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Weak Convergence Is Not Strong Convergence For Amenable Groups

Joseph M. Rosenblatt and George A. Willis

Abstract. Let G be an infinite discrete amenable group or a non-discrete amenable group. It is shown how to construct a net (f_α) of positive, normalized functions in $L_1(G)$ such that the net converges weak* to invariance but does not converge strongly to invariance. The solution of certain linear equations determined by colorings of the Cayley graphs of the group are central to this construction.

1 Introduction

Let G be a locally compact Hausdorff group, and let λ denote a left-invariant Haar measure on the group. Let $P(G)$ denote the positive functions f in $L_1(G)$ such that $\int_G f\,d\lambda = 1$. For $f \in L_1(G)$ and $F \in L_\infty(G)$, denote by $\langle f, F\rangle$ the integral $\int_G f(x)\overline{F(x)}\,d\lambda(x)$. The convex set $P(G)$ has all of the means on $L_\infty(G)$ in its weak* closure. In particular, if G is amenable, and ϕ is a left-invariant mean on $L_\infty(G)$, then there is a net (f_α) in $P(G)$ such that (f_α) converges to ϕ in the weak* topology *i.e.* $\lim_{\alpha\to\infty}\langle f_\alpha, F\rangle = \phi(F)$ for any $F \in L_\infty(G)$. Thus, for any $F \in L_\infty(G)$ and any $g \in G$, we have $\lim_{\alpha\to\infty}\langle f_\alpha - {}_g f_\alpha, F\rangle = 0$. This is the condition of weak* convergence to invariance of Day. In Day [2], he used this to show that amenability is equivalent to his condition of strong convergence to invariance: G is amenable if and only if there exists a net (f_α) in $P(G)$ such that for all $g \in G$, we have $\lim_{\alpha\to\infty}\|f_\alpha - {}_g f_\alpha\|_1 = 0$. This is proved by a use of the principle in functional analysis that the weak and strong closures of a convex set are the same, but it is not saying that weak* convergence to invariance is the same as strong convergence to invariance. Indeed, in Day [3], he points out that the failure of the equivalence of these topologies is exactly why the functional analytic argument is needed. See in addition Namioka [5] where the connection between weak* convergence to invariance and strong convergence to invariance is clearly presented. Namioka [5] also gives the first easily understood proof of the equivalence of amenability and the existence of Følner sets. See also Greenleaf [4] and Patterson [6] for background about amenable groups.

Despite so much having been written on amenable groups, there does not seem to have been a construction of nets which are weak* converging to invariance but not converging strongly to invariance. There is something to prove here since Sine [7] showed that for sequences, weak* convergence to invariance actually does imply strong convergence to invariance. This article will construct nets which are weak*

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converging to invariance but not converging strongly to invariance, by considering some aspects of Cayley graphs on the group which are equivalent to amenability of the group.

2 Configurations in Discrete Groups

Our object is to construct nets which are weak* converging to invariance but not converging strongly to invariance. Consider first the case of *discrete groups*. It is clear that what one has to do is to show that there is some $x \in G$ and some $\delta > 0$ such that given any $\epsilon > 0$, functions $F_1, \dots, F_m \in L_\infty(G)$ and elements $g_1, \dots, g_n \in G$, there is some $f \in P(G)$ with $|\langle f - {}_{g_j}f, F_i\rangle| < \epsilon$ for $i = 1, \dots, m$ and $j = 1, \dots, n$, but $\|f - {}_x f\|_1 \geq \delta$. Then, varying over the directed set of α given by choices of ϵ, $\{F_i : i = 1, \dots, m\}$ and $\{g_j : i = j, \dots, n\}$ in the usual manner gives a net converging weak* to invariance, but such that strong invariance fails since $\|f_\alpha - {}_x f_\alpha\|_1 \geq \delta$ for all α. It is also clear that the largest possible value for δ would be $\delta = 2$. In carrying out such a construction, there is no harm in replacing the functions $\{F_i : 1 = 1, \dots, m\}$ by an arbitrary finite partition $\{E_1, \dots, E_m\}$ of G. Also, being optimistic one could look to show that there is some $x \in G$, such that for any finite partition $\{E_i : i = 1, \dots, m\}$ of G, and any choice of $\{g_j : j = 1, \dots, n\}$, there is some $f \in P(G)$ such that for all i and j, one has $\langle f - {}_{g_j}f, 1_{E_i}\rangle = 0$, while also $\|f - {}_x f\|_1 = 2$. It turns out that this is possible in any infinite amenable group.

