PROOF OF THE FIXED POINT THEOREMS OF POINCARÉ AND BIRKHOFF

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Introduction. In 1912, shortly before his death, Poincaré **(8)** conjectured the following theorem in his investigation of the restricted problem of three bodies.

POINCARÉ'S LAST GEOMETRIC THEOREM. *Given a ring $0 < a \leqslant r \leqslant b$ in the r, θ plane and a homeomorphic, area-preserving mapping T of the ring onto itself under which points on $r = a$ advance and those on $r = b$ regress, there will exist at least two points of the ring invariant under T.*

Poincaré was able to prove this theorem in only a few special cases. Shortly thereafter, Birkhoff was able to give a complete proof in **(2)** and in, **(3)** he gave a generalization of the theorem, dropping the assumption that the transformation was area-preserving. Birkhoff's proofs were very ingenious; however, they did not use standard topological arguments.

Also in 1912, Brouwer **(4)** announced and proved his Plane Translation Theorem. In 1928, Kerékjártó **(6)** showed that there existed a close connection between the Brouwer Plane Translation Theorem and the Poincaré Last Geometric Theorem. He gave topological proofs of both theorems.

Kerékjártó's proof, although clearer than Brouwer's original proof, is still rather hard to follow. Several authors, Scherer **(9)**, Teraska **(11)**, and Sperner **(10)**, have given simpler proofs of the Brouwer Plane Translation Theorem, but no simplification of the proof of the Poincaré Last Geometric Theorem has appeared.

The purpose of the present paper is to give a simpler proof of the Poincaré Last Geometric Theorem and its generalization by Birkhoff along the lines of **(9–10)**. It would be very interesting to prove that either the Brouwer or the Poincaré theorem is a consequence of the other, but the present author has been unable to do so. Our aim is to prove the following theorem:

POINCARÉ–BIRKHOFF THEOREM. *Consider a sense-preserving homeomorphism T of the (x, y) plane onto itself, which satisfies the following four conditions:*

1. *If $T(p) = q$, then $T(p^{(n)}) = q^{(n)}$, where we use the notation that if $p = (x, y)$ is an arbitrary point in the plane, then $p^{(n)} = (x, y + 2n\pi)$, $n = 0, \pm 1, \pm 2, \ldots$.*
2. *Points on the line $x = a$ are mapped on the same line with a smaller y coordinate.*
3. *There is a line $x = b$ with $b > a$ such that any arc from a point on the line $x = a$ to a point on the line $x = b$ intersects its image.*

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4. *For all points p, q of the plane $d(p, T(p)) > \epsilon$, and, if $d(p, q) < \delta_1$, then $d(T(p), T(q)) < \bar{\delta}$, and similarly for T^{-1}.*

Then there is a simple curve S, with points $p = (\bar{x}, \bar{y})$ such that $a < \bar{x} < b$, $T(S) \cap S = \emptyset$, and if $p \in S$ so is $p^{(n)}$.

Remark 1. The theorem clearly applies to an annulus in polar coordinates with $r = x, \theta = y$. We have formulated the theorem as above to make clear the assumptions necessary for the application of the theorem when $a = 0$.

Remark 2. From the way we have defined the mapping, it is clear that for an annulus A, the local degree $d(I - T, A, 0) = 0$. (See Cronin (**5**, p. 31) for the definition of local degree.) Thus, if we assume that the mapping is differentiable, since it is also a homeomorphism, it follows from Cronin (**5**, Theorem 7.2) that the existence of one fixed point assures the existence of a second. Birkhoff (**3**) proves this, assuming T only continuous.

The proof of the Poincaré–Birkhoff Theorem rests on the idea of a critical region, which we now discuss.

The critical region. The fundamental idea in the papers of Scherrer, Teraska, and Sperner for dealing with the Brouwer Plane Translation Theorem is that of the critical region. The idea was originated in Scherrer's paper.

