INTEGRALS INVOLVING E-FUNCTIONS

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I

§ 1. *Introductory.* The formula to be proved is

$$\prod_{r=1}^{m-1}\int_0^\infty e^{-t_r}E(\alpha,\beta::t_r)t^{r/m-1}\,dt_r\,e^{-b/(t_1t_2\dots t_{m-1})}E\left(\alpha,\beta::\frac{b}{t_1t_2\dots t_{m-1}}\right)$$
$$=\frac{\{\Gamma(\alpha)\Gamma(\beta)\}^m}{\Gamma(m\alpha)\Gamma(m\beta)}(2\pi)^{\frac{1}{2}m-\frac{1}{2}}m^{-\frac{1}{2}}e^{-mb^{1/m}}E(m\alpha,\ m\beta::mb^{1/m}),\ \dots\dots\dots(1)$$

where $b>0$.

In § 2 it will be shown that the function

$$w(z)\equiv e^{-z}E(\alpha,\beta::z)$$

satisfies the differential equation

$$z^2w''+z(z-\alpha-\beta+1)w'+(\alpha\beta-\alpha z-\beta z+z)w=0\ ;\ \dots\dots\dots(2)$$

while in § 3 formula (1) will be established by means of (2). The following formulae are also required :

$$\int_0^\infty e^{-t}t^{\gamma-1}\,E(\alpha,\beta::t)dt=\frac{\Gamma(\alpha)\Gamma(\beta)\Gamma(\alpha+\gamma)\Gamma(\beta+\gamma)}{\Gamma(\alpha+\beta+\gamma)},\ \dots\dots\dots(3)$$

where $R(\alpha+\gamma)>0$, $R(\beta+\gamma)>0$ [MacRobert, *Complex Variable*, p. 381] ;

$$\Gamma(n)\Gamma\left(n+\frac{1}{m}\right)\Gamma\left(n+\frac{2}{m}\right)\dots\Gamma\left(n+\frac{m-1}{m}\right)=(2\pi)^{\frac{1}{2}m-\frac{1}{2}}m^{\frac{1}{2}-mn}\Gamma(mn)\ ;\ \dots\dots\dots(4)$$

$$E(\alpha,\beta::z)=\sum_{\alpha,\beta}\Gamma(\beta-\alpha)\Gamma(\alpha)z^\alpha F(\alpha\ ;\ \alpha-\beta+1\ ;\ z)\ ;\ \dots\dots\dots(5)$$

$$F(\alpha\ ;\ \rho\ ;\ z)=e^zF(\rho-\alpha\ ;\ \rho\ ;\ -z).\ \dots\dots\dots(6)$$

§ 2. *The Differential Equation.* If $y=E(\alpha,\beta::z)$ it satisfies the equation

$$z^2y''=z(z+\alpha+\beta-1)y'-\alpha\beta y\ \dots\dots\dots(7)$$

[MacRobert, *Complex Variable*, p. 349].

Now let $w=e^{-z}y$, so that $y=e^zw$; then $z^2(w''+2w'+w)=z(z+\alpha+\beta-1)(w'+w)-\alpha\beta w$, from which (2) follows. Other solutions of (2) are

$$e^{-z}z^\alpha F(\alpha\ ;\ \alpha-\beta+1\ ;\ z)=z^\alpha F(1-\beta\ ;\ \alpha-\beta+1\ ;\ -z),$$

and

$$e^{-z}z^\beta F(\beta\ ;\ \beta-\alpha+1\ ;\ z)=z^\beta F(1-\alpha\ ;\ \beta-\alpha+1\ ;\ -z).$$

§ 3. *Proof of the Multiple Integral.* Let $F(b)$ denote the L.H.S. of (1). Then, if

$$w(z)=e^{-z}E(\alpha,\beta::z),$$

$$F(b)=\prod_{r=1}^{m-1}\int_0^\infty e^{-t_r}E(\alpha,\beta::t_r){t_r}^{r/m-1}\,dt_r\,w\left(\frac{b}{t_1t_2\dots t_{m-1}}\right),$$

$$F'(b)=\prod_{r=1}^{m-1}\int_0^\infty e^{-t_r}E(\alpha,\beta::t_r){t_r}^{r/m-2}\,dt_r\,w'\left(\frac{b}{t_1t_2\dots t_{m-1}}\right),$$

and

$$F''(b)=\prod_{r=1}^{m-1}\int_0^\infty e^{-t_r}E(\alpha,\beta::t_r){t_r}^{r/m-3}\,dt_r\,w''\left(\frac{b}{t_1t_2\dots t_{m-1}}\right).$$

