On the projective geometry of paths

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INTRODUCTION.

An affine connection in an n-dimensional manifold X_n defines a system of paths, but conversely a connection is not defined uniquely by a system of paths. It was shown by H. Weyl[1] that any two affine connections whose components are related by an equation of the form

$$(1) \qquad {}'\Gamma^h_{ji} = \Gamma^h_{ji} + p_j A^h_i + p_i A^h_j,$$

where A^h_i is the unit affinor[2], give the same system of paths. In the geometry of a system of paths, a particular parameter on the paths, called the *projective normal parameter*, plays an important part. This parameter, which is invariant under a transformation of connection (1), was introduced by J. H. C. Whitehead[3]. It can be defined by means of a Schwarzian differential equation and it is determined up to linear fractional transformations[4]. In § 1 this method is briefly discussed.

In §2 another method of treating the projective geometry of paths is given, based upon the introduction of homogeneous coordinates in an n-dimensional manifold[5]. Instead of one parameter two homogeneous parameters u^0, u^1 are introduced on each path. This leads to a set of coefficients of a projective connection on each path. Then a preferred system of projective parameters is obtained by putting

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[1] H. Weyl, "Zur Infinitesimalgeometrie: Einordnung der projektiven und der konformen Auffassung," *Göttinger Nachrichten* (1921), pp. 99-112.

[2] In this paper the term "affinor" is used instead of "tensor."

[3] J. H. C. Whitehead, "The representation of projective spaces," *Ann. of Math.*, 32 (1931), pp. 327-360.

[4] L. Berwald, "On the projective geometry of paths," *Ann. of Math.*, 37 (1936), pp. 879-898.

[5] D. van Dantzig, "Theorie des projektiven Zusammenhangs n-dimensionaler Räume," *Math. Annalen* 106 (1932), pp. 400-454. See also J. A. Schouten and J. Haantjes, "Zur allgemeinen projektiven Differentialgeometrie," *Compositio Math.* 3 (1935), pp. 1-51. This paper is referred to as A. P. D.

these coefficients equal to zero. Such a preferred system is determined up to linear homogeneous transformations with constant coefficients. Hence the ratio $p = u^1/u^0$ is a non-homogeneous parameter, which is defined up to linear fractional transformations. In § 3 it is shown that the parameter p is a projective normal parameter.

§ 1. PATHS AND AFFINE CONNECTIONS.

1. *Paths in L_n.*

We consider an n-dimensional manifold L_n, in which a symmetrical affine connection Γ^h_{ji} is given. The coordinates of a point are denoted by ξ^h $(h, \ldots, m = 1, \ldots, n)$. A coordinate transformation is given by a set of n analytic functions

$$\xi^{h'} = \xi^{h'}(\xi^1, \ldots, \xi^n), \tag{1.1}$$

whose functional determinant is different from zero for all points under consideration.

By a *path* we mean a curve $\xi^h = \xi^h(t)$, where $\xi^h(t)$ is a solution of the following system of differential equations

$$\frac{d^2\xi^h}{dt^2} + \Gamma^h_{ji}\frac{d\xi^j}{dt}\frac{d\xi^i}{dt} = \beta\frac{d\xi^h}{dt}, \tag{1.2}$$

β being a function of t. Thus the paths are autoparallel curves. It is possible to introduce a new parameter $s = s(t)$ on each curve, such that the differential equations take the form

$$\frac{d^2\xi^h}{ds^2} + \Gamma^h_{ji}\frac{d\xi^j}{ds}\frac{d\xi^i}{ds} = 0. \tag{1.3}$$

The parameter s is called an *affine parameter* of the system of paths. On each path it is determined up to an arbitrary linear transformation $s' = as + b$, a and b being arbitrary constants.

2. *Projective transformations of an affine connection.*

An affine connection defines a system of paths, but a system of paths (1.2) does not define a symmetrical connection uniquely. For the equations (1.2) remain unaltered if we put in the place of Γ^h_{ji} the functions

$${}'\Gamma^h_{ji} = \Gamma^h_{ji} + p_j A^h_i + p_i A^h_j, \tag{1.4}$$

where p_i is an arbitrary covariant vector and A^h_j denotes the unit affinor. Such a transformation of connection is called a *projective transformation of the affine connection*[1]. In general it changes the parameter s.

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[1] H. Weyl, *l. c.*

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The object with components

$$\Pi^h_{ji} = \Gamma^h_{ji} - \frac{1}{n+1} (A^h_j \, \Gamma^l_{il} + A^h_i \, \Gamma^l_{jl}) \tag{1.5}$$

is unaltered by projective transformations of connection. These Π^h_{ji}, which satisfy the identity $\Pi^h_{jh} = 0$ are called the *Thomas parameters*[1].