Throughout this paper, the mapping T means a sense-preserving homeomorphism of the Euclidean plane Π onto itself without fixed points. If G is a bounded open set whose boundary J is a Jordan curve, and if

$$\bar{G} \cap T(\bar{G}) = J \cap T(J) \neq \emptyset,$$

we speak of G as a critical region or, more suggestively, as a region that touches its image.

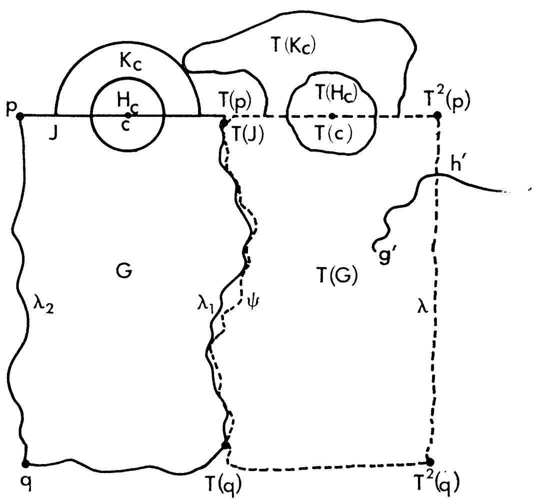

FIGURE 1

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The basic properties of a critical region are contained in the following statement, which is a summary of Scherrer's remarks.

THEOREM 1 OR THEOREM OF SCHERRER. *If G is a region that touches its image, and the Jordan curve J is the boundary of G, then:*

(a) *There is an arc on J and one on $T(J)$ which have the same end points. Let us call them $T(p)$ and $T(q)$. These arcs consists of all the points of intersection of J and $T(J)$, and all points of J and $T(J)$ that are not arcwise accessible from infinity in $\Pi - (J \cup T(J))$. For convenience we shall call these arcs the overlapping arcs (the arcs λ_1 and ψ of Figure 1). In case $J \cap T(J)$ reduces to a single point, the overlapping arcs reduce to this point.*

(b) *The inverse image of the overlapping arc of $T(J)$ is an arc of J, which we shall call the obstruction arc of J (the arc $[p, q]$ of Figure 1). The obstruction arc of J and the overlapping arc of J do not intersect. They are separated by two arcs ($[p, T(p)]$ and $[q, T(q)]$ of Figure 1), which only intersect their images in one end point, and which we shall call translation arcs.*

The proof of part (a) of the theorem is based on the following lemma about curves, which is proved in Lefschetz (**7**, p. 346).

LEMMA 1. *Let J be a Jordan curve in the sphere S_2 and U one of the two regions into which J divides the sphere S_2. Let λ be an arc in U joining two distinct points p and q of J. Then λ divides U into two distinct regions. Further, if λ_1 and λ_2 are the two arcs of J with the common end points p and q, then the boundaries of the two regions are $J_1 = \lambda_2 \cup \lambda$ and $J_2 = \lambda_1 \cup \lambda$.*

The proof of part (a) now proceeds readily. When $J \cap T(J)$ is a single point, the proof is trivial because there are no fixed points. Thus we assume that there is more than one point of intersection.

Take any point $g' \in T(G)$. Since it is exterior to G, it can be connected to the point of infinity p_∞, by an arc that does not touch J. There is a last point h' on this arc that belongs to $T(J)$. The point h' is arcwise accessible from p_∞ and is not in J. Starting from h' we may go clockwise and counterclockwise on $T(J)$ until we reach points on J (the points $T(p)$, $T(q)$ of Figure 1). Call this arc of $T(J)$, λ. Then λ is in the exterior of G and only touches J in the points $T(p)$ and $T(q)$. By Lemma 1, λ divides the exterior of G into two disjoint regions, each of which has λ in its boundary. One of these two regions contains p_∞. Further, by Lemma 1, its boundary J_1 is of the form $\lambda_2 \cup \lambda$, where λ_2 is an arc of J connecting the points $T(p)$ and $T(q)$. The other such arc is λ_1. Thus, every point of J_1 is arcwise accessible from p_∞ since, by the Schoenflies Extension Theorem (see Wilder (**12**, p. 94)), the exterior U_1 of J_1 is homeomorphic to a closed two-cell. Further, $T(J) - \lambda = \psi$ is not in U_1, for by Lemma 1, if it were it would divide U_1 into two regions, only one of which would contain p_∞, so that either λ or λ_2 would not be arcwise accessible from p_∞, which is a contradiction. Hence, ψ is not accessible from p_∞. ψ is the overlapping arc in $T(J)$, and λ_1 is the overlapping arc in J. This completes the proof of part (a).