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Hence, from (2),

$$b^2F''(b) = \prod_{r=1}^{m-1} \int_0^\infty e^{-t_r} E(\alpha, \beta :: t_r) t_r^{r/m-1} dt_r$$
$$\times \left\{ \frac{-b}{t_1 \dots t_{m-1}} \left(\frac{b}{t_1 \dots t_{m-1}} - \alpha - \beta + 1 \right) w' - \alpha\beta w + \frac{b}{t_1 \dots t_{m-1}} (\alpha + \beta - 1) w \right\}.$$

Therefore

$$b^2F''(b) - (\alpha+\beta-1)bF'(b) + \alpha\beta F(b) = L + M, \quad \text{.........................(8)}$$

where

$$L = -b^2 \prod_{r=1}^{m-1} \int_0^\infty e^{-t_r} E(\alpha, \beta :: t_r) t_r^{r/m-3} dt_r\, w',$$

and

$$M = (\alpha+\beta-1)b \prod_{r=1}^{m-1} \int_0^\infty e^{-t_r} E(\alpha, \beta :: t_r) t_r^{r/m-2} dt_r\, w.$$

Now in L change the order of integration so that the first integral becomes the last; then

$$L = b \prod_{r=2}^{m-1} \int_0^\infty e^{-t_r} E(\alpha, \beta :: t_r) t_r^{r/m-2} dt_r \times \int_0^\infty e^{-t_1} E(\alpha, \beta :: t_1) t_1^{1/m-1} \frac{\partial w}{\partial t_1} dt_1.$$

Here integrate by parts, noting that w vanishes at $t_1 = 0$, and get

$$L = b \prod_{r=2}^{m-1} \int_0^\infty e^{-t_r} E(\alpha, \beta :: t_r) t_r^{r/m-2} dt_r \times - \int_0^\infty e^{-b/(t_1 \dots t_{m-1})} E\left(\alpha, \beta :: \frac{b}{t_1 \dots t_{m-1}}\right)$$
$$\times \left[\left(\frac{1}{m} - 1\right) e^{-t_1} E(\alpha, \beta :: t_1) t_1^{1/m-2} + t_1^{1/m-1} \frac{d}{dt_1} \{e^{-t_1} E(\alpha, \beta :: t_1)\} \right] dt_1.$$

On substitituting $b/(\lambda t_2 \dots t_{m-1})$ for t_1, this becomes

$$L = -\left(\frac{1}{m} - 1\right) b^{1/m} \prod_{r=2}^{m-1} \int_0^\infty e^{-t_r} E(\alpha, \beta :: t_r) t_r^{(r-1)/m-1} dt_r$$
$$\times \int_0^\infty e^{-\lambda} E(\alpha, \beta :: \lambda)\, \lambda^{(m-1)/m-1}\, w\left(\frac{b}{\lambda t_2 \dots t_{m-1}}\right) d\lambda$$
$$- b^{1/m+1} \prod_{r=2}^{m-1} \int_0^\infty e^{-t_r} E(\alpha, \beta :: t_r) t_r^{(r-1)/m-2} dt_r$$
$$\times \int_0^\infty e^{-\lambda} E(\alpha, \beta :: \lambda)\, \lambda^{(m-1)/m-2}\, w'\left(\frac{b}{\lambda t_2 \dots t_{m-1}}\right) d\lambda$$
$$= \left(1 - \frac{1}{m}\right) b^{1/m} F(b) - b^{1/m+1} F'(b).$$

Similarly

$$M = (\alpha+\beta-1) b^{1/m} F(b).$$

Hence (8) can be written

$$b^2 F''(b) - (\alpha+\beta-1-b^{1/m}) b F'(b) + \{\alpha\beta - (\alpha+\beta-1/m) b^{1/m}\} F(b) = 0 \quad \text{...........(9)}$$

