslant b\leqslant Ca$, we have

$$\begin{aligned}\int_0^t |J_{12}|\,\mathrm{d}s &\leqslant C\int_0^t \left(\|u(s)\|_{H^2}^2 + a^2\||\nabla|^{-1}\tau(s)\|_{H^3}^2\right)\,\mathrm{d}s + C\int_0^t \Big(\|u(s)\|_{H^2}^2\|\tau(s)\|_{H^2}\\ &\quad+\||\nabla|^{-1}u(s)\|_{H^2}\|\tau(s)\|_{H^2}\left(\|\nabla^2 u(s)\|_{H^1}+b\|\nabla\mathrm{tr}\,\tau(s)\|_{H^1}\right)\Big)\,\mathrm{d}s\\ &\leqslant C\mathcal{E}_{11}(t) + C\int_0^t \||\nabla|^{-1}u(s)\|_{H^3}\left(\|u(s)\|_{H^2}+\|\tau(s)\|_{H^2}\right)\\ &\quad\times\left(\|\nabla^2 u(s)\|_{H^1}+b\|\mathrm{tr}\,\tau(s)\|_{H^2}\right)\mathrm{d}s\end{aligned}$$

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$$
\begin{aligned}
&\leqslant C\mathcal{E}_{11}(t) + C \sup_{0\leqslant s\leqslant t} \left(\||\nabla|^{-1}u(s)\|_{H^3}^2 + \|\tau(s)\|_{H^2}^2\right) \\
&\quad \times \left\{ \sup_{0\leqslant s\leqslant t} \left(\mathrm{e}^{as/2}\|\mathrm{tr}\,\tau(s)\|_{H^2}\right) \int_0^t b\cdot \mathrm{e}^{-(as/2)}\,\mathrm{d}s \right. \\
&\quad \left. + \left(\int_0^t (1+s)^{-2}\,\mathrm{d}s\right)^{1/2} \left(\int_0^t (1+s)^2\|\nabla^2 u(s)\|_{H^1}^2\,\mathrm{d}s\right)^{1/2} \right\} \\
&\leqslant C\mathcal{E}_{11}(t) + C\left(\mathcal{E}_1^{3/2}(t) + \mathcal{E}_2^{3/2}(t) + \mathcal{E}_3^{3/2}(t)\right).
\end{aligned}
$$

Here, we can deal with the remaining two terms J_2 and J_3 as follows:

$$
\begin{aligned}
\int_0^t |J_2|\,\mathrm{d}s &\leqslant C \int_0^t \|u(s)\|_{H^2}^2 \||\nabla|^{-1}\mathbb{P}\,\mathrm{div}\,\tau(s)\|_{H^2}\,\mathrm{d}s \\
&\leqslant C \sup_{0\leqslant s\leqslant t} \||\nabla|^{-1}\mathbb{P}\,\mathrm{div}\,\tau(s)\|_{H^2} \int_0^t \|u(s)\|_{H^2}^2\,\mathrm{d}s \leqslant C\mathcal{E}_1^{3/2}(t), \\
\int_0^t |J_3|\,\mathrm{d}s &\leqslant C \int_0^t \|\nabla u(s)\|_{H^2} \||\nabla|^{-1}\mathbb{P}\,\mathrm{div}\,\tau(s)\|_{H^2}\,\mathrm{d}s \\
&\leqslant \frac{1}{2}\int_0^t \||\nabla|^{-1}\mathbb{P}\,\mathrm{div}\,\tau(s)\|_{H^2}^2\,\mathrm{d}s + C\int_0^t \|\nabla u(s)\|_{H^2}^2\,\mathrm{d}s \\
&\leqslant \frac{1}{2}\int_0^t \||\nabla|^{-1}\mathbb{P}\,\mathrm{div}\,\tau(s)\|_{H^2}^2\,\mathrm{d}s + C\mathcal{E}_{11}(t).
\end{aligned}
$$

Substituting the above estimates of J_1, J_2 and J_3 into (3.7), we obtain (3.5) immediately. □

With the help of lemmas 3.2 and 3.3, we infer the following lemma.

LEMMA 3.4. *Under the assumptions of proposition 3.1, it holds that*

$$
\mathcal{E}_1(t) \leqslant C\mathcal{E}(0) + C\left(\mathcal{E}_1^{3/2}(t) + \mathcal{E}_2^{3/2}(t) + \mathcal{E}_3^{3/2}(t)\right). \tag{3.8}
$$

Proof. Now, adding $(2\eta/(1-C\eta))\times$(3.5) to (3.2), and choosing η suitably small, one obtains that

$$
\begin{aligned}
&\sup_{0\leqslant s\leqslant t} \left(\||\nabla|^{-1}u(s)\|_{H^3}^2 + \||\nabla|^{-1}\tau(s)\|_{H^3}^2 \right. \\
&\quad \left. -\frac{2\eta}{1-C\eta}\sum_{k=0}^{2}\int |\nabla|^{-1}\nabla^k u\cdot |\nabla|^{-1}\nabla^k\mathbb{P}\,\mathrm{div}\,\tau\,\mathrm{d}x\right) \\
&\quad + \int_0^t \left(2\|u(s)\|_{H^3}^2 + 2a\||\nabla|^{-1}\tau(s)\|_{H^3}^2 + \frac{\eta}{1-C\eta}\||\nabla|^{-1}\mathbb{P}\,\mathrm{div}\,\tau(s)\|_{H^2}^2\right)\mathrm{d}s \\
&\quad \leqslant C\mathcal{E}(0) + C\left(\mathcal{E}_1^{3/2}(t) + \mathcal{E}_2^{3/2}(t) + \mathcal{E}_3^{3/2}(t)\right).
\end{aligned}
$$