By replacing Γ^h_{ji} in (1.2) by Π^h_{ji} we get the same set of curves. The parameter s corresponding to Π^h_{ji}, that is the parameter s for which the differential equations have the form

$$\frac{d^2 \xi^h}{ds^2} + \Pi^h_{ji} \frac{d\xi^j}{ds} \frac{d\xi^i}{ds} = 0, \tag{1.6}$$

is called *the projective parameter of T. Y. Thomas.* It does not change under projective transformations of connection. But since the Π^h_{ji} are not transformed under a transformation of coordinates like the components of an affine connection, this projective parameter is *not a scalar.*

3. *The projective normal parameter.*

The curvature affinor of the affine connection Γ^h_{ji} is defined by

$$R_{kji}{}^{h} = 2\partial_{[k} \, \Gamma^h_{j]i} + 2 \, \Gamma^h_{[k \,|\, l \,|} \, \Gamma^l_{j]i}, \qquad \left(\partial_j = \frac{\partial}{\partial \xi^j}\right) \tag{1.7}$$

where the square brackets mean alternation with respect to the indices k and j (for example, $2\partial_{[k} \, w_{i]} = \partial_k \, w_i - \partial_i \, w_k$). Contraction gives the affinor

$$R_{ji} = R_{hji}{}^{h}. \tag{1.8}$$

A *projective normal parameter* π on the paths (1.2) is now defined by means of a differential equation of the form

$$\{\pi, s\} = \frac{2}{n-1} R_{hi} \frac{d\xi^h}{ds} \frac{d\xi^i}{ds}, \tag{1.9}$$

where s is an affine parameter belonging to the connection Γ^h_{ji} and $\{\pi, s\}$ stands for the Schwarzian derivative

$$\{\pi, s\} = \frac{\dfrac{d^3\pi}{ds^3}}{\dfrac{d\pi}{ds}} - \frac{3}{2} \left(\frac{\dfrac{d^2\pi}{ds^2}}{\dfrac{d\pi}{ds}} \right)^2. \tag{1.10}$$

[1] T. Y. Thomas, "On the projective and equiprojective geometry of paths," *Proc. Nat. Acad. Sci., U.S.A.* **11** (1925), pp. 199-203; "A projective theory of affinely connected manifolds," *Math. Zeitschrift* **25** (1926), pp. 723-733.

By (1.9) π is defined as function of s up to linear fractional transformations. It can be proved that a projective normal parameter of a system of paths has the following properties[1]:

(*a*) It is not altered by transformations of coordinates, which means that π is a scalar.

(*b*) It is not altered by projective transformations of the connection.

If this parameter $\pi(s)$ is introduced in the differential equations (1.3) these equations take the form[2]

$$\frac{d^2 \xi^h}{d\pi^2} + \Gamma^h_{ji} \frac{d\xi^j}{d\pi} \frac{d\xi^i}{d\pi} + 2\alpha \frac{d\xi^h}{d\pi} = 0, \tag{1.11}$$

where α satisfies the equation

$$2\alpha \left(\frac{d\pi}{ds}\right)^2 = \frac{d^2 \pi}{ds^2}. \tag{1.12}$$

From this equation and (1·9) (1·10) it follows by differentiation

$$\frac{d\alpha}{d\pi} + \alpha^2 - \frac{1}{n-1} R_{hi} \frac{d\xi^h}{d\pi} \frac{d\xi^i}{d\pi} = 0. \tag{1.13}$$

It can be shown that conversely the equations (1.11) and (1.13) determine a projective normal parameter π.

§ 2. PATHS AND PROJECTIVE CONNECTIONS.

1. *Paths in H_n.*

We introduce in the n-dimensional manifold homogeneous coordinates x^κ, ($\kappa, \ldots, \tau = 0, 1, \ldots, n$), and subject these coordinates to the set of transformations

$$x^{\kappa'} = x^{\kappa'}(x^0, \ldots, x^n), \tag{2.1}$$

where the $x^{\kappa'}$ are homogeneous analytic functions of the first degree in the x^κ, such that the functional determinant is different from zero for all points under consideration. Such an n-dimensional manifold with homogeneous coordinates is called a *generalized projective space*[3] and is denoted by H_n. A particular property of an H_n is that the coordinates x^κ of a point transform like the components of a projective contravariant vector, for we have from (2.1) according to Euler's condition of homogeneity

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[1] J. H. C. Whitehead, *l. c.*; L. Berwald, *l. c.*, p. 882.

[2] J. H. C. Whitehead, *l. c.*, p. 338; L. Berwald, *l. c.*, p. 884.

[3] D. van Dantzig, *l. c.*

(2.2) $$x^{\kappa'} = x^{\kappa}\,\partial_{\kappa}\,x^{\kappa'} = \mathscr{A}^{\kappa'}_{\kappa}\,x^{\kappa},\ \ \partial_{\mu} = \frac{\partial}{\partial x^{\mu}},\ \ \mathscr{A}^{\kappa'}_{\kappa} = \partial_{\kappa}\,x^{\kappa'}.$$

A covariant derivative in the H_n is given by $(n+1)^3$ functions $\Pi^{\kappa}_{\mu\lambda}$ called *the coefficients of the projective connection.* These coefficients are homogeneous functions of x^{κ} of degree -1. From the transformation formula for the coefficients $\Pi^{\kappa}_{\mu\lambda}$