Next, in (2) put $z = m\lambda^{1/m}$, replace α and β by $m\alpha$ and $m\beta$, and get

$$\lambda^2 \frac{d^2 w}{d\lambda^2} - (\alpha+\beta-1-\lambda^{1/m}) \lambda \frac{dw}{d\lambda} + \left\{ \alpha\beta - \left(\alpha+\beta-\frac{1}{m}\right) \lambda^{1/m} \right\} w = 0,$$

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which is (9) with w in place of $F(b)$ and λ in place of b. Thus

$$F(b) = Ae^{-mb^{1/m}}(mb^{1/m})^{m\alpha}\ F(m\alpha\ ;\ m\alpha - m\beta + 1\ ;\ mb^{1/m})$$
$$+ Be^{-mb^{1/m}}(mb^{1/m})^{m\beta} F(m\beta\ ;\ m\beta - m\alpha + 1\ ;\ mb^{1/m}).$$

Now $F(b)$ is symmetrical in α and β, and so are the coefficients of A and B. Therefore, if $A = f(\alpha, \beta)$, it follows that $B = f(\beta, \alpha)$.

Let it be assumed that $\alpha < \beta$, multiply by $b^{-\alpha}$ and let $b \to 0$; then, as $1/m + \beta - \alpha > 0$,

$$\prod_{r=1}^{m-1} \int_0^\infty e^{-t_r} E(\alpha, \beta : : t_r) t_r^{r/m - \alpha - 1}\, dt_r\ \Gamma(\alpha)\Gamma(\beta - \alpha) = Am^{m\alpha},$$

or, by (3),

$$\prod_{r=1}^{m-1} \frac{\Gamma(\alpha)\ \Gamma(\beta)\ \Gamma(r/m)\ \Gamma(\beta - \alpha + r/m)}{\Gamma(\beta + r/m)}\ \Gamma(\alpha)\ \Gamma(\beta - \alpha) = Am^{m\alpha}.$$

Hence, by (4),

$$A = \frac{\{\Gamma(\alpha)\Gamma(\beta)\}^m}{\Gamma(m\alpha)\Gamma(m\beta)} (2\pi)^{\frac{1}{2}m - \frac{1}{2}} m^{-\frac{1}{2}}\, \Gamma(m\alpha)\ \Gamma\{m(\beta - \alpha)\},$$

from which (1) follows.

If, on the other hand, $\beta < \alpha$, multiply by $b^{-\beta}$, let $b \to 0$, and the same results are obtained. The case $\alpha = \beta$ can be derived by continuity.

II

§ 1. *Introductory.* It is here proposed to establish the formula

$$\int_0^\infty e^{-n\lambda} \lambda^{nk-1} E\left(p\ ;\ \alpha_r : \ q\ ;\ \rho_s : \ \frac{x}{\lambda^n}\right) d\lambda = \frac{1}{(2\pi)^{\frac{1}{2}n - \frac{1}{2}} \sqrt{(n)}}\ E(p + n\ ;\ \alpha_r : \ q\ ;\ \rho_s : \ x), \quad \ldots(1)$$

where $R(k) > 0$, n is a positive integer, and

$$\alpha_{p+\nu+1} = k + \frac{\nu}{n},\ \nu = 0, 1, 2, \ldots, n - 1.$$

In § 2 the subsidiary formula

$$\int_0^\infty \int_0^\infty \ldots \int_0^\infty e^{-F}\, G\, dx_1\, dx_2 \ldots dx_{n-1} = (2\pi)^{\frac{1}{2}n - \frac{1}{2}}\, n^{-\frac{1}{2}}\, e^{-nb^{1/n}}, \quad \ldots\ldots\ldots\ldots\ldots\ldots(2)$$

where $b > 0$ and

$$F = x_1 + x_2 + \ldots + x_{n-1} + b/(x_1 x_2 \ldots x_{n-1}),$$
$$G = x_1^{1/n - 1}\, x_2^{2/n - 1} \ldots x_{n-1}^{(\ -1)/n - 1},$$

will be established ; in § 3 formula (1) will be derived by means of (2).