Since

$$\sum_{k=0}^{2}\left|\int |\nabla|^{-1}\nabla^k u\cdot|\nabla|^{-1}\nabla^k\mathbb{P}\,\mathrm{div}\,\tau\,\mathrm{d}x\right|\leqslant C\||\nabla|^{-1}u\|_{H^2}^2+C\|\tau\|_{H^2}^2,$$

then there exist two positive constants c_1 and c_2, independent of t, a and b, such that

$$\begin{aligned}c_1&\left(\||\nabla|^{-1}u(s)\|_{H^3}^2+\||\nabla|^{-1}\tau(s)\|_{H^3}^2\right)\\&\leqslant\||\nabla|^{-1}u(s)\|_{H^3}^2+\||\nabla|^{-1}\tau(s)\|_{H^3}^2\\&\quad-\frac{2\eta}{1-C\eta}\sum_{k=0}^{2}\int|\nabla|^{-1}\nabla^k u\cdot|\nabla|^{-1}\nabla^k\mathbb{P}\,\mathrm{div}\,\tau\,\mathrm{d}x\\&\leqslant c_2\left(\||\nabla|^{-1}u(s)\|_{H^3}^2+\||\nabla|^{-1}\tau(s)\|_{H^3}^2\right),\end{aligned}$$

provided that η is suitably small. Therefore, we get (3.8) immediately. □

3.2. Estimate of $\mathcal{E}_2(t)$

According to the estimate of $\mathcal{E}_1(t)$ in § 3.1, we see that it is necessary to prove some time decay estimates as $\mathcal{E}_2(t)$.

Lemma 3.5. *Under the assumptions of proposition 3.1, it holds that*

$$\begin{aligned}\mathcal{E}_{21}'(t)\leqslant C\left(\|u\|_{H^2}+\|\tau\|_{H^2}\right)&\left(\|\nabla^2u\|_{L^2}+\|\nabla\mathbb{P}\,\mathrm{div}\,\tau\|_{L^2}+b\|\nabla\mathrm{tr}\,\tau\|_{H^1}\right)\\&\times\left(\|\nabla^2u\|_{H^1}+\|\nabla\mathbb{P}\,\mathrm{div}\,\tau\|_{L^2}\right),\end{aligned}\tag{3.9}$$

where C is a positive constant independent of t, a and b, and

$$\mathcal{E}_{21}'(t):=\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\nabla u\|_{H^1}^2+2\|\mathbb{P}\,\mathrm{div}\,\tau\|_{H^1}^2\right)+\|\nabla^2u\|_{H^1}^2+2a\|\mathbb{P}\,\mathrm{div}\,\tau\|_{H^1}^2.\tag{3.10}$$

Proof. Applying the operator ∇^{k+1} $(k=0,\,1)$ to the first equation of system (PTT) and $\nabla^k\mathbb{P}\,\mathrm{div}$ $(k=0,\,1)$ to the second equation of system (PTT), we see that

$$\begin{cases}\nabla^{k+1}u_t+\nabla^{k+1}(u\cdot\nabla u)-\nabla^{k+1}\triangle u+\nabla^{k+1}\nabla p=\nabla^{k+1}\,\mathrm{div}\,\tau,\\\nabla^k\mathbb{P}\,\mathrm{div}\,\tau_t+a\nabla^k\mathbb{P}\,\mathrm{div}\,\tau+\nabla^k\mathbb{P}\,\mathrm{div}\,(u\cdot\nabla\tau+Q(\tau,\nabla u)+b(\mathrm{tr}\,\tau)\tau)\\\quad=\dfrac{1}{2}\nabla^k\triangle u.\end{cases}\tag{3.11}$$

Taking the L^2 inner product of the first equation of (3.11) with $\nabla^{k+1}u$ and the second equation of (3.11) with $2\nabla^k\mathbb{P}\,\mathrm{div}\,\tau$, we arrive at

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\nabla u\|_{H^1}^2+2\|\mathbb{P}\,\mathrm{div}\,\tau\|_{H^1}^2\right)+\|\nabla^2u\|_{H^1}^2+2a\|\mathbb{P}\,\mathrm{div}\,\tau\|_{H^1}^2\\&\quad=\sum_{k=0,1}\int\left(\nabla^{k+1}\,\mathrm{div}\,\tau\cdot\nabla^{k+1}u+\nabla^k\triangle u\cdot\nabla^k\mathbb{P}\,\mathrm{div}\,\tau\right)\mathrm{d}x\\&\qquad-\sum_{k=0,1}\int\nabla^{k+1}\left(u\cdot\nabla u\right)\cdot\nabla^{k+1}u\,\mathrm{d}x\end{aligned}$$

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$$
\begin{aligned}
&- \sum_{k=0,1} 2 \int \nabla^k \mathbb{P} \operatorname{div}\left(u \cdot \nabla \tau + Q(\tau, \nabla u)\right) \cdot \nabla^k \mathbb{P} \operatorname{div} \tau \, \mathrm{d}x \\
&- \sum_{k=0,1} 2b \int \nabla^k \mathbb{P} \operatorname{div}\left((\operatorname{tr} \tau)\tau\right) \cdot \nabla^k \mathbb{P} \operatorname{div} \tau \, \mathrm{d}x \\
&:= K_1 + K_2 + K_3 + K_4. \qquad (3.12)
\end{aligned}
$$

Observe that the divergence-free and symmetry conditions, and by using integration by parts, we deduce that $K_1 = 0$. For the second term K_2, we readily have

$$
|K_2| \leqslant C \|u\|_{H^2} \|\nabla^2 u\|_{L^2} \|\nabla^2 u\|_{H^1}.
$$

We now turn to the toughest term K_3. By taking advantage of lemma 2.1, Hölder's inequality and Gagliardo–Nirenberg–Sobolev inequality, we can conclude that

$$
\begin{aligned}
|K_3| &\leqslant C \|\nabla u\|_{L^6} \|\nabla \tau\|_{L^{3/2}} \|\mathbb{P} \operatorname{div} \tau\|_{L^6} + \|\tau\|_{L^3} \|\nabla^2 u\|_{L^2} \|\mathbb{P} \operatorname{div} \tau\|_{L^6} \\
&\quad + \|\nabla u\|_{L^\infty} \|\nabla^2 \tau\|_{L^2} \|\nabla \mathbb{P} \operatorname{div} \tau\|_{L^2} + \|\nabla^2 u\|_{L^3} \|\nabla \tau\|_{L^6} \|\nabla \mathbb{P} \operatorname{div} \tau\|_{L^2} \\
&\quad + \|\tau\|_{L^\infty} \|\nabla^3 u\|_{L^2} \|\nabla \mathbb{P} \operatorname{div} \tau\|_{L^2} \\
&\leqslant C \|\tau\|_{H^2} \|\nabla^2 u\|_{H^1} \|\nabla \mathbb{P} \operatorname{div} \tau\|_{L^2}.
\end{aligned}
$$