(2.3) $$\Pi^{\kappa'}_{\mu'\lambda'} = \mathscr{A}^{\kappa'\,\mu\,\lambda}_{\kappa\,\mu'\,\lambda'}\,\Pi^{\kappa}_{\mu\lambda} + \mathscr{A}^{\kappa'}_{\kappa}\,\partial_{\mu'}\,\mathscr{A}^{\kappa}_{\lambda'};$$
$$\mathscr{A}^{\kappa'\,\mu\,\lambda}_{\kappa\,\mu'\,\lambda'} = \mathscr{A}^{\kappa'}_{\kappa}\,\mathscr{A}^{\mu}_{\mu'}\,\mathscr{A}^{\lambda}_{\lambda'};\ \ \mathscr{A}^{\lambda}_{\lambda'} = \partial_{\lambda'}\,x^{\lambda};$$

it follows that the

(2.4) $$\Pi^{\kappa}_{\mu\lambda}\,x^{\mu}$$

transform like the components of a projective affinor. Hereafter homogeneous projective affinors will be called *projectors.*

In an H_n the equations $x^{\kappa} = f^{\kappa}(t)$ and $x^{\kappa} = \rho(t) f^{\kappa}(t)$ define the same curve. From this it follows that the differentials dx^{κ} define the same direction as $\rho dx^{\kappa} + x^{\kappa}\,d\rho$. In other words, the vectors

$$\frac{dx^{\kappa}}{dt} \ \text{and}\ \rho\,\frac{dx^{\kappa}}{dt} + \frac{d\rho}{dt}\,x^{\kappa}$$

define the same direction. We restrict ourselves to symmetrical connections with the property that the projector (2.4) is zero, hence

(2.5) $$\Pi^{\kappa}_{\mu\lambda} = \Pi^{\kappa}_{\lambda\mu},\ \ \Pi^{\kappa}_{\mu\lambda}\,x^{\mu} = 0.$$

A result of the hypothesis $\Pi^{\kappa}_{\mu\lambda}\,x^{\mu} = 0$ is that there exists a *displacement* for a direction in its own direction. For if the homogeneous vector v^{κ} satisfies the relation

(2.6) $$dx^{\mu}\,\nabla_{\mu}\,v^{\kappa} :: v^{\kappa},$$

then it satisfies the relation, which is obtained from (2.6) by putting $\rho dx^{\mu} + x^{\mu}\,d\rho$ instead of dx^{μ}. Thus in this case a *path* can be defined as an *autoparallel curve.* In a more general H_n autoparallel curves need not exist.

If the curve $x^{\kappa} = x^{\kappa}(t)$ is a path (autoparallel curve), then the vector

$$\frac{dx^{\mu}}{dt}\,\nabla_{\mu}\,\frac{dx^{\kappa}}{dt}$$

has the same direction as $\dfrac{dx^{\kappa}}{dt}$. Therefore, we find for the differential equation of the paths

(2.7) $$\frac{dx^{\mu}}{dt}\,\nabla_{\mu}\,\frac{dx^{\kappa}}{dt} = \frac{d^2 x^{\kappa}}{dt^2} + \Pi^{\kappa}_{\mu\lambda}\frac{dx^{\mu}}{dt}\,\frac{dx^{\lambda}}{dt} = \alpha x^{\kappa} + \beta\,\frac{dx^{\kappa}}{dt}.$$

In these equations α and β depend on t. The equations (2.7) define $\infty^{2(n-2)}$ paths, such that through each point in a certain region of H_n there is a unique path in each direction.

The *projector of curvature* is defined by

(2.8) $$N_{\nu\mu\lambda}^{\cdot\cdot\cdot\kappa} = 2\partial_{[\nu}\Pi_{\mu]\lambda}^{\kappa} + 2\,\Pi_{[\nu|\rho|}^{\kappa}\Pi_{\mu]\lambda}^{\rho}.$$

Transvection with x^ν gives

(2.9) $$N_{\nu\mu\lambda}^{\cdot\cdot\cdot\kappa}x^\nu = x^\nu\partial_\nu\Pi_{\mu\lambda}^{\kappa} - x^\nu\partial_\mu\Pi_{\nu\lambda}^{\kappa} = -\Pi_{\mu\lambda}^{\kappa} + \Pi_{\mu\lambda}^{\kappa} = 0.$$

2. *The transformations of the projective connection, which leave the system of paths invariant.*

A projective connection, satisfying (2.5), defines a system of paths, but conversely the system of paths does not define uniquely a symmetrical projective connection. Indeed, the transformation of connection[1]

(2.10) $$'\Pi_{\mu\lambda}^{\kappa} = \Pi_{\mu\lambda}^{\kappa} + Z_{\mu\lambda}x^\kappa + z_\mu \mathcal{A}_\lambda^\kappa + z_\lambda \mathcal{A}_\mu^\kappa,$$

where $Z_{\mu\lambda}$ and z_μ are arbitrary projectors, homogeneous of degree -2 and -1 respectively and $\mathcal{A}_\lambda^\kappa$ denotes the unit projector, leaves the system of paths, defined by (2.7), invariant. The restriction (2.5) gives