§ 2. *Proof of the Subsidiary Formula.* Let the L.H.S. of (2) be denoted by $F(b)$: then

$$F'(b) = \int_0^\infty \int_0^\infty \ldots \int_0^\infty e^{-F}\, G\left(\frac{-1}{x_1 x_2 \ldots x_{n-1}}\right) dx_1\, dx_2 \ldots dx_{n-1}.$$

Here change the order of integration so that the first integral becomes the last, and put

$$x_1 = \frac{b}{\mu x_2 x_3 \ldots x_{n-1}}$$

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in the innermost integral: then

$$F'(b) = -\int_0^\infty \int_0^\infty \cdots \int_0^\infty e^{-F_1} G_1 \mu \, dx_2 \ldots dx_{n-1} \frac{d\mu}{\mu^2 x_2 \ldots x_{n-1}},$$

where

$$F_1 = x_2 + x_3 + \ldots + x_{n-1} + \mu + b/(\mu x_2 \ldots x_{n-1})$$

and

$$G_1 = \left(\frac{b}{\mu x_2 \ldots x_{n-1}}\right)^{1/n-1} x_2^{2/n-1} \ldots x_{n-1}^{(n-1)/n-1},$$

$$= b^{1/n-1} x_2^{1/n} x_3^{2/n} \ldots x_{n-1}^{(n-2)/n} \mu^{-1/n+1}.$$

Hence, on replacing $x_2, x_3, \ldots, x_{n-1}, \mu$ by $x_1, x_2, \ldots, x_{n-1}$ respectively, we have

$$F'(b) = -b^{1/n-1} F(b).$$

Thus

$$F(b) = A e^{-nb^{1/n}}.$$

To determine A let $b \to 0$: then

$$\Gamma\left(\frac{1}{n}\right) \Gamma\left(\frac{2}{n}\right) \ldots \Gamma\left(\frac{n-1}{n}\right) = A;$$

so that

$$A = (2\pi)^{\frac{1}{2}n-\frac{1}{2}} n^{-\frac{1}{2}},$$

from which (2) follows.

§ 3. *Proof of the Integral.* If the formula

$$\int_0^\infty e^{-\lambda} \lambda^{k-1} E\left(p;\ \alpha_r:\ q;\ \rho_s:\ \frac{x}{\lambda}\right) d\lambda = E(p+1;\ \alpha_r:\ q;\ \rho_s:\ x), \quad \ldots\ldots\ldots\ldots(3)$$

where $\alpha_{p+1} = k$, $R(k) > 0$ [MacRobert, *Phil. Mag.*, Ser. 7, XXXI, p. 255], is applied repeatedly to itself, it becomes

$$\int_0^\infty \int_0^\infty \cdots \int_0^\infty e^{-(x_0+x_1+\ldots+x_{n-1})} x_0^{k-1} x_1^{k+1/n-1} \ldots x_{n-1}^{k+(n-1)/n-1}$$

$$\times E\left(p;\ \alpha_r:\ q;\ \rho_s:\ \frac{x}{x_0 \ldots x_{n-1}}\right) dx_0 \ldots dx_{n-1} = E(p+n;\ \alpha_r:\ q;\ \rho_s:\ x),$$

where the α's are those given in (1). Now change the order of integration so that the first integral becomes the last, and put

$$x_0 = \frac{\sigma}{x_1 x_2 \ldots x_{n-1}}.$$

Then the L.H.S. becomes

$$\int_0^\infty \int_0^\infty \cdots \int_0^\infty e^{-(x_1+x_2+\ldots+x_{n-1})} x_1^{1/n-1} x_2^{2/n-1} \ldots x_{n-1}^{(n-1)/n-1} dx_1 \ldots dx_{n-1}$$

$$\times \int_0^\infty e^{-\sigma/(x_1 x_2 \ldots x_{n-1})} \sigma^{k-1} E\left(p;\ \alpha_r:\ q;\ \rho_s:\ \frac{x}{\sigma}\right) d\sigma.$$

Here change the order of integration so that the last integral becomes the first, apply (2), and get

$$(2\pi)^{\frac{1}{2}n-\frac{1}{2}} n^{-\frac{1}{2}} \int_0^\infty e^{-n\sigma^{1/n}} \sigma^{k-1} E\left(p;\ \alpha_r:\ q;\ \rho_s:\ \frac{x}{\sigma}\right) d\sigma.$$

Finally, put $\sigma = \lambda^n$ and so obtain (1).