For the last term K_4, we easily get that

$$
|K_4| \leqslant Cb \|\nabla \operatorname{tr} \tau\|_{H^1} \|\tau\|_{H^2} \|\nabla \mathbb{P} \operatorname{div} \tau\|_{L^2}.
$$

Combining the above estimates, we get inequality (3.9). □

Next, we recover the dissipation estimate for $\mathbb{P} \operatorname{div} \tau$.

LEMMA 3.6. *Under the assumptions of proposition 3.1, it holds that*

$$
\begin{aligned}
\mathcal{E}_{22}'(t) &\leqslant \left(\|\nabla^2 u\|_{H^1}^2 + a^2 \|\mathbb{P} \operatorname{div} \tau\|_{L^2}^2\right) + C\left(\|u\|_{H^2} + \|\tau\|_{H^2}\right) \\
&\quad \times \left(\|\nabla^2 u\|_{L^2} + \|\nabla \mathbb{P} \operatorname{div} \tau\|_{L^2} + b \|\nabla \operatorname{tr} \tau\|_{H^1}\right) \|\nabla^2 u\|_{H^1}, \qquad (3.13)
\end{aligned}
$$

where C is a positive constant independent of t, a and b, and

$$
\mathcal{E}_{22}'(t) := -\frac{\mathrm{d}}{\mathrm{d}t} \int \nabla u \cdot \nabla \mathbb{P} \operatorname{div} \tau \, \mathrm{d}x + \frac{1}{2} \|\nabla \mathbb{P} \operatorname{div} \tau\|_{L^2}^2. \qquad (3.14)
$$

Proof. Applying the operator $\nabla \mathbb{P}$ to the first equation of system (PTT), we obtain the equation that

$$
\nabla u_t + \nabla \mathbb{P}(u \cdot \nabla u) - \nabla \triangle u = \nabla \mathbb{P} \operatorname{div} \tau.
$$

Taking the L^2 inner product with $\nabla \mathbb{P} \operatorname{div} \tau$, we find that

$$
\begin{aligned}
\|\nabla \mathbb{P} \operatorname{div} \tau\|_{L^2}^2 &= \int \nabla u_t \cdot \nabla \mathbb{P} \operatorname{div} \tau \, \mathrm{d}x + \int \nabla \mathbb{P}\left(u \cdot \nabla u\right) \cdot \nabla \mathbb{P} \operatorname{div} \tau \, \mathrm{d}x \\
&\quad - \int \nabla \triangle u \cdot \nabla \mathbb{P} \operatorname{div} \tau \, \mathrm{d}x \\
&:= L_1 + L_2 + L_3. \qquad (3.15)
\end{aligned}
$$

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To deal with the first term L_1, we use integration by parts to get

$$L_1 = \frac{\mathrm{d}}{\mathrm{d}t}\int \nabla u\cdot\nabla\mathbb{P}\,\mathrm{div}\,\tau\,\mathrm{d}x - \int \nabla u\cdot\nabla\mathbb{P}\,\mathrm{div}\,\tau_t\,\mathrm{d}x := L_{11} + L_{12}.$$

Applying the operator $\mathbb{P}\,\mathrm{div}$ to the second equation of system (PTT) yields

$$\mathbb{P}\,\mathrm{div}\,\tau_t + a\mathbb{P}\,\mathrm{div}\,\tau + \mathbb{P}\,\mathrm{div}\,(u\cdot\nabla\tau + Q(\tau,\nabla u) + b(\mathrm{tr}\,\tau)\tau) = \frac{1}{2}\triangle u,$$

from which it follows that

$$L_{12} = -\int \triangle u\cdot\left(\mathbb{P}\,\mathrm{div}\,(u\cdot\nabla\tau + Q(\tau,\nabla u) + b(\mathrm{tr}\,\tau)\tau) + a\mathbb{P}\,\mathrm{div}\,\tau - \frac{1}{2}\triangle u\right)\mathrm{d}x.$$

According to lemma 2.1, we have

$$\begin{aligned}|L_{12}| \leqslant{}& C\left(\|\nabla^2 u\|_{L^2}^2 + a^2\|\mathbb{P}\,\mathrm{div}\,\tau\|_{L^2}^2\right) + C\|\nabla^2 u\|_{L^2}\left(\|u\|_{L^\infty}\|\nabla\mathbb{P}\,\mathrm{div}\,\tau\|_{L^2}\right.\\ &\left.+\|\nabla u\|_{L^\infty}\|\nabla\tau\|_{L^2} + \|\tau\|_{L^\infty}\|\nabla^2 u\|_{L^2} + b\|\tau\|_{H^2}\|\nabla\mathrm{tr}\,\tau\|_{H^1}\right).\end{aligned}$$

Next, we treat the second term L_2 and the last term L_3. We infer that

$$\begin{aligned}|L_2| &\leqslant C\left(\|u\|_{L^\infty}\|\nabla^2 u\|_{L^2} + \|\nabla u\|_{L^\infty}\|\nabla u\|_{L^2}\right)\|\nabla\mathbb{P}\,\mathrm{div}\,\tau\|_{L^2},\\ |L_3| &\leqslant C\|\nabla^3 u\|_{L^2}\|\nabla\mathbb{P}\,\mathrm{div}\,\tau\|_{L^2} \leqslant \frac{1}{2}\|\nabla\mathbb{P}\,\mathrm{div}\,\tau\|_{L^2}^2 + C\|\nabla^3 u\|_{L^2}^2.\end{aligned}$$

Utilizing the above estimates in (3.15), it is easy to get (3.13). □

In view of lemmas 3.5 and 3.6, we obtain the following lemma.