Consider for the rest of this section the case of a discrete group G. The general case will be treated in the next section. In order to carry out the construction outlined above, it is very convenient to look at colorings of Cayley graphs on G. We are not assuming that G is finitely generated, but given any finite set $\{g_j : j = 1, \dots, n\} \subset G$, there is a Cayley graph, denoted by $(G, \{g_j\})$, with vertices being the elements of G and the directed edges being from h_1 to h_2, for $h_1, h_2 \in G$, if $h_2 = g_j h_1$ for some $j = 1, \dots, n$. Suppose also that a partition $\{E_i : 1, \dots, m\}$ of G has been fixed. It will be useful to speak of this as corresponding to a coloring of the vertices of the Cayley graph $(G, \{g_j\})$, so that all vertices in E_i are given the same distinctive color. This provides us then with a class of finite sets which we call *configurations*. Each configuration C is an $(n+1)$-tuple of colors $(C_0, C_1, \dots, C_n)$ with each C_k being one of the m colors. For C to be a *realized configuration*, it is necessary that there is some (not necessarily unique) set of elements $x_0, x_1, \dots, x_n \in G$, such that each x_j is the color C_j, $j = 0, \dots, n$, and for each $j = 1, \dots, n$, we have $x_j = g_j x_0$. The element x_0 is called a *base point* of the configuration and the elements x_j, $j = 1, \dots, n$, are called *branch points* of C. In this case, we say that $(x_0, x_1, \dots, x_n)$ realizes the configuration C. For each configuration C and each j between 1 and n, $x_j(C)$ will denote the set of all elements of G which appear as the j-th branch point in an occurrence of C. Thus $x_j(C)$ consists of all points y of color C_j such that $x_0 = g_j^{-1} y$ is of color C_0, and $g_k x_0$ is of color C_k for all $k = 1, \dots, n$.

There are only a finite number of possible configurations, but there may be an infinite number of tuples $(x_0, x_1, \dots, x_n) \in G^{n+1}$ which have the same configuration. Also, the configurations can be thought of as refining the partition $\{E_i : i = 1, \dots, m\}$ if one further partitions each E_i into the subsets consisting of all base points x with $(x, g_1 x, \dots, g_n x)$ having the configuration C.

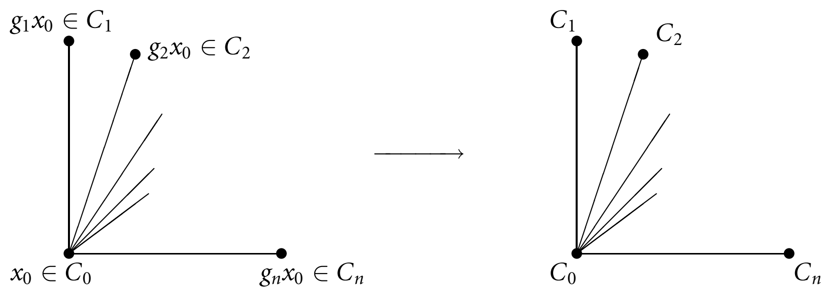

Figure 1: The configuration in the Cayley graph with base point x_0

Now given $f \in l_1(G)$, and a configuration C as above, let f_C denote the sum $\sum\{f(x) : x \in x_0(C)\}$. Clearly then $\sum\{f(x) : x \in E_i\} = \sum\{f_C : x_0(C) \subset E_i\}$. This is because for any two configurations C and C', either $x_0(C)$ and $x_0(C')$ are disjoint or $C = C'$. In this spirit, we have the following proposition. In this proposition, the equations (2) here are called the *configuration equations*. They can be considered abstractly without reference to the function $f \in l_1(G)$ if one takes the values (f_C) simply as the unknowns in the equations. The configuration equations depend on the choices of $\{E_i : i = 1, \dots, m\}$ and $\{g_j : j = 1, \dots, n\}$. There are n configuration equations, one corresponding to each generator g_j. The number of variables in the configuration equations is equal to the number of configurations, there being one variable f_C for each configuration C.

Proposition 2.1 *Given $f \in l_1(G)$, we have for all $i = 1, \dots, m$ and $j = 1, \dots, n$,*

$$\langle f - {}_{g_j}f, 1_{E_i}\rangle = 0, \tag{1}$$

if and only if for all $i = 1, \dots, m$ and $j = 1 \dots, n$

$$\sum\{f_C : x_0(C) \subset E_i\} = \sum\{f_C : x_j(C) \subset E_i\}. \tag{2}$$

Proof We have observed that $\langle f, 1_{E_i}\rangle = \sum\{f(x) : x \in E_i\} = \sum\{f_C : x_0(C) \subset E_i\}$. But in the same spirit, $\langle {}_{g_j}f, 1_{E_i}\rangle = \sum\{f(g_j^{-1}x) : x \in E_i\} = \sum\{f_C : x_j(C) \subset E_i\}$. Indeed, if $y \in x_j(C) \subset E_i$, then $g_j^{-1}y \in x_0(C)$. Also, if $y \in E_i$, then $g_j^{-1}y \in x_0(C)$ for some C with $y \in x_j(C)$. Furthermore, for any two configurations C and C', either $x_j(C)$ and $x_j(C')$ are disjoint or $C = C'$. Thus,

$$\begin{aligned}\sum\{f(g_j^{-1}x) : x \in E_i\} &= \sum\{f(z) : z \in x_0(C) \quad \text{and } x_j(C) \subset E_i\}\\ &= \sum\{f_C : x_j(C) \subset E_i\}.\end{aligned}$$ ■

Remark 2.2 a) The configuration equations are just saying that for the i-th color and the j-th branch, the sum of all f_C with base point of the i-th color should be

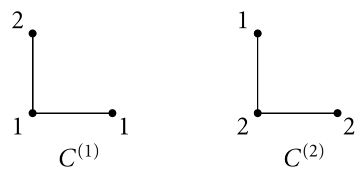

Figure 2: The configuration set corresponding to the partition $\mathcal{E}$.

the same as the sum of all f_C with j-th branch point of that same color. As such, the configuration equations derived from a function $f \in l_1(G)$ depend only on (f_C), and not on the individual values of f at points in $x_0(C)$.

b) Given a solution (f_C) to (2) in Proposition 2.1, one can construct the $f \in l_1(G)$ to go with them. See the end of Proposition 2.4.