(2.11) $$Z_{\mu\lambda} = Z_{\lambda\mu}, \quad Z_{\mu\lambda}x^\mu + z_\lambda = 0, \quad z_\mu x^\mu = 0.$$

The projectors $Z_{\mu\lambda}$ and z_λ can be chosen in such a way that the contracted projector of curvature $'N_{\mu\lambda} = {}'N_{\kappa\mu\lambda}^{\cdot\cdot\cdot\kappa}$ of the new connection $'\Pi_{\mu\lambda}^{\kappa}$ vanishes. On calculation we find[2]

(2.12) $$Z_{\mu\lambda} - \nabla_\mu z_\lambda + z_\mu z_\lambda = -\frac{1}{n^2-1}(n\,N_{\kappa\mu\lambda}^{\cdot\cdot\cdot\kappa} + N_{\kappa\lambda\mu}^{\cdot\cdot\cdot\kappa}),$$

so that

(2.13) $$(n+1)\nabla_{[\mu}z_{\lambda]} = N_{\kappa[\mu\lambda]}^{\cdot\cdot\cdot\kappa} = N_{[\mu\lambda]}.$$

This equation is easily shown to be integrable by use of Bianchi's identity, and determines z_λ but for a gradient vector. From (2.12) and (2.13) we obtain the following theorem.

A system of paths (2.7) *determines a symmetrical projective connection with*

(2.14) $$\Pi_{\mu\lambda}^{\kappa}x^\mu = 0, \quad N_{\mu\lambda} = 0$$

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[1] A. P. D., p. 32.

[2] A P. D., p. 33.

up to a transformation of the form (2.10), *where* $Z_{\mu\lambda}$ *and* z_λ *satisfy, besides* (2.11) *the following equations*

(2.15a) $$Z_{\mu\lambda} - \nabla_\mu z_\lambda + z_\mu z_\lambda = 0,$$

(2.15b) $$\nabla_{[\mu} z_{\lambda]} = 0.$$

The equation (2.15b) means that z_λ is a gradient vector, $z_\lambda = \partial_\lambda z$.

3. *The projective parameter p.*

Let us now introduce two homogeneous parameters on the paths. A path $x^\kappa = x^\kappa(t)$ may also be given by the equations

(2.16) $$x^\kappa = x^\kappa(u^0, u^1) = x^\kappa(u^a), \quad (a, \ldots, g = 0, 1)$$

where the $x^\kappa(u^0, u^1)$ are homogeneous functions of degree 1[1]. Then u^a and σu^a determine the same point on the curve. A transformation of homogeneous parameters is given by a set of functions

(2.17) $$u^{a'} = u^{a'}(u^0, u^1)$$

homogeneous of degree 1. From (2.16) and (2.17) it follows from Euler's condition of homogeneity

(2.18) $$x^\kappa = B^\kappa_a u^a, \quad B^\kappa_a = \partial_a x^\kappa, \quad \partial_a = \frac{\partial}{\partial u^a}$$

(2.19) $$u^{a'} = B^{a'}_a u^a, \quad B^{a'}_a = \partial_a u^{a'}.$$

The vectors B^κ_0 and B^κ_1 have the same direction as $\frac{dx^\kappa}{dt}$, from which it follows that both B^κ_0 and B^κ_1 can be expressed linearly in terms of x^κ and $\frac{dx^\kappa}{dt}$

(2.20) $$B^\kappa_a = p_a \frac{dx^\kappa}{dt} + q_a x^\kappa.$$

These equations may be solved for x^κ and $\frac{dx^\kappa}{dt}$, giving (c.*f.* 2.18),

(2.21) $$\frac{dx^\kappa}{dt} = \mathfrak{r}^a B^\kappa_a, \quad x^\kappa = u^a B^\kappa_a.$$

From (2.20) it follows by covariant differentiation, in consequence of (2.7), that $B^\mu_c \nabla_\mu B^\kappa_a$ is a linear expression in x^κ and $\frac{dx^\kappa}{dt}$, and therefore, by (2.21), a linear expression in B^κ_0 and B^κ_1.
Hence the differential equations of the paths take the form

(2.22) $$B^\mu_c \nabla_\mu B^\kappa_b = \partial_c B^\kappa_b + \Pi^\kappa_{\mu\lambda} B^{\mu\lambda}_{cb} = \Gamma^a_{cb} B^\kappa_a, \quad B^{\mu\lambda}_{cb} = B^\mu_c B^\lambda_b,$$

[1] The equations are obtainable in this form by putting $t = u^1/u^0$ in $x^\kappa = x^\kappa(t)$ and multiplying by an arbitrary homogeneous function of degree 1.

