# Preliminaries

A Fano variety, named after Gino Fano, is a proper variety $X$ whose anti-canonical bundle $\omega_X^{-1}$ is ample. This class of varieties is central to several mathematical fields, including *higher dimensional geometry*. In fact, while originally people were mostly interested in smooth Fano manifolds, from the viewpoint of *minimal model program*, it became natural to consider Fano varieties with mild singularities, as they are one of the three building blocks of an arbitrary variety, up to birational equivalence.

A Fano variety may have multiple "optimal" birational models, and birational maps to connect different models are complex. This complexity makes the birational geometry of Fano varieties a fascinating but challenging topic. An important related question involves understanding the limits of a family of Fano varieties, which often present numerous possibilities. So some kind of *stability condition* needs to be added. However, for higher dimensional varieties, Mumford's geometric invariant theory (GIT) (Mumford et al., 1994) is not an ideal framework because it depends on a choice of embeddings (see Wang and Xu, 2014). Therefore, researchers seek for a more intrinsic theory.

Another deep question about Fano varieties is whether it admits a Kähler–Einstein metric. This traces back to the long tradition in people's study on Einstein metrics, with the Kähler condition added in the complex setting. More precisely, recall that a Kähler–Einstein metric on a compact manifold $X$ if the Kähler form $\omega$ satisfies the Einstein equation:

$$\mathrm{Ric}(\omega) = \lambda \cdot \omega\,, \tag{0.1}$$

where $\lambda$ is a constant. If we take the class of (0.1), then

$$[\mathrm{Ric}(\omega)] = c_1(X) = -K_X = \lambda \cdot [\omega]\,.$$

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If $\lambda < 0$, this is established independently in Aubin (1978) and Yau (1978). When $\lambda = 0$, this follows from the solution of the Calabi conjecture in Yau (1978). Moreover, these two results are generalized to the case that $X$ contains canonical singularities in Eyssidieux et al. (2009). See Guedj and Zeriahi (2017) for a comprehensive study of singular Kähler–Einstein metrics.

The remaining case $\lambda > 0$ is subtler, as in this case, a Kähler–Einstein metric does not always exist. This fact was known for a long time, for example, Matsushima (1957) shows that a Kähler–Einstein Fano manifold $X$ satisfies Aut(X) is reductive, but finding out a sharp geometric condition to characterize the existence of Kähler–Einstein metrics is challenging. A similar question for a vector bundle $E$ was extensively studied, which is to search the right condition to characterize the existence of Hermitian–Einstein metrics. The solution, called the *Hitchin–Kobayashi correspondence*, says it is equivalent to the slope stability of $E$; see Narasimhan and Seshadri (1965), Donaldson (1985), Uhlenbeck and Yau (1986), and Donaldson (1987). Inspired by this, in Mabuchi (1986), the *K-energy function*, on the space $\mathcal{H}$ of Kähler metrics with the same class, was defined, and it is shown that a Kähler metric $\omega$ satisfies (0.1) if and only if it is a minimizer of the K-energy function. Moreover, using the convexity of the K-energy function, it is shown in Bando and Mabuchi (1987) that a Kähler–Einstein metric, if exists, is unique up to an element in the connected component of $\mathrm{Aut}(X)$.

In order to understand the existence of a Kähler–Einstein metric, one must address this infinite-dimensional minimizing problem, ideally using geometric constructions. In Ding and Tian (1992), the (*generalized*) *Futaki invariant* was introduced to attack the problem. It is defined for a one-parameter group (normal) degeneration $X_0$ of $X$, called a *test configuration*, as the *Futaki invariant* $\mathrm{Fut}(X_0)$ for $\mathbb{G}_m \curvearrowright X_0$ introduced earlier in Futaki (1983). Moreover, they showed that the existence of a Kähler–Einstein metric $\omega$ on $X$ implies the non-negativity of $\mathrm{Fut}(X_0)$, because the test configuration induces a ray emitting from $\omega$, and the Futaki invariant is the derivative of the K-energy along this ray. This significantly expands the range of geometric tests that can be applied, as previously Futaki only considered the product case. The natural question is whether these tests are sufficient. In Tian (1997), it was proved that the existence of a minimizer was implied by a suitably defined *properness* of the K-energy function, and it was also conjectured that all tests as above provided a sufficient condition for the properness. Not long after that, it was realized in Donaldson (2002) that the Futaki invariant can be defined completely using algebraic terms, and more generally for all polarized varieties. Thus the proposed geometric tests are indeed algebraic, confirming the speculation by Yau in the 1980s. The notion is called *K-stability*. There are a lot of later developments in the analytic theory, but now we switch our discussion to the algebro-geometric theory.