Translation arcs. To prove part (b) of the Scherrer theorem, we must first prove a lemma about translation arcs. By a translation arc, following Brouwer, we mean a simple arc L with end points p and $T(p)$ such that

$$L \cap T(L) = T(p).$$

Although various proofs of this lemma have been given, Sperner (**10**, p.9) states that the proof given by Brouwer is not rigorous, and I find the proofs given by both Sperner (**10**) and Kerékjártó (**6**, especially Lemma 1) long and hard to follow.

LEMMA 2. *If L is a translation arc, then* $L \cap T^n(L) = \emptyset$, $|n| \geqslant 2$.

Proof. We give the proof for $n = 2$; an analogous proof applies for $|n| \geqslant 2$. If $T^2(L) \cap L \neq \emptyset$, then we use the idea of the local degree of a mapping (see Cronin (**5**, Chapter 1)) to show that there is a fixed point, contrary to our assumptions.

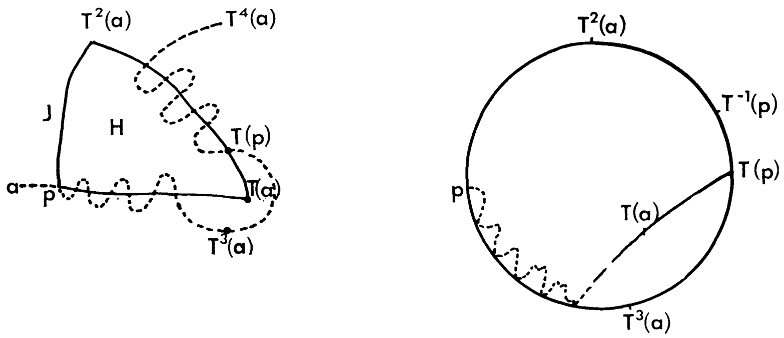

FIGURE 2

Let $[a, T(a)]$ be a translation arc L, and $[T(a), T^2(a)]$, $[T^2(a), T^3(a)]$, $[T^3(a), T^4(a)]$ be its iterated images. If p is the first point on $T^2(L)$ that intersects L, we consider the Jordan curves $J \equiv [p, T(a), T^2(a), p]$ belonging to $[a, T(a), T^2(a), T^3(a)]$ and its image $T(J)$.

Let the region interior to J be denoted by H. It follows from Cronin's proof of the Brouwer Fixed Point Theorem (Cronin (**5**, p. 52)) that if the boundary J is mapped into itself without fixed points by a mapping F, then

$$d(I - F, H, 0) = 1.$$

It now follows from the Theorem on Invariance under Homotopy (**5**, p. 31) and the Existence Theorem (**5**, p. 32) that if we can deform $T(J)$ into J

continuously, while satisfying certain conditions, then T will have a fixed point interior to H. The conditions are that if t varies in the interval [0, 1] and $q \in J$, with $V_t(q)$ its image in the deformation, then (1) $V_t(q)$ is continuous in t and q, (2) $\overrightarrow{qV_t(q)} \neq 0$ for any t or q, (3) $V_0(q) \in J$, and (4) $V_1(q) = T(q)$.

In order to demonstrate the homotopy, we first use the following lemma (See Lefschetz (**7**, p. 345).)