LEMMA 3.7. *Under the assumptions of proposition 3.1, it holds that*

$$\mathcal{E}_2(t) \leqslant C\mathcal{E}(0) + C\mathcal{E}_1(t) + C\left(\mathcal{E}_1^{3/2}(t) + \mathcal{E}_2^{3/2}(t) + \mathcal{E}_3^{3/2}(t)\right). \tag{3.16}$$

Proof. As above two lemmas, together with (3.9) and $\eta\times$ (3.13), using the facts that $0 \leqslant a \leqslant 1$ and $0 \leqslant b \leqslant Ca$, and choosing η suitably small, this ensures that

$$\begin{aligned}\mathcal{E}_2'(t) :={}& \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\{\|\nabla u\|_{H^1}^2 + 2\|\mathbb{P}\,\mathrm{div}\,\tau\|_{H^1}^2 - 2\eta\int \nabla u\cdot\nabla\mathbb{P}\,\mathrm{div}\,\tau\,\mathrm{d}x\right\}\\ &+\frac{1}{2}\|\nabla^2 u\|_{H^1}^2 + a\|\mathbb{P}\,\mathrm{div}\,\tau\|_{H^1}^2 + \frac{\eta}{2}\|\nabla\mathbb{P}\,\mathrm{div}\,\tau\|_{L^2}^2\\ \leqslant{}& C\left(\|\nabla^2 u\|_{H^1} + \|\nabla\mathbb{P}\,\mathrm{div}\,\tau\|_{L^2}\right)\left(\|u\|_{H^2} + \|\tau\|_{H^2}\right)\\ &\times\left(\|\nabla^2 u\|_{L^2} + \|\nabla\mathbb{P}\,\mathrm{div}\,\tau\|_{L^2} + b\|\nabla\mathrm{tr}\,\tau\|_{H^1}\right).\end{aligned}$$

Since

$$\left|\int \nabla u\cdot\nabla\mathbb{P}\,\mathrm{div}\,\tau\,\mathrm{d}x\right| \leqslant \|\nabla u\|_{L^2}^2 + \|\nabla\mathbb{P}\,\mathrm{div}\,\tau\|_{L^2}^2,$$

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then we have

$$\begin{aligned}
&\frac{1}{2}\left(\|\nabla u\|_{H^1}^2 + 2\|\mathbb{P}\operatorname{div}\tau\|_{H^1}^2\right)\\
&\quad\leqslant \|\nabla u\|_{H^1}^2 + 2\|\mathbb{P}\operatorname{div}\tau\|_{H^1}^2 - 2\eta\int \nabla u\cdot\nabla\mathbb{P}\operatorname{div}\tau\,\mathrm{d}x\\
&\quad\leqslant \frac{3}{2}\left(\|\nabla u\|_{H^1}^2 + 2\|\mathbb{P}\operatorname{div}\tau\|_{H^1}^2\right).
\end{aligned}$$

The key to the proof is a weighted energy estimate. By multiplying the time weight $(1+t)^2$ and integrating directly in time, we obtain that

$$\begin{aligned}
&\sup_{0\leqslant s\leqslant t}(1+s)^2\left(\|\nabla u(s)\|_{H^1}^2 + 2\|\mathbb{P}\operatorname{div}\tau(s)\|_{H^1}^2\right)\\
&\qquad + \int_0^t (1+s)^2\left(\|\nabla^2 u(s)\|_{H^1}^2 + \|\nabla\mathbb{P}\operatorname{div}\tau(s)\|_{L^2}^2 + a\|\mathbb{P}\operatorname{div}\tau(s)\|_{H^1}^2\right)\mathrm{d}s\\
&\quad\leqslant C\mathcal{E}(0) + 2\int_0^t (1+s)\left(\|\nabla u(s)\|_{H^1}^2 + 2\|\mathbb{P}\operatorname{div}\tau(s)\|_{H^1}^2\right)\mathrm{d}s\\
&\qquad + C\int_0^t (1+s)^2\left(\|\nabla^2 u(s)\|_{H^1} + \|\nabla\mathbb{P}\operatorname{div}\tau(s)\|_{L^2}\right)\left(\|u(s)\|_{H^2} + \|\tau(s)\|_{H^2}\right)\\
&\qquad \times\left(\|\nabla^2 u(s)\|_{L^2} + \|\nabla\mathbb{P}\operatorname{div}\tau(s)\|_{L^2} + b\|\nabla\operatorname{tr}\tau(s)\|_{H^1}\right)\mathrm{d}s\\
&\quad\leqslant C\mathcal{E}(0) + C\mathcal{E}_1^{1/2}(t)\mathcal{E}_2^{1/2}(t) + C\mathcal{E}_1^{1/2}(t)\int_0^t (1+s)^2(\|\nabla^2 u(s)\|_{H^1}^2\\
&\qquad + \|\nabla\mathbb{P}\operatorname{div}\tau(s)\|_{L^2}^2)\mathrm{d}s + Cb^2\mathcal{E}_1^{1/2}(t)\int_0^t (1+s)^2\|\nabla\operatorname{tr}\tau(s)\|_{H^1}^2\,\mathrm{d}s\\
&\quad\leqslant C\mathcal{E}(0) + C\mathcal{E}_1^{1/2}(t)\mathcal{E}_2^{1/2}(t) + C\left(\mathcal{E}_1^{3/2}(t) + \mathcal{E}_2^{3/2}(t) + \mathcal{E}_3^{3/2}(t)\right).
\end{aligned}$$