## Characterizations of K-stability

The earlier attempt to study K-stability algebraically is using the framework of GIT. However, in Odaka (2013b), it was first observed that K-stability notion relates to the minimal model program. This surprising connection became more explicit in Li and Xu (2014), where minimal model program was used to show that testing K-stability for all test configurations is equivalent to only testing it in the case $X_0$ is a klt Fano variety, that is, the test configuration is *special*. In particular, this confirms Tian's definition of K-stability is equivalent to Donaldson's for any Fano variety. Li and Xu (2014) is the first one in a sequence of works, which show that K-stability can be equivalently defined in several different ways, but to establish the equivalences is highly nontrivial.

In Berman (2016), inspired by the work of Ding (1988) that introduced the *Ding energy functional* whose minimizers are also Kähler–Einstein metrics, Berman shows that this functional yields the algebraic notion of *Ding invariants* for test configurations and uses it to define *Ding stability*. In analytic studies, the Ding functional has the advantage that it requires less regularity than K-energy. Similarly, in the algebraic side, Ding invariants behave better than Futaki invariants in various operations, especially in an approximating process. This was first observed in Fujita (2018), where it is proved that Ding invariants $\mathbf{D}(\mathcal{F})$ can be extended to all filtrations. The extension from test configurations to general filtrations can be regarded as an algebraic analog to the operation of taking completion with respect to suitable norms for the infinite-dimensional space of Kähler metrics. Besides, it gives more flexibility to test the stability, and it also yields a right ambient space for taking limits. In particular, this is a necessary step to construct a canonical test object.

Further foundational properties for invariants of filtrations are obtained in Blum and Jonsson (2020), using the theory of Okounkov bodies. In fact, one can skip the notion of K-stability and only focus on Ding stability to use it to build the entire algebraic theory. Nevertheless, following Li and Xu (2014), it was shown by Fujita (2019b) and Berman, Boucksom, and Jonsson that K-stability and Ding-stability are equivalent for Fano varieties, as they are the same when test on special test configurations. In Xu and Zhuang (2020), it is noticed that for a filtration $\mathcal{F}$, one may define base ideals

$$I_{m,\lambda} = \text{the base ideal of } (\mathcal{F}^{\lambda} H^0(-mK_X) \subseteq H^0(-mK_X))\,,$$

and $\mathbf{D}(\mathcal{F})$ can be defined using the slope $\mu$ such that $\mathrm{lct}(X, I_{\bullet}^{(\mu)}) = 1$, where $I_{\bullet}^{(\mu)} = \{I_{m,m\mu}\}$. This yields a conceptually more satisfying definition of $\mathbf{D}(\mathcal{F})$.

Another key progress is to test the stability using *valuations*. In Fujita (2019b) and Li (2017), they defined a new type of invariants, called the *Fujita–Li invariant*,

$$\mathrm{FL}(v) = A_X(v) - S_X(v)\,,$$

where $A_X(v)$ is the *log discrepancy* and $S_X(v)$ is the *expected vanishing order*. The Fujita–Li invariant is markedly easier to calculate, and when $v$ arises from a special test configuration, $\mathrm{FL}(v)$ is equal to the Ding invariant (as well as the Futaki invariant) of the test configuration. The *Fujita–Li criterion*, independently established in Fujita (2019b) and Li (2017), says that $\mathrm{FL}(v)$ gives an equivalent characterization of the notions of Ding stability.

From the Fujita–Li criterion, one easily sees the *stability threshold*

$$\delta(X) = \inf_{v} \delta_X(v), \quad \text{where } \delta_X(v) := \tfrac{A_X(v)}{S_X(v)},$$

gives a quantitative measure of how stable $X$ is. When $\delta(X) \leq 1$, by Berman et al. (2021) and Cheltsov et al. (2019), this invariant indeed has an analytic explanation

$$\delta(X) = \sup\{\, t \mid \mathrm{Ric}(\omega) \geq t \cdot \omega \text{ for a Kähler form } \omega\,\}.$$

To further advance the algebraic theory, the question of whether there is a divisorial valuation computing $\delta(X)$ plays a central role. We will come back to this topic in the “Minimizers of $\delta$” section.