the $\Pi^{\kappa}_{\mu\lambda}$ being the coefficients of one of the projective connections, determined by the system of paths, which satisfy the equations (2.14). From (2.22) it follows that the functions Γ^a_{cb} are homogeneous in u^a of degree -1. When we transform the parameters u^a by any transformation (2.17) the coefficients Γ^a_{cb} transform according to the equations

$$(2.23) \qquad \Gamma^{a'}_{c'b'} = B^{a'}_{a}{}^{c\,b}_{c'b'}\,\Gamma^a_{cb} + B^{a'}_{a}\,\partial_{c'}\,B^a_{b'}; \quad (B^a_{b'} = \partial_{b'}\,u^a).$$

The Γ^a_{cb} transform therefore like the coefficients of a projective connection in an H_1. Moreover we have from (2.22)

$$(2.24) \qquad \Gamma^a_{cb}\,u^c = 0; \quad \Gamma^a_{cb} = \Gamma^a_{bc}.$$

It is well known that a necessary and sufficient condition for the existence of a system of parameters u^a, such that all of the Γ^a_{cb} are zero, is that

$$(2.25) \qquad M_{\dot{d}\dot{c}\dot{b}}{}^{a} = 2\,\partial_{[d}\,\Gamma^a_{c]b} + 2\,\Gamma^a_{[d|e|}\,\Gamma^e_{c]b}$$

be zero. The transvection of $M_{\dot{d}\dot{c}\dot{b}}{}^{a}$ and u^d is (*c.f.* (2.9))

$$(2.26) \qquad u^d\,M_{\dot{d}\dot{c}\dot{b}}{}^{a} = u^d\,\partial_d\,\Gamma^a_{cb} - u^d\,\partial_c\,\Gamma^a_{db} = 0.$$

The quantity $M_{\dot{d}\dot{c}\dot{b}}{}^{a}$ is skew symmetrical in the indices d and c. Therefore, the rank of $M_{\dot{d}\dot{c}\dot{b}}{}^{a}$ with respect to the indices d and c must be either 2 or 0. The equations (2.26) express that the rank is less than 2, hence the rank is 0. Thus we have

$$(2.27) \qquad M_{\dot{d}\dot{c}\dot{b}}{}^{a} = 0.$$

This means that there exists a system of parameters u^a for which $\Gamma^a_{cb} = 0$ and from the transformation (2.23) of Γ^a_{cb} it follows that this system is determined up to linear homogeneous transformations with constant coefficients. The non-homogeneous parameter

$$(2.28) \qquad p = \frac{u^1}{u^0}$$

is then determined up to linear fractional transformations. We call this parameter a *projective parameter*. In § 3 it is proved that p is a projective normal parameter.

We have to prove first that the parameter p is unaltered by a transformation of connection (2.10), where $Z_{\mu\lambda}$ and z_λ satisfy the conditions (2.11) and (2.15). From (2.15) we have

$$(2.29) \qquad z_\mu = \partial_\mu\,z, \;\; Z_{\mu\lambda} = \nabla_\mu\,z_\lambda - z_\mu\,z_\lambda = \nabla_\mu\,\nabla_\lambda\,z - (\partial_\mu\,z)\,(\partial_\lambda\,z)$$

and the function z is homogeneous of degree zero. By (2.22) such a transformation of connection causes the following transformation of Γ^a_{cb}

$$(2.30)\qquad \begin{aligned}'\Gamma^a_{cb} &= \Gamma^a_{cb} + Z_{\mu\lambda} B^{\mu\lambda}_{cb} u^a + z_\mu B^\mu_c B^a_b + z_\mu B^\mu_b B^a_c \\ &= \Gamma^a_{cb} + Z_{cb} u^a + z_c B^a_b + z_b B^a_c, \quad B^a_b \begin{cases} = 1, & a = b \\ = 0, & a \neq b \end{cases}\end{aligned}$$

where

$$(2.31)\qquad \begin{aligned} z_c &= B^\mu_c \partial_\mu z = \partial_c z \\ Z_{cb} &= B^{\mu\lambda}_{cb} (\partial_\mu \partial_\lambda z - \Pi^\kappa_{\mu\lambda} \partial_\kappa z) - z_c z_b = \partial_c z_b - \Gamma^a_{cb} z_a - z_c z_b. \end{aligned}$$

For a preferred system of parameters ($\Gamma^a_{cb} = 0$) we have therefore

$$(2.32)\qquad '\Gamma^a_{cb} = (\partial_c z_b - z_c z_b) u^a + z_c B^a_b + z_b B^a_c.$$

The equations (2.24) hold also for $'\Gamma^a_{cb}$; from which it follows, as we have seen above, that the projector of curvature $'M_{\dot d \dot c \dot b}{}^a$ belonging to $'\Gamma^a_{cb}$ vanishes. There exist, therefore, systems of parameters $u^{a'}$ for which $'\Gamma^{a'}_{c'b'} = 0$. We shall now show that one of these systems of parameters can be obtained by a transformation of parameters of the following form

$$(2.33)\qquad \begin{aligned} u^{1'} &= \rho(u^a) u^1 \\ u^{0'} &= \rho(u^a) u^0, \end{aligned}$$

where $\rho(u^a)$ is a homogeneous function of degree zero. From (2.33) it follows