LEFSCHETZ'S LEMMA. *Let $J_1 J_2$ be two Jordan curves in the Euclidean plane* Π, *and let U_i be the interior of J_i. If $U_1 \cap U_2 \neq \emptyset$, then the infinite component V of* $\Pi - (J_1 \cup J_2)$ *has for its boundary a Jordan curve Q contained in $J_1 \cup J_2$.*

In our case, the curves J and $T(J)$ have the common arc

$$Z \equiv [T(p), T^2(a), p],$$

oriented the same in each Jordan curve. Since the mapping is sense preserving, all points on one side of Z are interior to both J and $T(J)$. Therefore:

(1) the assumptions of the above lemma are fulfilled, and hence by the Schoenflies Extension Theorem we may map the curve Q of the lemma and its interior onto a circle and its interior;

(2) the arc Z is on the circumference of the circle (see second half of Figure 2).

Since $p \in [T^2(a), T^3(a)]$, $T^{-1}(p) \in [T(a), T^2(a)]$ and so is $T(p)$. We must consider two separate cases.

Case 1. The orientation of $T^{-1}(p)$, $T(p)$ in $[T(a), T^2(a)]$ is

$$[T(a), T(p), T^{-1}(p), T^2(a)].$$

This is illustrated in Figure 2. Since $[T^{-1}(p), p]$ of J is mapped onto $[p, T(p)]$ of $T(J)$ and these two arcs only meet in p, it is readily seen that, if we set up an arbitrary homeomorphism between the points q' of the arc $[p, T^3(a), T(p)]$ of $T(J)$ and the points q of the arc $(p, T(a), T(p)]$ of J and set $q = q'$ on Z, then, because of the convexity of the circle, the desired homotopy is given by $q(1 - t) + tq' = V_t(q)$.

Case 2. The proof is completed by showing that the orientation

$$[T(a), T^{-1}(p), T(p), T^2(a)]$$

cannot occur (see Figure 3). Assume it does occur. Since $[T^2(a), p]$ only

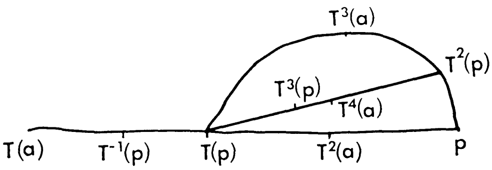

FIGURE 3

touches $[a, T(a)]$ in p, $[T^3(a), T(p)]$ only touches $[T(a), T^2(a)]$ in $T(p)$, and $[T^4(a), T^2(p)]$ only touches $[T^2(a), T^3(a)]$ in $T^2(p)$. Hence

$$J = [T(p), T^2(a), p, T^2(p), T^3(a), T(p)]$$

is a Jordan curve, whose totality of points in common with its image

$$T(J) = [T^2(p), T^3(a), T(p), T^3(p), T^4(a), T^2(p)]$$

is exactly the arc $Z = [T^2(p), T^3(a), T(p)]$, oriented the same in J and $T(J)$. Further, since the mapping is sense preserving, points on the same size of Z are interior to both J and $T(J)$. Thus if U is the interior of J, either $\bar{U} \subseteq T(\bar{U})$ or $\bar{U} \subseteq T^{-1}(\bar{U})$. Hence, by the Brouwer Fixed Point Theorem, there is a fixed point, which is a contradiction and completes the proof.

With the aid of Lemma 2, we can readily establish part (b) of the Theorem of Scherrer. We first show:

LEMMA 3. *If G is a critical region, $\bar{G} \cap T^n(\bar{G}) = \emptyset$ for $|n| \geqslant 2$.*

Proof. Assume $a \in \bar{G} \cap T^n(\bar{G})$; then $T^{-n}(a)$ and a belong to $\bar{G}$. If J is the boundary of G, choose a point $T(b)$ in $J \cap T(J)$. (If a or T^{-n}(a) belongs to $J \cap T(J)$, use one of them.) Let L be any arc in $\bar{G}$ that contains the points a, $T^{-n}(a)$, b, and $T(b)$, and that intersects J at most in these four points. L is then a translation arc, such that $T^n(L) \cap L \neq \emptyset$, which contradicts Lemma 2 and establishes Lemma 3.