It is observed by Blum, Liu, and Xu in Blum et al. (2022a) that any valuation induced by the irreducible special fiber of a weakly special test configuration precisely corresponds to an lc place of a $\mathbb{Q}$-complement. We call these valuations *weakly special*. The latter description using $\mathbb{Q}$-complements makes them more transparent to study in birational geometry. For instance, one can show when $\delta(X) < \frac{n+1}{n}$, $\delta(X)$ can be approximated by $\delta_X(E_i)$ for a sequence of weakly special divisors $E_i$. This yields an explicit explanation of the Fujita–Li criterion.

When $X$ admits a torus $\mathbb{T}$-action, we need the notion of *reduced* stability, as defined in Hisamoto (2016), given by invariants module the equivalence of the torus orbit. This is necessary when treating K-polystability.

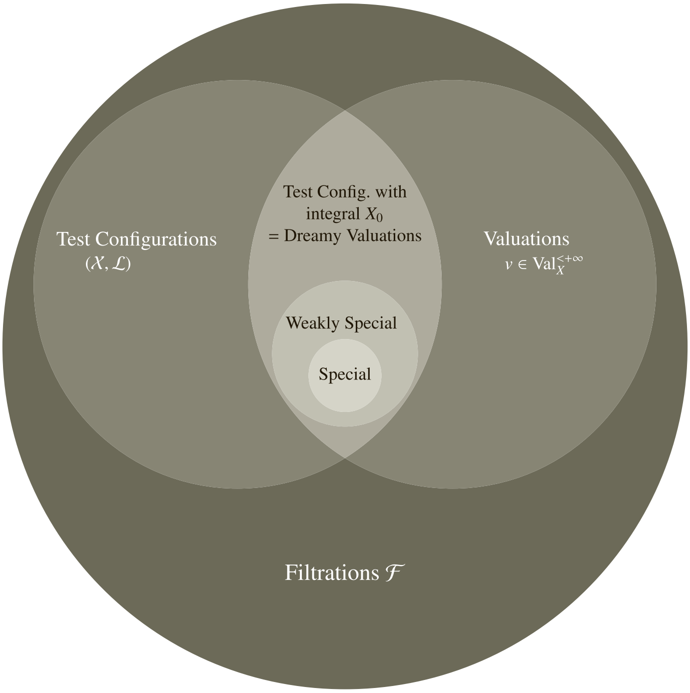


Figure 0.1 Test stability by different objects

## Minimizers of $\delta$

A key question in K-stability theory is to understand minimizers of $\delta(X)$ in the space $\mathrm{Val}(X)$ of valuations. The aim is to show that when $\delta(X) < \frac{n+1}{n}$, one can find a divisor $E$ such that $\delta(X) = \delta_X(E)$. Such a divisor $E$ yields a special test configuration minimizing the normalized Futaki invariant, which is an *optimal destabilization*. This can be regarded as an algebro-geometric analog to the regularity question for the minimizer of a functional in geometric partial differential equation. It is a key technical step to several central geometric questions.

One of them is the question of characterizing the existence of Kähler–Einstein metrics. As we explained, we need to understand whether the geometric construction of test configurations provides enough tests to the existence of a minimizer of the K-energy functional or Ding-energy functional, namely the *Yau–Tian–Donaldson conjecture*.

The Yau–Tian–Donaldson conjecture was first proved for smooth Fano manifolds (see Chen et al. 2015a, 2015b, 2015c; Tian 2015; and Székelyhidi 2016). A main recipe is to show that a sequence of Kähler–Einstein Fano manifolds or log smooth Fano pairs admits a Kähler–Einstein limit. Unfortunately, for now the smoothness assumption is essential to the existence of the Kähler–Einstein limit. The algebraic analog is that a sequence of K-stable Fano varieties admits a K-(poly)stable limit. We will see in the next section that the existence of a minimizer $E$ for $\delta_X(\cdot)$ plays a central role in showing this.