$$(2.34)\qquad \begin{aligned} B^{a'}_a &\stackrel{*}{=} \rho \delta^{a'}_b (B^b_a + u^b \partial_a \log \rho) \\ B^a_{a'} &\stackrel{*}{=} \rho^{-1} \delta^b_{a'} (B^a_b - u^a \partial_b \log \rho) \end{aligned}$$

where the $\delta^{a'}_b$ denote the generalized Kronecker symbols[1]. Substitution in the transformation formula (2.23) for $'\Gamma^a_{cb}$ gives

$$(2.35)\qquad \begin{aligned} '\Gamma^{a'}_{c'b'} \stackrel{*}{=} \rho^{-1} \delta^{a'}_a \delta^c_{c'} \delta^b_{b'} [&'\Gamma^a_{cb} + \{- \partial_c \partial_b \log \rho + {}'\Gamma^e_{cb} \partial_e \log \rho \\ &- (\partial_c \log \rho)(\partial_b \log \rho)\} u^a - \partial_b \log \rho\, B^a_c - \partial_c \log \rho\, B^a_b]. \end{aligned}$$

By substituting the expression (2.32) for $'\Gamma^a_{cb}$, we get

$$(2.36)\qquad \begin{aligned} '\Gamma^{a'}_{c'b'} \stackrel{*}{=} \rho^{-1} \delta^{a'}_a \delta^c_{c'} \delta^b_{b'} [&\{\partial_c z_b - z_c z_b - \partial^c \partial_b \log \rho - (\partial_c \log \rho)(\partial_b \log \rho) \\ &+ z_c \partial_b \log \rho + z_b \partial_c \log \rho\} u^a + (z_b - \partial_b \log \rho) B^a_c + (z_c - \partial_c \log \rho) B^a_b] \end{aligned}$$

and from this equation we see that we get

$$(2.37)\qquad '\Gamma^{a'}_{c'b'} \stackrel{*}{=} 0$$

by putting

$$(2.38)\qquad \log \rho = z, \quad \partial_c \log \rho = \partial_c z = z_c.$$

[1] The sign $\stackrel{*}{=}$ means that the equation holds with respect to the coordinate system or systems used in the equation itself; it need not hold with respect to another system.

Thus it is possible to get a preferred system of homogeneous parameters $u^{a'}$ belonging to $'\Gamma^a_{cb}$ from a preferred system of parameters belonging to Γ^a_{cb} by a transformation of the form (2.33). From (2.28) and (2.33) it follows that

$$\frac{u^{1'}}{u^{0'}} = \frac{u^1}{u^0} = p. \tag{2.39}$$

Hence, the projective parameter p remains unaltered under a transformation (2.10) of the projective connection. It is, therefore, determined by the system of paths up to linear fractional transformations.

§ 3. THE TWO PARAMETERS p AND π.

1. *Introduction of non-homogeneous coordinates in H_n.*

In this paragraph we shall prove that the projective parameters p and π are "identical," in other words that p is a projective normal parameter.

In order to compare the parameters p and π, we have to introduce[1] non-homogeneous coordinates in H_n. A system of non-homogeneous coordinates in H_n is given by a set of n functions

$$\xi^h = \xi^h(x^0, \ldots, x^n), \tag{3.1}$$

homogeneous of degree zero, whose functional matrix

$$\| \mathcal{E}^h_\lambda \|, \text{ where } \mathcal{E}^h_\lambda = \partial_\lambda \xi^h, \tag{3.2}$$

has rank n. From (3.1) it follows by Euler's condition of homogeneity that

$$x^\mu \mathcal{E}^h_\mu = 0. \tag{3.3}$$

Moreover we introduce a projective covariant vector field q_λ of degree -1, such that

$$q_\lambda x^\lambda = 1. \tag{3.4}$$

But for this equation, q_λ may be chosen at will. This vector q_λ enables us to define the inverse of $\mathcal{E}^h_\lambda$. We define the quantity $\mathcal{E}^\kappa_i$ by means of the equations

$$\begin{aligned} \mathcal{E}^\kappa_i \mathcal{E}^h_\kappa &= A^h_i \qquad \text{(unit affinor)} \\ \mathcal{E}^\kappa_i q_\kappa &= 0. \end{aligned} \tag{3.5}$$

Multiplication with $\mathcal{E}^i_\lambda$ gives

$$\mathcal{E}^\kappa_\lambda = \mathcal{E}^\kappa_i \mathcal{E}^i_\lambda = \mathcal{A}^\kappa_\lambda - x^\kappa q_\lambda. \tag{3.6}$$

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[1] A. P. D., p. 11.