Note that in the above inequality we have used the facts that

$$\begin{aligned}
&\int_0^t (1+s)\left(\|\nabla u(s)\|_{H^1}^2 + 2\|\mathbb{P}\operatorname{div}\tau(s)\|_{H^1}^2\right)\mathrm{d}s\\
&\quad\leqslant C\left\{\left(\int_0^t \left(\|u(s)\|_{L^2}^2 + \||\nabla|^{-1}\mathbb{P}\operatorname{div}\tau(s)\|_{L^2}^2\right)\mathrm{d}s\right)^{1/2}\right.\\
&\qquad\left. + \left(\int_0^t \left(\|\nabla^2 u(s)\|_{L^2}^2 + \|\nabla\mathbb{P}\operatorname{div}\tau(s)\|_{L^2}^2\right)\mathrm{d}s\right)^{1/2}\right\}\\
&\qquad\times\left(\int_0^t (1+s)^2\left(\|\nabla^2 u(s)\|_{L^2}^2 + \|\nabla\mathbb{P}\operatorname{div}\tau(s)\|_{L^2}^2\right)\mathrm{d}s\right)^{1/2}\\
&\quad\leqslant C\mathcal{E}_1^{1/2}(t)\mathcal{E}_2^{1/2}(t),
\end{aligned}$$

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and

$$b^2 \int_0^t (1+s)^2 \|\mathrm{tr}\,\tau(s)\|_{H^2}^2 \,\mathrm{d}s$$
$$\leqslant Cb^2 \sup_{0\leqslant s\leqslant t} \left(\mathrm{e}^{as}\|\mathrm{tr}\,\tau(s)\|_{H^2}^2\right) \int_0^t (1+s)^2 \,\mathrm{e}^{-as}\,\mathrm{d}s \leqslant C\mathcal{E}_3(t).$$

Since $0 \leqslant a \leqslant 1$ and $0 \leqslant b \leqslant Ca$, then $\lim_{a\to 0}((1-\mathrm{e}^{-at})/a) = \lim_{a\to 0}\mathrm{e}^{-at} \leqslant 1$, we have

$$\begin{aligned}
&b^2 \int_0^t (1+s)^2 \,\mathrm{e}^{-as}\,\mathrm{d}s \\
&= -\frac{b^2}{a}\left((1+t)^2\,\mathrm{e}^{-at} - 1 - 2\int_0^t (1+s)\,\mathrm{e}^{-as}\,\mathrm{d}s\right) \\
&= -\frac{b^2}{a}\left((1+t)^2\,\mathrm{e}^{-at} - 1 + \frac{2}{a}\left((1+t)\,\mathrm{e}^{-at} - 1 - \int_0^t \mathrm{e}^{-as}\,\mathrm{d}s\right)\right) \\
&= -\frac{b^2}{a}((1+t)\,\mathrm{e}^{-at}-1) - \frac{2b^2}{a^2}((1+t)\,\mathrm{e}^{-at}-1) + \frac{2b^2}{a^2}\frac{(1-\mathrm{e}^{-at})}{a} \\
&\leqslant C.
\end{aligned}$$

Combining the above estimates, and recalling the definition of $\mathcal{E}_2(t)$, it is easy to deduce estimate (3.16). □

3.3. Estimate of $\mathcal{E}_3(t)$

Since what we obtained above is the dissipation estimate for $\mathbb{P}\,\mathrm{div}\,\tau$, but rather than τ. In order to control the nonlinear term $b(\mathrm{tr}\,\tau)\tau$, we need further and more arguments. Here, we shall make full use of the structure of system (PTT) to obtain $\|\mathrm{tr}\,\tau\|_{H^3}$ decays exponentially.

LEMMA 3.8. *Under the assumptions of proposition 3.1, it holds that*

$$\mathcal{E}_3(t) \leqslant \mathcal{E}(0)\exp\left\{C\mathcal{E}_2^{1/2}(t) + C\mathcal{E}_3^{1/2}(t)\right\}, \tag{3.17}$$

where C is a positive constant independent of t, a and b.

Proof. Applying the operator tr to the second equation of (PTT), and using the fact that $\mathrm{div}\,u = 0$, we may write

$$(\mathrm{tr}\,\tau)_t + (a + b\mathrm{tr}\,\tau)\,\mathrm{tr}\,\tau + u\cdot\nabla\mathrm{tr}\,\tau = 0,$$

with the following standard estimate:

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\|\mathrm{tr}\,\tau\|_{H^2}^2 + a\|\mathrm{tr}\,\tau\|_{H^2}^2 \\
&= \sum_{k=0}^{2}\int \nabla^k\left(u\cdot\nabla\mathrm{tr}\,\tau\right)\cdot\nabla^k(\mathrm{tr}\,\tau)\,\mathrm{d}x - b\sum_{k=0}^{2}\int \nabla^k\left((\mathrm{tr}\,\tau)^2\right)\cdot\nabla^k(\mathrm{tr}\,\tau)\,\mathrm{d}x \\
&\leqslant C\left(\|\nabla u\|_{L^\infty} + \|\nabla^2 u\|_{L^3} + b\|\mathrm{tr}\,\tau\|_{H^2}\right)\|\mathrm{tr}\,\tau\|_{H^2}^2.
\end{aligned}$$

Consequently, the above inequality reads

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\mathrm{e}^{at}\|\mathrm{tr}\,\tau\|_{H^2}^2\right) \leqslant C\left(b\|\mathrm{tr}\,\tau\|_{H^2} + \|\nabla u\|_{L^\infty} + \|\nabla^2 u\|_{L^3}\right)\left(\mathrm{e}^{at}\|\mathrm{tr}\,\tau\|_{H^2}^2\right).$$

Gronwall's lemma now provides the bound

$$\begin{aligned}\mathrm{e}^{at}\|\mathrm{tr}\,\tau(t)\|_{H^2}^2 \leqslant \|\mathrm{tr}\,\tau_0\|_{H^2}^2 \exp\left\{C\int_0^t \left(\|\nabla u(s)\|_{L^\infty} + \|\nabla^2 u(s)\|_{L^3}\right)\mathrm{d}s\right.\\ \left.+Cb\int_0^t \|\mathrm{tr}\,\tau(s)\|_{H^2}\,\mathrm{d}s\right\}. \qquad (3.18)\end{aligned}$$