To solve the Yau–Tian–Donaldson conjecture for all Fano varieties including singular ones, one can apply a different set of analytic tools, for example, the pluripotential theory, to characterize the existence of a Kähler–Einstein metric. This is called the *variational approach*, and it requires less regularity than the aforementioned Riemannian geometry method. Initiated by Berman, Boucksom, and Jonsson in Berman et al. (2021), and completed by Li, Tian, and Wang in Li et al. (2022) and Li (2022), it is proved that uniform K-stability gives a necessary and sufficient condition for the existence of a (weak) Kähler–Einstein metric (in the case when the automorphism group is discrete). To complete the solution, one needs to show the equivalence between uniform K-stability and K-stability, which immediately follows from the existence of a minimizer $E$ in the case when $\delta(X) = 1$.

The proof of a minimizer $E$ consists of two steps.

Since $\delta(X)$ can be approximated by $\delta_X(E_i)$ for a sequence of divisors $E_i$ which are weakly special, as we mentioned before, one can apply Birkar (2019) to conclude that, all these valuations are lc places of a bounded family of complements. Then after passing to an infinite subsequence, we can assume all $E_i$ are lc places of *one* complement. So after possibly passing to an infinite subsequence again, we may assume the rescaling $\frac{1}{A_X(E_i)}\mathrm{ord}_{E_i}$ has a limit $v$, which is a quasi-monomial valuation and satisfies $\delta(X) = \delta_X(v)$. This was proved in Blum et al. (2022a).

To obtain a divisorial valuation, it is noticed in Li and Xu (2018) that for $R = \bigoplus_{m\in r\cdot\mathbb{N}} H^0(-mK_X)$, if $\mathrm{Gr}_v R$ is finitely generated, then for a rational perturbation of $w = c \cdot \mathrm{ord}_E$, $\mathrm{Gr}_v R \cong \mathrm{Gr}_w R$, and

$$\delta(X) = \delta_X(v) = \delta_X(w),$$

that is, any small rational perturbation yields a divisor that computes $\delta(X)$. The finite generation of $\mathrm{Gr}_v R$ was first proved by Liu, Xu, and Zhuang in Liu et al. (2022), and later stronger results were given in Xu and Zhuang (2023). In both proofs, the key is to prove the birational geometry statement that a *special valuation* has the sought-after finite generation properties. Then one verifies that any minimizer $v$ is special.

Step 1: Turn solving the PDE to a variational problem, so the solution is given by the minimizer of an suitable functional

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Step 2: Enlarge the ambient space of solutions, extending the functional, and finding the minimizer in the larger space (weak solution)

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Step 3: Proving the regularity of the weak solution

Figure 0.2 Solving a PDE by variational method

Step 1: Minimize the normalized Ding invariant

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Step 2: Include all filtrations, extend the Ding functional

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Step 2': Focus on the space $\mathrm{Val}_X$ of all valuations, and minimize the function $\delta_X(\cdot)$

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Step 3: Show the minimizer of $\delta_X(\cdot)$ is a special divisorial valuation

Figure 0.3 Optimal destabilization

We draw a flowchart to compare solving a partial differential equation, for example the Kähler–Einstein problem, with the optimal destabilization in algebraic K-stability theory (Figures 0.2 and 0.3).

## Moduli of Fano Varieties

One major application of K-stability is that it provides an approach to parametrizing Fano varieties. The concept of a family of higher dimensional varieties $X \to S$ (or more generally a family of log pairs $(X, \Delta) \to S$), is rather subtle and it has been addressed in Kollár (2023). Then to make it a well-behaved moduli functor, one needs to add a natural polarization, for example, $\omega_{X/S}$ or $\omega_{X/S}^{-1}$ is relatively ample. In the case of $\omega_{X/S}$ being ample, the functor is called the *KSB moduli* (or *KSBA* moduli), and it has been investigated in detail in Kollár (2023).

In the case of $\omega_X^{-1}$ being ample, one major obstacle is that, as shown in elementary examples, the Fano condition alone is not enough to make the family behave well, especially when one looks at degenerations. Only until the notion of K-stability was introduced, pioneers looked at the moduli problem for Fano varieties again. The progress of using K-stability to construct a moduli space intertwined with the improving of understanding the notion itself. After around a decade's work, it is finally settled that with the K-stability assumption on the fibers, the moduli functor, called the *K-moduli stack*, behaves very satisfactorily; for example, it admits a *projective good* moduli space, namely the *K-moduli space*.