III

§ 1. *Introductory.* It is proposed to establish the formula

$$\int_0^\infty e^{-t}\, t^{\kappa-1} E(\gamma, \delta : : t)\, E(p\; ;\; \alpha_r :\; q\; ;\; \rho_s :\; x/t)\, dt$$
$$= \Gamma(\gamma)\,\Gamma(\delta)\, E(p+2\; ;\; \alpha_r :\; q+1\; ;\; \rho_s :\; x), \qquad \text{.....................(1)}$$

where $\alpha_{p+1} = \gamma + \kappa$, $\alpha_{p+2} = \delta + \kappa$, $\rho_{q+1} = \gamma + \delta + \kappa$ and $R(\gamma+\kappa) > 0$, $R(\delta+\kappa) > 0$.

The following formulae are required in the proof :

$$\int_0^\infty \mu^{\rho_{q+1}-\alpha_{p+1}-1}(1+\mu)^{-\rho_{q+1}} E\{p\; ;\; \alpha_r :\; q\; ;\; \rho_s :\; (1+\mu)x\}\, d\mu$$
$$= \Gamma(\rho_{q+1} - \alpha_{p+1})\, E(p+1\; ;\; \alpha_r :\; q+1\; ;\; \rho_s :\; x), \qquad \text{..............(2)}$$

where $R(\alpha_{p+1}) > 0$, $R(\rho_{q+1} - \alpha_{p+1}) > 0$ (**1**) ;

$$\int_0^\infty e^{-\lambda}\lambda^{\alpha_{p+1}-1} E(p\; ;\; \alpha_r :\; q\; ;\; \rho_s :\; x/\lambda)\, d\lambda = E(p+1\; ;\; \alpha_r :\; q\; ;\; \rho_s :\; x), \qquad \text{...........(3)}$$

where $R(\alpha_{p+1}) > 0$, (**2**).

$$E(\alpha, \beta : : x) = \Gamma(\alpha) \int_0^\infty e^{-\lambda}\lambda^{\beta-1}(1+\lambda/x)^{-\alpha}\, d\lambda, \qquad \text{...........................(4)}$$

where $R(\beta) > 0$, (**3**).

It is assumed that x is real and positive.

The proof of (1) is given in § 2. An integral involving a product of three E-functions is discussed in § 3.

§ 2. *Proof of the Formula.* On applying (2) to (3) it is seen that

$$\int_0^\infty \mu^{\delta-1}(1+\mu)^{-\gamma-\delta-\kappa}\, d\mu \int_0^\infty e^{-\lambda}\lambda^{\delta+\kappa-1} E\{p\; ;\; \alpha_r :\; q\; ;\; \rho_s :\; (1+\mu)\, x/\lambda\}\, d\lambda$$
$$= \Gamma(\delta)\, E(p+2\; ;\; \alpha_r :\; q+1\; ;\; \rho_s :\; x),$$

where $R(\gamma+\kappa) > 0$, $R(\delta+\kappa) > 0$, $R(\delta) > 0$.

Now put $\lambda = (1+\mu)t$ and change the order of integration : then the double integral becomes

$$\int_0^\infty e^{-t}\, t^{\delta+\kappa-1} E(p\; ;\; \alpha_r :\; q\; ;\; \rho_s :\; x/t)\, dt \int_0^\infty e^{-\mu t}\mu^{\delta-1}(1+\mu)^{-\gamma}\, d\mu .$$

In the inner integral put $\mu = \xi/t$ and it becomes

$$t^{-\delta}\int_0^\infty e^{-\xi}\xi^{\delta-1}(1+\xi/t)^{-\gamma}\, d\xi = \{1/\Gamma(\gamma)\}\, t^{-\delta} E(\gamma, \delta : : t),$$

by (4). From this (1) follows.