It is then obvious that

$$\begin{aligned}&\int_0^t \left(\|\nabla u(s)\|_{L^\infty} + \|\nabla^2 u(s)\|_{L^3}\right)\mathrm{d}s\\ &\leqslant C\left(\int_0^t (1+s)^{-2}\,\mathrm{d}s\right)^{1/2}\left(\int (1+s)^2\|\nabla^2 u(s)\|_{H^1}^2 ds\right)^{1/2} \leqslant C\mathcal{E}_2^{1/2}(t).\end{aligned}$$

In fact, due to $0 \leqslant b \leqslant Ca$, we have

$$\begin{aligned}b\int_0^t \|\mathrm{tr}\,\tau(s)\|_{H^2}\,\mathrm{d}s &\leqslant b\sup_{0\leqslant s\leqslant t}\left(\mathrm{e}^{as/2}\|\mathrm{tr}\,\tau(s)\|_{H^2}\right)\int_0^t \mathrm{e}^{-(as/2)}\,\mathrm{d}s\\ &\leqslant \frac{2b}{a}\mathcal{E}_3^{1/2}(t) \leqslant C\mathcal{E}_3^{1/2}(t).\end{aligned}$$

Returning to (3.18), and combining the estimates above, we get (3.17) immediately. □

3.4. Closure of the estimates

With the help of the estimates of $\mathcal{E}_i(t)$ $(i = 1, 2, 3)$, we start to prove proposition 3.1.

Proof of proposition 3.1. Together with all the estimates of $\mathcal{E}_i(t)$ $(i = 1, 2, 3)$, this gives that

$$\begin{aligned}&\mathcal{E}_1(t) + \mathcal{E}_2(t) + \mathcal{E}_3(t)\\ &\quad\leqslant C^*\left(\mathcal{E}(0) + \mathcal{E}_1^{3/2}(t) + \mathcal{E}_2^{3/2}(t) + \mathcal{E}_3^{3/2}(t)\right)\\ &\quad\quad + C^*\mathcal{E}(0)\exp\left\{C\mathcal{E}_2^{1/2}(t) + C\mathcal{E}_3^{1/2}(t)\right\}.\end{aligned}$$

Using the bootstrap assumption (3.1), we then get

$$\begin{aligned}&C^*\left(\mathcal{E}(0) + \mathcal{E}_1^{3/2}(t) + \mathcal{E}_2^{3/2}(t) + \mathcal{E}_3^{3/2}(t)\right) \leqslant C^*\delta_0 + C\delta_0^{3/2},\\ &C^*\mathcal{E}(0)\exp\left\{C\mathcal{E}_2^{1/2}(t) + C\mathcal{E}_3^{1/2}(t)\right\} \leqslant 2C^*\delta_0.\end{aligned}$$

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Hence,

$$\mathcal{E}_1(t)+\mathcal{E}_2(t)+\mathcal{E}_3(t)\leqslant 3C^*\delta_0+C\delta_0^{3/2}<4C^*\delta_0,$$

provided that δ_0 is sufficiently small. This completes the proof of proposition 3.1. □

4. Proofs of main results

Thus far, we have established the uniform estimates for smooth solutions with respect to t, a and b. In this section, we pay attention to the proofs of theorems 1.1, 1.7 and 1.8. Indeed, proposition 3.1 in § 3 leads us to prove theorem 1.1.

Proof of theorem 1.1. In contrast to the Oldroyd-B system, however, there is no derivative in the additional term $b(\operatorname{tr}\tau)\tau$ of system (PTT). The proof of local well-posedness for system (PTT) is similar to that of the Oldroyd-B system, and will thus be omitted. Then there exists a positive time T such that, for any $0\leqslant t\leqslant T$, we have

$$\mathcal{E}_1(t)+\mathcal{E}_2(t)+\mathcal{E}_3(t)\leqslant 4C^*\delta_0. \tag{4.1}$$

Let T^* denote the maximal time of existence of solutions with (4.1) holds true. We claim that $T^*=+\infty$. Thanks to proposition 3.1, we have

$$\mathcal{E}_1(t)+\mathcal{E}_2(t)+\mathcal{E}_3(t)< 4C^*\delta_0. \tag{4.2}$$

A standard continuous argument leads to $T^*=+\infty$, which completes the proof of theorem 1.1. □

Next, we employ once again proposition 3.1, combined with Aubin–Lions lemma (see lemma 2.2), to complete the proof of theorem 1.7.

Proof of theorem 1.7. For any $T>0$, let (u,τ) be a solution of system (PTT) in the time interval $[0,T]$ with the initial data $(|\nabla|^{-1}u_0,|\nabla|^{-1}\tau_0)$ belongs to H^3. We immediately deduce from proposition 3.1 that the following uniform estimates:

$$\begin{aligned}&\|(|\nabla|^{-1}u,|\nabla|^{-1}\tau)(t)\|_{H^3}^2\\&\quad+\int_0^t\left(\|u(s)\|_{H^3}^2+a\||\nabla|^{-1}\tau(s)\|_{H^3}^2\right)\mathrm{d}s\leqslant C\mathcal{E}(0),\quad\forall\, t\in[0,T].\end{aligned}$$

Furthermore, being able to use the structure of system (PTT), we get

$$\||\nabla|^{-1}u_t(\cdot,t)\|_{H^1}\leqslant C\mathcal{E}(0),\quad \||\nabla|^{-1}\tau_t(\cdot,t)\|_{H^2}\leqslant C\mathcal{E}(0),\quad\forall\, t\in[0,T].$$

The application of the above uniform estimates and lemma 2.2 implies that there exists a subsequence still denoted by (u,τ) such that for $0\leqslant b\leqslant Ca$, when (a,b)