To show the *K-moduli stack* is of finite type, one only needs to show that if we fix the numerical invariants, the functor is bounded and open. Since the volume $(\omega_{X_t}^{-1})^n$ is a constant in a family, we can simply fix it. Then to get the boundedness, Jiang (2020) shows that one can reduce it to the boundedness results established in Birkar (2019, 2021). Later, Xu and Zhuang (2021), applying deeper local results, reduced it to the earlier boundedness result proved by Hacon, McKernan, and Xu in Hacon et al. (2014). The openness is confirmed by Blum, Liu,and Xu in Blum et al. (2022a) as well as in Xu (2020), by showing that the invariants that test the K-stability, for example stability threshold or normalized volume, are constructible for the Zariski topology. One key recipe in both proofs is the boundedness of complement proved in Birkar (2019).

What distinguishes the K-moduli stack with other functors of families of Fano varieties, is it admits a projective good moduli space. For an algebraic stack, admitting a good moduli space is delicate, which implies strong properties of the stack. In Alper et al. (2023), Alper, Halpern-Leistner, and Heinloth show that two valuative criteria, called $S$-completeness and $\Theta$-reductivity, imply the existence of a separated good moduli space. This can be viewed as the Artin stack analog to the result of Keel and Mori (1997) on the existence of separated coarse moduli space for a Deligne–Mumford stack. For families of K-semistable Fano varieties, these two criteria are verified by Alper, Blum, Halpern-Leistner, and Xu in Alper et al. (2020b), on the basis of earlier works studying families of K-semistable Fano varieties by Li, Wang, and Xu in Li et al. (2021) and by Blum and Xu (2019).

Following Halpern-Leistner's work on instability theory, one knows the properness of the good moduli space follows from the existence of a $\Theta$-*stratification* on the stack of all Fano varieties. It is shown by Blum, Halpern-Leistner, Liu, and Xu in Blum et al. (2021) that this can be deduced from the existence of a divisor $E$, such that $\delta(X) = \frac{A_{X,\Delta}(E)}{S(E)}$, that is, the $\delta(X)$-minimizing problem we discussed in the Moduli of Fano Varieties section.

Finally, the projectivity of the good moduli space is obtained by establishing the ampleness of the *Chow–Mumford (CM)* ($\mathbb{Q}$)-*line bundle*. The CM line bundle can be defined for any family of Fano varieties as in Tian (1997), but it is not always positive, and the subtlety is to show that it is positive along the locus parametrizing K-semistable Fano varieties. The algebraic theory of establishing the connection between the K-stability of fibers and the positivity of the CM line bundle on the base was first developed in Codogni and Patakfalvi (2021) by applying the general K-stability theory to investigate the Harder–Narasimhan filtration on the base. This connection is elaborated in Xu and Zhuang (2020), which completely addresses the positivity of the CM line bundle, by developing the notion of reduced uniform K-stability.

## K-stability for Explicit Fano Varieties

One active research topic is verifying whether an explicitly given Fano variety is K-(semi,poly)stable. In general, this is a quite challenging question. The case of smooth surfaces was solved in Tian (1990) decades ago, but in higher dimension, the knowledge is far from being complete. Nevertheless, several powerful tools have been developed.

The first one is estimating $\delta(X)$ by studying the singularity in $|-K_X|_{\mathbb{Q}}$. There have been a number of works (see, e.g., Tian 1987, 1990; Cheltsov 2008; Cheltsov and Shramov 2008; and so on and so forth), devoted to estimate the $\alpha$-invariant

$$\alpha(X) = \inf\left\{ \operatorname{lct}(X, D) \mid 0 \le D \sim_{\mathbb{Q}} -K_X \right\},$$

and the condition $\alpha(X) > \frac{n}{n+1}$ yields K-stability of Fano varieties as $\delta(X) \ge \frac{n+1}{n}\alpha(X)$. However, this approach is limited, because the inequality for the $\alpha$-invariant only gives a sufficient condition, but usually it is not necessary. To estimate the $\delta$-invariant, one can use the observation made in Fujita and Odaka (2018) and Blum and Jonsson (2020) that $\delta(X) = \lim_m \delta_m(X)$, where

$$\delta_m(X) = \inf\left\{ \operatorname{lct}(X, D) \mid m\text{-basis type divisor } D \sim_{\mathbb{Q}} -K_X \right\}.$$

A powerful approach to estimate $\delta(X)$ is established in Abban and Zhuang (2022), called the *Abban–Zhuang method*. It studies the multigraded linear series obtained by restricting a linear series along an admissible flag and uses the inversion of adjunction to obtain inequalities, which reduces the estimate of $\delta(X)$ to an estimate of log canonical thresholds of the multi-graded linear series on lower dimensional subvarieties. Besides the original application to Fano hypersurfaces in Abban and Zhuang (2022, 2023), it yields a long list of results for three-dimensional smooth Fano manifolds (see Araujo et al. 2023; Fujita 2023; Abban et al. 2022, 2023; Cheltsov et al. 2023, 2024; and many others).