§ 3. *Integral involving a Product of three E-functions.* The formula to be proved is

$$\int_0^\infty e^{-t}\, t^{\kappa-1} E(\alpha, \beta : : t)\, E(\gamma, \delta : : t)\, E(p\; ;\; \alpha_r :\; q\; ;\; \rho_s :\; x/t)\, dt$$
$$= \Gamma(\alpha)\,\Gamma(\beta)\,\Gamma(\gamma)\,\Gamma(\delta) \sum_{r=0}^{\infty} \frac{(\alpha\; ;\; r)(\delta ;\; r)}{r!}\, E\begin{pmatrix} \alpha_1, \ldots, \alpha_p, \alpha+\gamma+\kappa, \beta+\delta+\kappa, \alpha+\delta+\kappa+r\; ;\; x \\ \rho_1, \ldots, \rho_q, \alpha+\beta+\delta+\kappa+r,\; \alpha+\gamma+\delta+\kappa+r \end{pmatrix}, \qquad \text{...(5)}$$

where the constants are such that the integral converges.

The following formulae are required :

$$E(p\; ;\; \alpha_r :\; q\; ;\; \rho_s :\; x) = \frac{1}{2\pi i}\int \frac{\Gamma(\zeta)\prod_{r=1}^{p}\Gamma(\alpha_r - \zeta)}{\prod_{s=1}^{q}\Gamma(\rho_s - \zeta)}\, x^\zeta\, d\zeta, \qquad \text{.....................(6)}$$

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where the contour is of the usual type employed by Barnes;

$$\int_0^\infty e^{-t}\,t^{\kappa-1}\,E(\alpha,\beta::t)\,E(\gamma,\delta::t)\,dt$$
$$=\Gamma(\beta)\,\Gamma(\gamma)\,\Gamma(\alpha+\gamma+\kappa)\,\Gamma(\beta+\delta+\kappa)\sum_{r=0}^{\infty}\frac{\Gamma(\alpha+r)\,\Gamma(\delta+r)\,\Gamma(\alpha+\delta+\kappa+r)}{r!\,\Gamma(\alpha+\beta+\delta+\kappa+r)\,\Gamma(\alpha+\gamma+\delta+\kappa+r)},\quad\text{..............(7)}$$

provided that $R(\alpha+\gamma+\kappa)>0$, $R(\beta+\gamma+\kappa)>0$, $R(\alpha+\delta+\kappa)>0$, $R(\beta+\delta+\kappa)>0$, (**4**)

Now substitute from (6) in the L.H.S. of (5), and change the order of integration; then

$$\frac{1}{2\pi i}\int\frac{\Gamma(\zeta)\prod_{r=1}^{p}\Gamma(\alpha_r-\zeta)}{\prod_{s=1}^{q}\Gamma(\rho_s-\zeta)}\,x^\zeta\,d\zeta\int_0^\infty e^{-t}\,t^{\kappa-\zeta-1}\,E(\alpha,\beta::t)\,E(\gamma,\delta::t)\,dt$$
$$=\frac{1}{2\pi i}\int\frac{\Gamma(\zeta)\prod_{r=1}^{p}\Gamma(\alpha_r-\zeta)}{\prod_{s=1}^{q}\Gamma(\rho_s-\zeta)}\,x^\zeta\,d\zeta\,\Gamma(\beta)\,\Gamma(\gamma)\,\Gamma(\alpha+\gamma+\kappa-\zeta)\,\Gamma(\beta+\delta+\kappa-\zeta)$$
$$\times\sum_{m=0}^{\infty}\frac{\Gamma(\alpha+m)\,\Gamma(\delta+m)\,\Gamma(\alpha+\delta+\kappa+m-\zeta)}{m!\,\Gamma(\alpha+\beta+\delta+\kappa+m-\zeta)\,\Gamma(\alpha+\gamma+\delta+\kappa+m-\zeta)}.$$

On changing the order of integration and summation and applying (6), formula (5) is obtained.

REFERENCES

(**1**) MacRobert, T. M., *Phil. Mag.* (VII), **31**, 256 (1941).
(**2**) MacRobert, T. M., *loc. cit.*, p. 255.
(**3**) MacRobert, T. M., *Functions of a Complex Variable* (3rd ed. London, 1946), p. 348.
(**4**) MacRobert, T. M., *Quart. Journ. of Maths., Oxford*, **13**, 68, (1942).