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goes to (0, 0), then

$$
\begin{aligned}
|\nabla|^{-1}u &\to |\nabla|^{-1}\widetilde{u} \text{ strongly in } C([0,T]; H^{3-s}_{loc}),\\
|\nabla|^{-1}\tau &\to |\nabla|^{-1}\widetilde{\tau} \text{ strongly in } C([0,T]; H^{3-s}_{loc}),
\end{aligned}
$$

with $s \in (0,\, 1/2)$, and

$$
\begin{aligned}
&a|\nabla|^{-1}\tau \to 0 \text{ strongly in } L^2((0,T); H^3_{loc}),\\
&b|\nabla|^{-1}((\operatorname{tr}\tau)\tau) \to 0 \text{ strongly in } L^2((0,T); H^3_{loc}).
\end{aligned}
$$

This allows us to pass to the limit in system (PTT) with $0 \leqslant b \leqslant Ca$ when $(a,\, b)$ goes to (0, 0) and to conclude that the limit $(\widetilde{u},\, \widetilde{\tau})$ is indeed a solution of system (OB). This completes the proof of theorem 1.7. □

Finally, we use energy estimate again to show the rate of convergence for any positive time.

Proof of theorem 1.8*.* Let $(u,\, \tau)$ and $(\widetilde{u},\, \widetilde{\tau})$ be two solutions of system (PTT) and system (OB) with the same initial data, respectively. Indeed, according to theorem 1.1, we have, for any $t \geqslant 0$,

$$
\begin{aligned}
&\|u(t)\|^2_{H^2} + \|\tau(t)\|^2_{H^2}\\
&\quad + \int_0^t \left(\|u(s)\|^2_{H^3} + a\|\tau(s)\|^2_{H^2} + (1+s)^2\|\nabla^2 u(s)\|^2_{H^1}\right) \mathrm{d}s \leqslant C\mathcal{E}(0),
\end{aligned}
\tag{4.3}
$$

and

$$
\|\widetilde{u}(t)\|^2_{H^2} + \|\widetilde{\tau}(t)\|^2_{H^2} + \int_0^t \|\widetilde{u}(s)\|^2_{H^3}\,\mathrm{d}s \leqslant C\mathcal{E}(0). \tag{4.4}
$$

Define

$$
v := u - \widetilde{u}, \quad \sigma := \tau - \widetilde{\tau}.
$$

We shall focus on the rate of convergence of $(u,\, \tau)$ towards $(\widetilde{u},\, \widetilde{\tau})$ in H^1 norm. To prove this, we write the evolution equation for $(v,\, \sigma)$ that

$$
\begin{cases}
v_t + u\cdot\nabla v + v\cdot\nabla\widetilde{u} - \triangle v + \nabla p - \nabla\widetilde{p} = \operatorname{div}\sigma,\\
\sigma_t + u\cdot\nabla\sigma + v\cdot\nabla\widetilde{\tau} + a\tau + b(\operatorname{tr}\tau)\tau + Q(\sigma,\nabla u) + Q(\widetilde{\tau},\nabla v) = D(v),\\
\operatorname{div} v = 0,\\
v|_{t=0} = 0,\ \sigma|_{t=0} = 0.
\end{cases}
\tag{4.5}
$$

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Taking the L^2 inner product of the first equation of system (4.5) with v and the second equation of system (4.5) with σ, because of $\operatorname{div} v = 0$ and $\sigma_{ij} = \sigma_{ji}$, we get

$$
\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\|v\|_{L^2}^2 + \|\sigma\|_{L^2}^2\right) + \|\nabla v\|_{L^2}^2 \\
&\quad = -\int ((v\cdot\nabla\widetilde{u})\cdot v + (v\cdot\nabla\widetilde{\tau})\cdot\sigma)\,\mathrm{d}x - a\int \tau\cdot\sigma\,\mathrm{d}x - b\int (\operatorname{tr}\tau)\tau\cdot\sigma\,\mathrm{d}x \\
&\qquad - \int (Q(\sigma,\nabla u)\cdot\sigma + Q(\widetilde{\tau},\nabla v)\cdot\sigma)\,\mathrm{d}x \\
&\quad := M_1 + M_2 + M_3 + M_4.
\end{aligned}
$$

Using the energy estimates (4.3) and (4.4), we have

$$
\begin{aligned}
|M_1| &\leqslant C\left(\|\nabla\widetilde{u}\|_{L^3}\|v\|_{L^2} + \|\nabla\widetilde{\tau}\|_{L^3}\|\sigma\|_{L^2}\right)\|v\|_{L^6} \\
&\leqslant \frac{1}{4}\|\nabla v\|_{L^2}^2 + C\left(\|v\|_{L^2}^2 + \|\sigma\|_{L^2}^2\right), \\
|M_2| &\leqslant Ca\|\tau\|_{L^2}\|\sigma\|_{L^2} \leqslant a^2 + C\|\sigma\|_{L^2}^2, \\
|M_3| &\leqslant Cb\|\tau\|_{L^3}\|\tau\|_{L^6}\|\sigma\|_{L^2} \leqslant b^2 + C\|\sigma\|_{L^2}^2, \\
|M_4| &\leqslant C\|\sigma\|_{L^2}^2\|\nabla u\|_{L^\infty} + C\|\sigma\|_{L^2}\|\nabla v\|_{L^2}\|\widetilde{\tau}\|_{L^\infty} \\
&\leqslant \frac{1}{4}\|\nabla v\|_{L^2}^2 + C\|\sigma\|_{L^2}^2 + C\|\nabla^2 u\|_{H^1}\|\sigma\|_{L^2}^2.
\end{aligned}
$$

Together with the above estimates, this enables us to conclude that

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left(\|v\|_{L^2}^2 + \|\sigma\|_{L^2}^2\right) + \|\nabla v\|_{L^2}^2 \\
&\quad \leqslant \left(a^2 + b^2\right) + C\left(\|v\|_{L^2}^2 + \|\sigma\|_{L^2}^2\right) + C\|\nabla^2 u\|_{H^1}\|\sigma\|_{L^2}^2.
\end{aligned} \tag{4.6}
$$