2. *The induced affine connection.*

We shall now prove that the quantities

$$\Gamma^h_{ji} = \mathcal{E}^{h\mu\lambda}_{\kappa ji}\, \Pi^\kappa_{\mu\lambda} - \mathcal{E}^{\mu\lambda}_{ji}\, \partial_\mu \mathcal{E}^h_\lambda, \quad (\mathcal{E}^{h\mu\lambda}_{\kappa ji} = \mathcal{E}^h_\kappa\, \mathcal{E}^\mu_j\, \mathcal{E}^\lambda_i) \tag{3.7}$$

are the coefficients of an affine connection, which gives the same system of paths as the projective connection $\Pi^\kappa_{\mu\lambda}$. The system of geodesics, defined by the projective connection $\Pi^\kappa_{\mu\lambda}$, is given by the differential equations (2.7). If $x^\kappa = x^\kappa(t)$ is the equation of a path in homogeneous coordinates, then the non-homogeneous equation is given by

$$\xi^h = \xi^h(x^\kappa(t)) = \xi^h(t). \tag{3.8}$$

From this equation we have

$$\begin{aligned} \frac{d\xi^h}{dt} &= \mathcal{E}^h_\lambda \frac{dx^\lambda}{dt}, \\ \frac{d^2\xi^h}{dt^2} &= \mathcal{E}^h_\lambda \frac{d^2x^\lambda}{dt^2} + \frac{dx^\mu}{dt}\frac{dx^\lambda}{dt}\, \partial_\mu \mathcal{E}^h_\lambda. \end{aligned} \tag{3.9}$$

Consequently

$$\begin{aligned} \frac{d^2\xi^h}{dt^2} + \Gamma^h_{ji}\frac{d\xi^j}{dt}\frac{d\xi^i}{dt} &= \mathcal{E}^h_\kappa \left(\frac{d^2x^\kappa}{dt^2} + \Pi^\kappa_{\rho\sigma}\, \mathcal{E}^{\rho\sigma}_{\mu\lambda}\frac{dx^\mu}{dt}\frac{dx^\lambda}{dt}\right) \\ &\quad + (\partial_\mu \mathcal{E}^h_\lambda - \mathcal{E}^{\rho\sigma}_{\mu\lambda}\, \partial_\rho \mathcal{E}^h_\sigma)\frac{dx^\mu}{dt}\frac{dx^\lambda}{dt}. \end{aligned} \tag{3.10}$$

The transvection $\Pi^\kappa_{\mu\lambda}\, x^\mu$ is zero by (2.14). Hence

$$\Pi^\kappa_{\rho\sigma}\, \mathcal{E}^{\rho\sigma}_{\mu\lambda} = \Pi^\kappa_{\mu\lambda}. \tag{3.11}$$

Furthermore we have from (3.6) and the definition of $\mathcal{E}^h_\mu$

$$\mathcal{E}^{\rho\sigma}_{\mu\lambda}\, \partial_\rho \mathcal{E}^h_\sigma = \partial_\mu \mathcal{E}^h_\lambda + q_\mu\, \mathcal{E}^h_\lambda + q_\lambda\, \mathcal{E}^h_\mu. \tag{3.12}$$

Substituting these expressions in (3.10) we see from (2.7) that the right hand side is proportional to $\frac{d\xi^h}{dt}$. Hence the non-homogeneous differential equations for the geodesics are

$$\frac{d^2\xi^h}{dt^2} + \Gamma^h_{ji}\frac{d\xi^j}{dt}\frac{d\xi^i}{dt} = a\,\frac{d\xi^h}{dt}. \tag{3.13}$$

These equations show that the Γ^h_{ji} defined by (3.7) transform as the coefficients of an affine connection. This connection is called *the induced affine connection*. It defines the same system of paths as the projective connection.

If we choose another projective covariant vectorfield q_λ, then

(3.7) defines another affine connection, but this connection gives the same system of paths and can, therefore, be obtained from the connection Γ^h_{ji} by a projective transformation of connection (1.4).

The curvature tensor of the affine connection Γ^h_{ji} is defined by formula (1.7), namely

$$R_{kji}{}^{h} = 2\,\partial_{[k}\Gamma^h_{j]i} + 2\,\Gamma^h_{[k|l|}\Gamma^l_{j]i}. \tag{3.14}$$

When the expressions (3.7) for Γ^h_{ji} are substituted in the above equations, we find, after some calculation,

$$R_{kji}{}^{h} = \mathfrak{E}^{h\nu\mu\lambda}_{\kappa kji} N_{\nu\mu\lambda}{}^{\kappa} - 2q_{[kj}\,A^h_{i]} + 2\,A^h_{[k}\,q_{j]i}, \tag{3.15}$$

where

$$q_{ji} = \mathfrak{E}^{\nu\mu}_{ji}\,\nabla_\nu q_\mu. \tag{3.16}$$

The projector of curvature $N_{\nu\mu\lambda}{}^{\kappa}$ has according to (2.9) and (2.14) the properties

$$N_{\nu\mu\lambda}{}^{\kappa}x^\nu = 0, \quad N_{\mu\lambda} = 0. \tag{3.17}$$

Contraction of (3.15) with respect to the indices k and h gives therefore

$$R_{ji} = -\,2q_{[ji]} + nq_{ji} - q_{ji}, \tag{3.18}$$

$$R_{(ji)} = (n-1)\,q_{(ji)}, \tag{3.19}$$

where the round brackets indicate symmetrization with respect to i and j.