IV

§ 1. *Introductory.* It is proposed to establish the formula

$$\int_0^\infty e^{-nt}\,t^{nk-1}\,E(n\gamma,n\delta::nt)\,E\left(p\,;\ \alpha_r:\ q\,;\ \rho_s:\ \frac{x}{t^n}\right)dt$$
$$=\frac{\Gamma(n\gamma)\,\Gamma(n\delta)}{(2\pi)^{\frac{1}{2}n-\frac{1}{2}}\sqrt{n}}\,E(p+2n\,;\ \alpha_r:\ q+n\,;\ \rho_s:\ x),\quad\text{....................(1)}$$

where n is a positive integer and

$$\alpha_{p+2\nu+1}=\gamma+k+\nu/n,\quad \alpha_{p+2\nu+2}=\delta+k+\nu/n,\quad \rho_{q+\nu+1}=\gamma+\delta+k+\nu/n,\quad\text{..............(1')}$$

$\nu=0,1,2,\ldots,n-1$, $R(k+\gamma)>0$, $R(k+\delta)>0$.

The two following formulae are required in the proof:

$$\int_0^\infty e^{-t}\,t^{k-1}\,E(\gamma,\delta::t)\,E\left(p\,;\ \alpha_r:\ q\,;\ \rho_s:\ \frac{x}{t}\right)dt=\Gamma(\gamma)\,\Gamma(\delta)\,E(p+2\,;\ \alpha_r:\ q+1\,;\ \rho_s:\ x),\ \text{...(2)}$$

where $R(\gamma+k)>0$, $R(\delta+k)>0$, and $\alpha_{p+1}=\gamma+k$, $\alpha_{p+2}=\delta+k$, $\rho_{q+1}=\gamma+\delta+k$, (**1**);

$$\prod_{r=1}^{n-1}\int_0^\infty e^{-t_r}\,E(\gamma,\delta::t_r)\,t_r^{\,r/n-1}\,dt_r\,e^{-b/(t_1t_2\ldots t_{n-1})}\,E\left(\gamma,\delta::\frac{b}{t_1t_2\ldots t_{n-1}}\right)$$
$$=\frac{\{\Gamma(\gamma)\,\Gamma(\delta)\}^n\,(2\pi)^{\frac{1}{2}n-\frac{1}{2}}}{\Gamma(n\gamma)\,\Gamma(n\delta)\,\sqrt{n}}\,e^{-nb^{1/n}}\,E(n\gamma,n\delta::nb^{1/n}),\ \text{.........(3)}$$

where $b>0$, (**2**).

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§ 2. *Proof of the Formula.* On applying (2) to itself $(n-1)$ times with

$$k+1/n,\ k+2/n, \ldots,\ k+(n-1)/n$$

in turn in place of k it is found that

$$\prod_{r=0}^{n-1} \int_0^\infty e^{-t_r}\, t_r^{k+r/n-1}\, E(\gamma, \delta :: t_r)\, dt_r\, E\left(p\ ;\ \alpha_r :\ q\ ;\ \rho_s : \frac{x}{t_0 t_1 \ldots t_{n-1}}\right)$$
$$= \{\Gamma(\gamma)\, \Gamma(\delta)\}^n\, E(p+2n\ ;\ \alpha_r :\ q+n\ ;\ \rho_s :\ x)$$

where the α's and ρ's are given by (1′).

Now change the order of integration so that the first integral becomes the last, put

$$t_0 = \frac{\lambda}{t_1 t_2 \ldots t_{n-1}},$$

and change the order of integration so that the last integral becomes the first. Then the L.H.S. becomes

$$\int_0^\infty \lambda^{k-1} E\left(p\ ;\ \alpha_r :\ q\ ;\ \rho_s : \frac{x}{\lambda}\right) d\lambda \prod_{r=1}^{n-1} \int_0^\infty e^{-t_r} t_r^{r/n-1} E(\gamma, \delta :: t_r)\, dt_r\, e^{-\lambda/(t_1 \ldots t_{n-1})}\, E\left(\gamma, \delta :: \frac{\lambda}{t_1 \ldots t_{n-1}}\right)$$
$$= \frac{\{\Gamma(\gamma)\, \Gamma(\delta)\}^n\, (2\pi)^{\frac{1}{2}n-\frac{1}{2}}}{\Gamma(n\gamma)\, \Gamma(n\delta)\, \sqrt{n}} \int_0^\infty e^{-n\lambda^{1/n}}\, \lambda^{k-1}\, E(n\gamma, n\delta :: n\lambda^{1/n})\, E\left(p\ ;\ \alpha_r :\ q\ ;\ \rho_s :\ \frac{x}{\lambda}\right) d\lambda,$$

by (3). On replacing λ by t^n formula (1) is obtained.