We now follow Terasaka **(11)** in completing the proof. Consider the arc $T^{-1}(\lambda_1)$ and its end points p and q. If one of the points (say p) were in λ_1 and the other point (say q) were in λ_2, then any arc in $T^{-1}(G) \cup p \cup q$ connecting p and q would intersect $T(G)$ (since by part (a) of the theorem, any point on λ_2 is arcwise accessible from p_∞ in $\Pi - (J \cup T(J))$, and points on λ_1 are not). Since by Lemma 3, $T^{-1}(G) \cap T(G) = \emptyset$, the above would be impossible. Other possibilities are similarly treated—such as when $T^{-1}(\lambda_1)$ and part of λ_2 form a Jordan curve with λ_1 in its interior. This shows that $T^{-1}(\psi)$ is either entirely in λ_1 or in λ_2.

However, if $T^{-1}(\psi)$ were in λ_1, since it contains all points common to $\bar{G}$ and $T^{-1}(\bar{G})$, this would mean that $T^{-1}(\bar{G})$ would be in the region H bounded by the Jordan curve $\lambda \cup \lambda_2$, and hence $T^{-1}(\lambda_2)$ would be also in this region. Thus $T^{-1}(H) \subseteq H$, and hence there would exist a fixed point. Since this is impossible, it follows that $T^{-1}(\psi) \subseteq \lambda_2$, which completes the proof of the Theorem of Scherrer.

Extension of the critical region. It follows from the Theorem of Scherrer that the arc $[p, q]$ of the critical region G and its image $[T(p), T(q)]$ do not intersect (see Figure 1). It is from this property that we shall construct the curves in the proof of the Birkhoff–Poincaré Theorem. To make the arcs longer we extend the critical region.

We shall now discuss how this extension is possible. Around every point c belonging to the translation arc $[p, T(p)]$ we may draw a circle H_c. (See Figure 1.) We may keep increasing the radius r of the circle H_c to r_c such that the open set $G \cup \tilde{H}_c$ is the largest possible critical region (where $\tilde{H}_c$ is an open circle of radius r_c around c). Since we shall consider only translation arcs that are either straight lines or the arc of a circumference of a circle, $G \cup \tilde{H}_c$ is clearly a region whose boundary is a Jordan curve. By the maximal semicircle K_c, or less precisely the semicircle K_c, we mean the open set $\tilde{H}_c - \bar{G}$. The principal property of a maximal semicircle that we wish to exploit can be stated as follows.

THEOREM 2. *If one of the translation arcs (say $[p, T(p)]$) of the critical region G is either a straight line or an arc of a circumference of a circle, then there is a point $c \in p, T(p))$ with maximal semicircle K_c such that K_c touches its image but does not intersect $T(\bar{G})$ or $T^{-1}(\bar{G})$. More precisely*

$$\bar{K}_c \cap T(\bar{G}) = T(\bar{K})_c \cap \bar{G} = \emptyset.$$

Before proving this theorem, we need one more lemma.

LEMMA 4. *Consider a translation arc L. For some integer N, let*

$$Q = \cup T^n(L), \qquad |n| \geqslant N.$$

If γ is a simple arc that intersects Q in only the end points p, q of a subarc $[p, q]$ that contains $T(p)$, then γ crosses its image $T(\gamma)$.

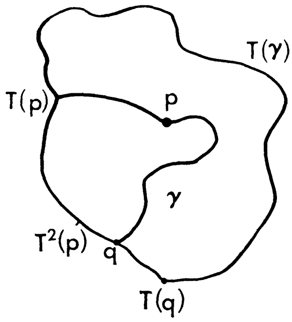

FIGURE 4

Proof. Suppose that γ does not cross its image. Then the Jordan curve $J = [p, T(p), q] \cup \gamma$ and its image $T(J) = [T(p), T^2(p), T(q)] \cup T(\gamma)$ intersect in exactly the arc $Z = [T(p), q]$, oriented the same in J and $T(J)$ (see Figure 4). Since the mapping is orientation preserving, points on the same side of Z are interior to both J and $T(J)$. Thus, since the boundaries do not cross, if U is the interior of J, then either $T(U) \subseteq U$ or $T^{-1}(U) \subseteq U$. Hence, by the Brouwer Fixed Point Theorem there is a fixed point, contrary to assumption.