Another approach is to use the existence of K-moduli and study deformations and degenerations of a K-stable variety (see Mabuchi and Mukai 1993). See Odaka et al. (2016) for two-dimensional examples; see Liu and Xu (2019) and Liu (2022) for higher dimensional examples. In Ascher et al. (2019, 2023a, 2023b), Ascher, DeVleming, and Liu develop a wall-crossing theory (see also Gallardo et al. 2021) that gives a geometric understanding to many birational maps between moduli spaces.

## The Organization of the Book

After the introductory Chapter 1, the book can be divided into two parts. From Chapter 2 to Chapter 6, it discusses the foundational theory of K-stability. From Chapter 7 to Chapter 9, it focuses on constructing the moduli space and showing it is a projective scheme.

In Chapter 1, we discuss preliminary results. That includes valuation theory, asymptotic invariants, and the construction of Okounkov bodies. We also list results from minimal model program and boundedness that we need later.

In Chapter 2, we will explain the original definition of K-stability using test configurations and its variant Ding stability. We show the invariants testing stability decrease, under a suitable minimal model program sequence. As a consequence, we conclude that K-stability is equivalent to Ding stability in the Fano setting. In fact, the latter stability notion is the foundation of the algebraic theory.

In Chapter 3, we introduce the view of studying K-stability using filtrations. We show that Ding invariants can be extended from test configurations to filtrations. We explain defining Ding invariants for filtrations by using graded sequences of its base ideals with a fixed slope.

In Chapter 4, we introduce the view of studying K-stability using valuations. That includes the definition of the Fujita–Li invariants. We also explain the theory of (weakly) special valuations, and use it to show the minimizers of the $\delta$-function are quasi-monomial. We will establish two applications: the first one is that the notion of K-semistability does not depend on the base field and it is equivalent to the equivariant K-semistability; then we introduce the *Abban–Zhuang method* and apply it to verify K-stability of any smooth Fano hypersurface with a large degree is K-stable.

In Chapter 5, we prove the higher rank finite generation theorem, which implies that there is always a divisorial valuation computing $\delta(X)$ when $\delta(X) < \frac{\dim X+1}{\dim X}$.

In Chapter 6, we introduce the notion of reduced uniform K-stability and use it to extend our theory to treat K-polystability.

In Chapter 7, we define the functor of families of Fano varieties. And we show that if we fix positive lower bounds of the volume and the stability threshold, the subfunctor is a finite-type global quotient stack.

In Chapter 8, we show that the K-moduli stack admits a good moduli space by verifying that it is S-complete and Θ-reductive. Moreover, we will prove that the K-moduli space is a proper algebraic space.

In Chapter 9, we define the CM line bundle and prove it is ample on the K-moduli space.

## Prerequisite

The algebraic theory of K-stability builds on the machinery of higher dimensional geometry. This book assumes the reader has basic familiarity with the subject. For example, the reader should have some knowledge of minimal model program as introduced in Kollár and Mori (1998) and we also need the results proved by Birkar, Cascini, Hacon, and McKernan in Birkar et al. (2010). Some results on asymptotic invariants are needed. Most of them are covered in Lazarsfeld (2004b). We also need boundedness-type theorems proved in Hacon et al. (2014), Birkar (2019), and Birkar (2021). This is sufficient to read Chapters 2–6. All the necessary higher dimensional geometry results are summarized in Chapter 1.

To read Chapters 7–9 for the construction of K-moduli spaces, we assume the reader has some knowledge on stacks. In particular, we will need results in Alper (2013), Alper et al. (2023), and Halpern-Leistner (2022) for *good moduli spaces*. We only briefly discuss the notion of a family of higher dimensional varieties or log pairs over an arbitrary base and refer to Kollár (2023) for the proofs. We also assume the semi-positivity for the pushforward of pluri-canonical bundles.