3. *The parameter p as independent variable.*

Let u^a be a preferred system of homogeneous parameters on the paths. Then the differential equations for the paths are (2.22)

$$\partial_c B^\kappa_b + \Pi^\kappa_{\mu\lambda}\,B^{\mu\lambda}_{cb} = 0. \tag{3.20}$$

If $x^\kappa = x^\kappa(u^0, u^1)$ is the equation of a path in homogeneous coordinates, then

$$\xi^h = \xi^h(x^\kappa(u^0, u^1)) = \xi^h(u^0, u^1) = \xi^h\left(1, \frac{u^1}{u^0}\right) = \xi^h(p) \tag{3.21}$$

is the equation of the same path in non-homogeneous coordinates with p as independent variable. Differentiation with respect to u^1 gives

$$\mathfrak{E}^h_\mu B^\mu_1 = (d_p\,\xi^h)\,\frac{1}{u^0}, \quad d_p\,\xi^h = \frac{d\xi^h}{dp}, \tag{3.22}$$

$$(\partial_\nu\,\mathfrak{E}^h_\mu)\,B^{\nu\mu}_{11} + \mathfrak{E}^h_\mu\,\partial_1 B^\mu_1 = (d^2_p\,\xi^h)\left(\frac{1}{u^0}\right)^2. \tag{3.23}$$

Consequently we have

$$(3.24) \quad d_p^2 \xi^h + \Gamma_{ji}^h (d_p \xi^j)(d_p \xi^i) = (u^0)^2 [\mathcal{E}_\mu^h \partial_1 B_1^\mu + (\partial_\mu \mathcal{E}_\lambda^h) B_{1\,1}^{\mu\,\lambda} + \Gamma_{ji}^h \mathcal{E}_{\mu\,\lambda}^{ji} B_{1\,1}^{\mu\,\lambda}].$$

From the equation (3.7) it follows by transvection with $\mathcal{E}_{\mu\,\lambda}^{j\,i}$ in consequence of (3.12)

$$(3.25) \qquad \Gamma_{ji}^h \mathcal{E}_{\mu\,\lambda}^{j\,i} = \mathcal{E}_\kappa^h \Pi_{\mu\lambda}^\kappa - \partial_\mu \mathcal{E}_\lambda^h - q_\mu \mathcal{E}_\lambda^h - q_\lambda \mathcal{E}_\mu^h.$$

Substituting this expression in (3.24) we get

$$(3.26) \quad \begin{aligned} d_p^2 \xi^h + \Gamma_{ji}^h (d_p \xi^j)(d_p \xi^i) &= (u^0)^2 [\mathcal{E}_\kappa^h (\partial_1 B_1^\kappa + \Pi_{\mu\lambda}^\kappa B_{1\,1}^{\mu\,\lambda}) - 2 q_\mu \mathcal{E}_\lambda^h B_{1\,1}^{\mu\,\lambda}] \\ &= -2 u^0 q_\mu B_1^\mu d_p \xi^h = -2 \alpha\, d_p \xi^h, \end{aligned}$$

according to (3.20). The coefficient α is a function of p and we find by differentiation

$$(3.27) \qquad \begin{aligned} d_p \alpha = d_p (u^0 q_\mu B_1^\mu) &= (u^0)^2 \frac{\partial}{\partial u^1} (q_\mu B_1^\mu) \\ &= (u^0)^2 B_{1\,1}^{\mu\,\lambda} (\partial_\mu q_\lambda - \Pi_{\mu\lambda}^\kappa q_\kappa) = (u^0)^2 B_{1\,1}^{\mu\,\lambda} \nabla_\mu q_\lambda. \end{aligned}$$

From (3.16) it follows, by multiplication with $\mathcal{E}_{\lambda\,\kappa}^{j\,i}$, that

$$(3.28) \quad \mathcal{E}_{\lambda\,\kappa}^{j\,i} q_{ji} = \mathcal{E}_{\lambda\,\kappa}^{\nu\,\mu} \nabla_\nu q_\mu = \nabla_\lambda q_\kappa - x^\nu q_\lambda \mathcal{E}_\kappa^\mu \nabla_\nu q_\mu - x^\mu q_\kappa \nabla_\lambda q_\mu = \nabla_\lambda q_\kappa + q_\lambda q_\kappa,$$

the components q_λ being homogeneous of degree -1. Substitution in (3.27) gives

$$(3.29) \qquad d_p \alpha = (u^0)^2 B_{1\,1}^{\mu\,\lambda} (\mathcal{E}_{\mu\,\lambda}^{j\,i} q_{ji} - q_\mu q_\lambda),$$

for which we may write according to (3.19), (3.22) and using the definition of α (3.26),

$$(3.30) \qquad d_p \alpha + \alpha^2 - \frac{1}{n-1} R_{ji} (d_p \xi^j)(d_p \xi^i) = 0.$$

These equations together with the differential equations (3.26) of the system of paths are identical with the equations (1.11) and (1.13) if we put $p = \pi$. This means that *the projective parameter* p defined in § 2 *is a projective normal parameter.*