We are now ready to prove Theorem 2.

Proof. For $c \in [p, T(p)]$ let A = all points c such that $\bar{K}_c \cap T(\bar{G}) \neq \emptyset$ and B = all points c such that $T(\bar{K}_c) \cap \bar{G} \neq \emptyset$ or equivalently $\bar{K}_c \cap T^{-1}(\bar{G}) \neq \emptyset$. We now prove three properties of the sets A and B.

Property 1. The sets A and B are closed. Consider a sequence of points $a_i \in A$ that approach a_0. Let $\bar{H}_j$ be a sequence of circles with centre a_0, whose radii decrease to r_{a_0}. Since clearly the radius r_{a_i} of $\tilde{H}_{a_i}$ is a continuous function of a_i, it follows that each $\bar{H}_j - G$ contains a $\bar{K}_{a_i}$ for sufficiently large i. Thus $(\bar{H}_j - G) \cap T(\bar{G}) \neq \emptyset$, and thus

$$\overline{K}_{a_0} \cap T(\overline{G}) = \bigcap_{j=1}^{\infty} (\overline{H}_j - G) \cap T(\overline{G}) \neq \emptyset,$$

by compactness. Hence $a_0 \in A$. A similar proof applies for the set B.

It follows by the uniform continuity of T (assumption 4 of the Poincaré–Birkhoff Theorem) that points near p are in B, and those near $T(p)$ are in A. Thus we have

Property 2. Neither of the sets A or B is empty.

Finally we have

Property 3. The sets A and B are disjoint. For if $c \in A \cap B$, $\bar{K}_c$ would contain points $\bar{p} \in T^{-1}(L)$ and $\bar{q} \in T(L)$, of the translation arc $L = [p, T(p)]$. Further, since $\tilde{H}_c \cup G$ is a critical region, it follows from Lemma 3 that $T^n(\bar{G}) \cap \bar{K}_c = \emptyset$ for $|n| \geqslant 2$. Hence K_c contains an arc γ connecting $\bar{p}$ and $\bar{q}$ that satisfies Lemma 4. Thus, by that lemma, γ crosses $T(\gamma)$, or equivalently K_c and $T(K_c)$ have common interior points, which contradicts our assumptions.

From these three properties it follows that A and B do not exhaust the points of the translation arc $[p, T(p)]$ for, if they did, they would be both open and closed and non-empty, and thus each would contain all the points of $[p, T(p)]$. This is clearly a contradiction, since they are disjoint. This completes the proof of Theorem 2.

Proof of the Poincaré–Birkhoff Theorem. The Brouwer Plane Translation Theorem readily follows from the Theorem of Scherrer and Theorem 2 (see, e.g., Terasaka (**11**, p. 69)), but we shall only use them to prove the Poincaré–Birkhoff Fixed Point Theorem.

Recalling the assumptions of the Poincaré–Birkhoff Theorem, we consider a point 1 on the line $x = a$, and its image $T(1)$. It readily follows from the proof of Theorem 2 that we can construct a semicircle S_1 about a point c of the line $[(1, T(1)]$ such that S_1 is a critical region (or touches its image). From the Theorem of Scherrer it follows that S_1 will have a free arc $[2, T(2)]$ on its boundary, and that $[2, T(2)]$ is part of the circumference of a circle. Now applying Theorem 2 to the arc $[2, T(2)]$ we can construct a semicircle S_2 over it such that S_2 touches its image. Thus S_2 will have a free arc $[3, T(3)]$ on its boundary, that is, an arc of a circumference of a circle. By repeated application of Theorem 2, and the Theorem of Scherrer, we may proceed indefinitely in this manner, obtaining an overlapping arc $l = [T(1), T(2), T(3), T(4), \ldots]$ and an obstruction arc $T^{-1}(l) = (1, 2, 3, \ldots]$. (See Figure 5.)