Similar arguments lead to

$$
\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\nabla v\|_{L^2}^2 + \|\nabla\sigma\|_{L^2}^2\right) + \|\nabla^2 v\|_{L^2}^2 \\
&\quad = -\int_0^t (\nabla(v\cdot\nabla\widetilde{u})\cdot\nabla v + \nabla(v\cdot\nabla\widetilde{\tau})\cdot\nabla\sigma)\,\mathrm{d}x - a\int_0^t \nabla\tau\cdot\nabla\sigma\,\mathrm{d}x \\
&\qquad - b\int_0^t \nabla\left((\operatorname{tr}\tau)\tau\right)\cdot\nabla\sigma\,\mathrm{d}x - \int_0^t (\nabla Q(\sigma,\nabla u)\cdot\nabla\sigma + \nabla Q(\widetilde{\tau},\nabla v)\cdot\nabla\sigma)\,\mathrm{d}x \\
&\qquad - \int_0^t ((\nabla u\cdot\nabla v)\cdot\nabla v + (\nabla u\cdot\nabla\sigma)\cdot\nabla\sigma)\,\mathrm{d}x \\
&\quad := N_1 + N_2 + N_3 + N_4 + N_5.
\end{aligned}
$$

Taking advantage of the energy estimates (4.3) and (4.4), we infer that

$$\begin{aligned}
N_1 &\leqslant C\left(\|\nabla\widetilde{u}\|_{L^3}\|\nabla v\|_{L^2}+\|\nabla\widetilde{\tau}\|_{L^3}\|\nabla\sigma\|_{L^2}\right)\|\nabla v\|_{L^6}\\
&\quad + C\left(\|\nabla^2\widetilde{u}\|_{L^2}\|\nabla v\|_{L^2}+\|\nabla^2\widetilde{\tau}\|_{L^2}\|\nabla\sigma\|_{L^2}\right)\|v\|_{L^\infty}\\
&\leqslant \frac{1}{4}\|\nabla v\|_{H^1}^2 + C\left(\|\nabla v\|_{L^2}^2+\|\nabla\sigma\|_{L^2}^2\right),\\
N_2 &\leqslant Ca\|\nabla\tau\|_{L^2}\|\nabla\sigma\|_{L^2}\leqslant a^2+C\|\nabla\sigma\|_{L^2}^2,\\
N_3 &\leqslant Cb\|\tau\|_{L^3}\|\nabla\tau\|_{L^6}\|\nabla\sigma\|_{L^2}\leqslant b^2+C\|\nabla\sigma\|_{L^2}^2,\\
N_4 &\leqslant C\|\nabla\sigma\|_{L^2}^2\|\nabla u\|_{L^\infty}+C\|\sigma\|_{L^6}\|\nabla^2 u\|_{L^3}\|\nabla\sigma\|_{L^2}\\
&\quad + C\|\nabla\sigma\|_{L^2}\|\nabla v\|_{L^6}\|\nabla\widetilde{\tau}\|_{L^3}+C\|\nabla\sigma\|_{L^2}\|\nabla^2 v\|_{L^2}\|\widetilde{\tau}\|_{L^\infty}\\
&\leqslant C\|\nabla^2 u\|_{H^1}\|\nabla\sigma\|_{L^2}^2+C\|\nabla^2 v\|_{L^2}\|\nabla\sigma\|_{L^2}\\
&\leqslant \frac{1}{4}\|\nabla v\|_{H^1}^2+C\|\nabla\sigma\|_{L^2}^2+C\|\nabla^2 u\|_{H^1}\|\nabla\sigma\|_{L^2}^2,\\
N_5 &\leqslant C\|\nabla u\|_{L^\infty}\left(\|\nabla v\|_{L^2}^2+\|\nabla\sigma\|_{L^2}^2\right)\\
&\leqslant C\|\nabla^2 u\|_{H^1}\left(\|\nabla v\|_{L^2}^2+\|\nabla\sigma\|_{L^2}^2\right).
\end{aligned}$$

Hence, the above estimates yield

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\nabla v\|_{L^2}^2+\|\nabla\sigma\|_{L^2}^2\right)+\|\nabla^2 v\|_{L^2}^2\\
&\quad\leqslant \frac{1}{2}\|\nabla v\|_{H^1}^2+\left(a^2+b^2\right)+C\left(\|\nabla v\|_{L^2}^2+\|\nabla\sigma\|_{L^2}^2\right)\\
&\qquad + C\|\nabla^2 u\|_{H^1}\left(\|\nabla v\|_{L^2}^2+\|\nabla\sigma\|_{L^2}^2\right). \qquad (4.7)
\end{aligned}$$

Together (4.6) with (4.7), this implies that

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left(\|v\|_{H^1}^2+\|\sigma\|_{H^1}^2\right)+\frac{1}{2}\|\nabla v\|_{H^1}^2\\
&\quad\leqslant \left(a^2+b^2\right)+C\left(1+\|\nabla^2 u\|_{H^1}\right)\left(\|v\|_{H^1}^2+\|\sigma\|_{H^1}^2\right).
\end{aligned}$$

As we have

$$\begin{aligned}
&\int_0^t \|\nabla^2 u(s)\|_{H^1}\,\mathrm{d}s\\
&\quad\leqslant C\left(\int_0^t (1+s)^2\|\nabla^2 u(s)\|_{H^1}^2\,\mathrm{d}s\right)^{1/2}\left(\int_0^t (1+s)^{-2}\,\mathrm{d}s\right)^{1/2}\leqslant C\mathcal{E}(0),
\end{aligned}$$

then Gronwall's lemma thus leads to

$$\|(v,\sigma)(t)\|_{H^1}^2\leqslant C\left(a^2+b^2\right)t\,\mathrm{e}^{C(t+1)},\quad \forall\, t\geqslant 0, \qquad (4.8)$$

for some positive constant C, independent of t, a and b. This completes the proof of theorem 1.8. □

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