REFERENCES

(1) Ragab, F. M., *Proc. Glasg. Math. Ass.*, **1**, 133 (1953).

(2) Ragab, F. M., *Proc. Glasg. Math. Ass.*, **1**, 129 (1953).

V

§ 1. *Introductory.* The formula to be proved is

$$\int_0^\infty \exp\left(-\frac{y^2}{4x} - x\right) x^{-\alpha-\beta-1}\, E(\alpha, \beta :: x)\, dx = 2\Gamma(\alpha)\, \Gamma(\beta)\, (\tfrac{1}{2}y)^{-\alpha-\beta}\, K_{\alpha-\beta}(y), \quad \ldots\ldots\ldots(1)$$

where $R(y^2) > 0$.

The following formulae will be required in the proof :

$$E(\alpha, \beta :: x) = \Gamma(\alpha) \int_0^\infty e^{-\lambda}\, \lambda^{\beta-1} \left(1 + \frac{\lambda}{x}\right)^{-\alpha} d\lambda, \quad \ldots\ldots\ldots(2)$$

where $R(\beta) > 0$, (**1**) ;

$$K_n(z) = \tfrac{1}{2}(\tfrac{1}{2}z)^n \int_0^\infty \exp\left(-T - \frac{z^2}{4T}\right) T^{-n-1}\, dT, \quad \ldots\ldots\ldots(3)$$

where $R(z^2) > 0$, (**2**) ;

$$\int_0^\infty \frac{K_n\{a\sqrt{(t^2+z^2)}\}}{(t^2+z^2)^{\frac{1}{2}n}}\, t^{2m+1}\, dt = \frac{2^m\, \Gamma(m+1)}{a^{m+1}\, z^{n-m-1}}\, K_{n-m-1}(az). \quad \ldots\ldots\ldots(4)$$

[This integral is due to Sonine and Gegenbauer, (**3**).]

§ 2. *Proof of the formula.* Substitute for the E-function on the L.H.S. of (1) from (2), and it becomes, if $R(\beta) > 0$,

$$\Gamma(\alpha) \int_0^\infty \exp\left(-\frac{y^2}{4x} - x\right) x^{-\alpha-\beta-1}\, dx \int_0^\infty e^{-\lambda}\, \lambda^{\beta-1} \left(1 + \frac{\lambda}{x}\right)^{-\alpha} d\lambda.$$

Here put $\lambda = x\xi$ and change the order of integration, so getting

$$\Gamma(\alpha)\int_0^\infty \xi^{\beta-1}(1+\xi)^{-\alpha}\,d\xi\int_0^\infty \exp\left\{-\frac{y^2}{4x}-x\,(1+\xi)\right\}x^{-\alpha-1}\,dx.$$

Next, put $x(1+\xi)=T$, and the expression becomes

$$\Gamma(\alpha)\int_0^\infty \xi^{\beta-1}\,d\xi\int_0^\infty \exp\left\{-\frac{y^2(1+\xi)}{4T}-T\right\}\frac{dT}{T^{\alpha+1}}=\Gamma(\alpha)\,\frac{2^{\alpha+1}}{y^\alpha}\int_0^\infty \xi^{\beta-1}\,\frac{K_\alpha\,\{y\surd(1+\xi)\}}{(1+\xi)^{\frac{1}{2}\alpha}}\,d\xi,$$

by (3).

Now replace ξ by t^2; then, on applying (4), formula (1) is obtained. The condition $R(\beta)>0$ may be removed by analytical continuation.

REFERENCES

(1) MacRobert, T. M., *Functions of a Complex Variable*, (3rd ed., London, 1946), p. 348.
(2) Gray, Matthews and MacRobert, *Bessel Functions*, p. 51.
(3) Watson, G. N., *Bessel Functions*, p. 417.

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