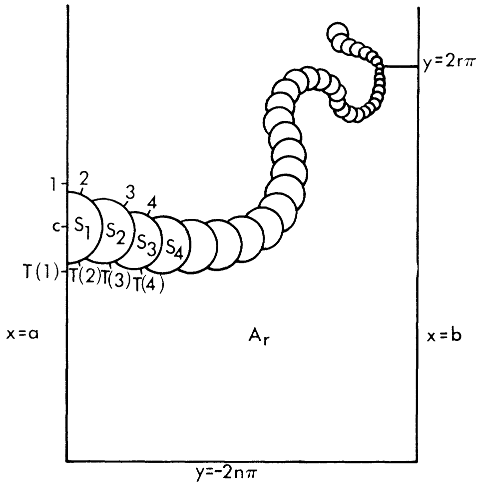

FIGURE 5

We now note, from the assumption $d(p, T(p)) > \epsilon$, that the chord in a semicircle connecting the two end points of a translation arc is greater than a fixed lower limit ϵ. Thus the radius of each semicircle is greater than $\epsilon/2$, and its area is greater than $\epsilon^2/8$. Hence in any finite region of the plane, there can only be a finite number of semicircles.

Since

$$\bigcup_{i=1}^{N} S_i$$

is a critical region, no curve contained in it can intersect its image. Thus it readily follows from condition 3 of the Poincaré–Birkhoff Theorem and Lemma 4 that none of the semicircles S_i, $i \geqslant 2$, can contain points of the lines $x = a$ or $x = b$. Further, since any curve very near to $x = a$ or $x = b$ can be made to touch these lines by the addition of a small straight arc m that does not intersect its own image, it follows that all semicircles S_i, $i \geqslant 2$, are for some fixed δ_2 interior to $a + \delta_2 < x < b - \delta_2$.

From the two previous paragraphs, it now follows, without loss of generality, that there is a $y = -2n\pi$ such that all S_i, $i \geqslant 1$, are above $y = -2n\pi$ (see Figure 5).

We now note that if $p \in l \equiv [T(1), T(2), T(3), \ldots]$, there exists a $\delta > 0$ such that no point of the sphere $S_{p\delta}$ of radius δ and centre p contains points of the obstruction arc $T^{-1}(l)$. This may be seen in the following way. Let

$$S = \bigcup_{i=1}^{\infty} S_i,$$

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and let k_i be the straight line connecting the centre of semi-circle S_i with that of S_{i+1}, and let

$$k = \bigcup_{i=1}^{\infty} k_i.$$

From the manner in which we have constructed the semicircles, it is clearly possible to draw a sphere of radius δ_3 (for some fixed δ_3) around each point of k such that the sphere is entirely in S. Further, any straight line from a point $p \in l$ to a point $q \in T^{-1}(l)$ must intersect k. From these considerations, it follows that the δ specified at the beginning of this paragraph exists.

Without loss of generality, we may then assume that for every integer r, part of the lines $x = a$, $y = -2n\pi$, $y = 2r\pi$, and l determine a Jordan region A_r, sufficiently indicated by the construction in Figure 5, such that, if $p \in A_r$, $S_{p\delta}$ does not intersect $T^{-1}(l)$.

In the following we shall use the definition of Ahlfors (**1**, p. 112): an open connected set of the plane is simply connected if its complement with respect to the extended plane is connected.

Let

$$A = \bigcup_{i=0}^{\infty} A_i \cup \{x, y | x > b\}$$

and let $K = \{x, y | (x, y + 2n\pi) \in A \text{ for all } n\}$. K is a periodic set in the sense that if $(x, y) \in K$, then $(x, y + 2n\pi) \in K$ for all n.

Since l is bounded away from $x = b$, K contains $x = b$ in its interior. Let T_k be a translation by $2\pi k$, i.e. $T_k(x, y) = (x, y + 2\pi k)$. Since each $T_k A$ is simply connected, and the union of intersecting sets is connected, it follows, since

$$K = \bigcap_{k=-\infty}^{\infty} T_k A,$$

that the complement of K is connected. Further, if K^* is the open component of K containing the line $x = b$, then K^* is simply connected, since any closed component K^{**} of K intersects the complement of K.

If $p \in A$, $S_{p\delta}$ does not intersect $T^{-1}(l)$, then, if $p \in K$, $S_{p\delta}$ does not intersect

$$\bigcap_{k=-\infty}^{\infty} T_k(T^{-1}(l)).$$

On the other hand since

$$K = \lim_{n \to \infty} \bigcap_{k=-n}^{n} T_k A,$$

it follows that the boundary of K is made up of translations of l, or limit points of such translations. Hence the boundary of K does not intersect the boundary of $T^{-1}(K)$. Since it readily follows that any boundary point of K^* is a boundary point of K, it follows that the boundary of K^* does not intersect the boundary of $T^{-1}(K^*)$.

If the boundary of K or K^* were a continuous periodic curve the theorem would then follow. If neither is a continuous curve we may construct a continuous periodic curve S arbitrarily close to the boundary of K^* in the following manner, and thus the theorem will still follow, since by uniform continuity S will not intersect $T^{-1}(S)$.

First by the mapping $M(z) = e^{-z}$, considering $x, y \in K^*$ as the complex variable $z = x + iy$, map K^* onto the bounded set $P = \text{Interior } M(K^*)$ containing the origin.

To show P is simply connected, we show that its complement in the plane is arcwise connected. Since the set $L = \{-\infty < x < \infty,\ 0 \leqslant y < 2\pi\}$ is mapped one-to-one onto the w-plane minus the origin by the map $w = e^{-z}$, it follows that any two points a and b in the w-plane and not in P have inverse images $\bar{a}, \bar{b}$ in $L \cap$ complement K^*. Since K^* is simply connected, $\bar{a}, \bar{b}$ may be connected by an arc B in the complement of K^*. The image of B by the map e^{-z} connects a and b in the w-plane and does not intersect P. Thus P is simply connected. Since P is simply connected, we may by the Riemann mapping theorem (Ahlfors (**1**, pp. 172–174)) map the interior of P onto the set $|u| < 1$ homeomorphically. Further any sequence $\{p_n\} \in P$ approaching a boundary point of P goes into a sequence $\{u_n\}$ such that $\lim_{n\to\infty} |u_n| \to 1$. The desired continuous periodic curve S is the inverse image of the curve $|u| = 1 - \epsilon$.

References

1. L. V. Ahlfors, *Complex analysis* (New York, 1953).

2. G. D. Birkhoff, *Proof of Poincaré's geometric theorem*, Trans. Amer. Math. Soc., *14* (1913). 14–22.

3. ——— *An extension of Poincaré's last geometric theorem*, Acta Math., *47* (1925), 297–311.

4. L. E. J. Brouwer, *Beweis des ebenen Translationssatzes*, Math. Ann., *72* (1912), 37–54.

5. J. Cronin, *Fixed points and topological degree in nonlinear analysis* (American Mathematical Society, 1964).

6. B. de Kerékjártó, *The plane translation theorem of Brouwer and the last geometric theorem of Poincaré*. Acta Sci. Math. Szeged., *4* (1928–29), 86–102.

7. S. Lefschetz, *Differential equations: geometric theory* (New York, 1957).

8. H. Poincaré, *Sur un théorème de geometrie*, Rend. Circ. Mat. Palermo, *33* (1913), 375–407.

9. W. Scherrer, *Translationen über einfach zusammenhängende Gebeite*, Viertelkschr. Naturf. Ges. Zürich, *70* (1925), 77–84.

10. E. Sperner, *Über die fixpunktfreien abbildungen der Ebene*, Hamburger Math. Einzelschr., *14* (1933), 1–47.

11. H. Terasaka, *Ein Beweis des Brouwerschen ebenen Translationssatzes*, Japan. J. Math., *7* (1930), 61–69.

12. R. L. Wilder, *Topology of manifolds* (American Mathematical Society, 1949).

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