Example 2.3 A simple example may help to illustrate what the configuration equations are. Let the group be the integers $\mathbb{Z}$. Let $\mathcal{E}$ be the partition whose two terms are E_1, the even integers, and E_2, the odd integers. Let $g_1 = 1$ and $g_2 = 2$. There are two realizable configurations, $C^{(1)} = (1, 2, 1)$ and $C^{(2)} = (2, 1, 2)$. Here we have used the index i of the color E_i to identify it. These are illustrated in Figure 2. Letting $c_i = f_{C^{(i)}}$ be the unknown value in the configuration equations which is corresponding to the configuration $C^{(i)}$, this example gives these configuration equations:

$$\begin{array}{cc} \text{From } g_1\text{:} & \text{From } g_2\text{:} \\ c_1 = c_2 & c_1 = c_1 \\ c_2 = c_1 & c_2 = c_2 \end{array}$$

Now, in general the configuration equations comprise a finite system of linear equations in the variables f_C with coefficients being either 0 or 1. We are interested in when there is a solution (f_C) such that each $f_C \geq 0$ and $\sum\{f_C : C \text{ is a configuration}\} = 1$. We call such a solution a *normalized solution of the configuration equations*. This is possible in general in amenable groups (and only in amenable groups).

Proposition 2.4 *There is a normalized solution of every possible instance of the configuration equations if and only if G is amenable.*

Proof Suppose that G is amenable. Let (f_α) be a net in $P(G)$ converging weak* to invariance. For a fixed choice of $\{E_i\}$ and $\{g_j\}$ as above, the values $\big((f_\alpha)_C\big)$ are normalized. Because of the weak* invariance, the values $\big((f_\alpha)_C\big)$ give approximate solutions of the configuration equations, with the error in the approximation tending to 0 as $\alpha \to \infty$. But the configuration equations form a finite linear system.

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Therefore, taking the limit as $\alpha \to \infty$, we get a normalized solution of the configuration equations. On the other hand, suppose there is a normalized solution for every instance of the configuration equations. Given fixed $\{E_i\}$ and $\{g_j\}$, let (f_C) be such a solution. Choose a single base point x_0^C for each realized configuration C. Let $f \in P(G)$ be the function such that $f(x_0^C) = f_C$ for every C, and $f = 0$ otherwise. Then by Proposition 2.1, for every i and j, we have $\langle f - {}_{g_j}f, 1_{E_i}\rangle = 0$. Since the choice of $\{E_i\}$ and $\{g_j\}$ is arbitrary, this shows that there is a net in $P(G)$ converging weak* to invariance, and hence that G is amenable. ■

Remark 2.5 The values which solve the configuration equations might be generated in other ways. For example, if μ is a positive, normalized finitely-additive measure defined on all subsets of G, and we have the equations $\mu(g_jE_i) = \mu(E_i)$ for all $i = 1, \dots, m$ and $j = 1, \dots, n$, then there is a solution, call it (f_C) to the corresponding instance of the configuration equations. In this case the values f_C would be the values $\mu\big(x_0(C)\big)$.

The proof of this proposition is actually showing the following.

Corollary 2.6 *Let G be an amenable group. Then for every possible instance of the configuration equations for some configurations $\{C^{(1)}, \dots, C^{(L)}\}$, there is a function $f \in P(G)$ such that the values $(f_{C^{(l)}} : l = 1, \dots, L)$ yield a normalized solution of the equations.*

This observation gives the following proposition.

Proposition 2.7 *Let G be an amenable group. Let $g_1, \dots, g_n \in G$ and $D_1, \dots, D_r$ be arbitrary subsets of G. Then there exists a positive, normalized function $f \in l_1(G)$, such that for all $i = 1, \dots, r$ and $j = 1, \dots, s$, we have $\langle f, 1_{g_jD_i}\rangle = \langle f, 1_{D_i}\rangle$.*

Proof Let the sets $E_i, i = 1, \dots, m$, be the atoms of the σ-algebra generated by $\{D_1, \dots, D_r\}$. Then apply Proposition 2.4 to the configuration equations given by $\{E_i : i = 1, \dots, m\}$ and $\{g_j : j = 1, \dots, n\}$. ■

Remark 2.8 The point of this proposition is that we could not get this exact invariance with a function $f \in P(G)$ if we were not dealing with finitely many sets and translations. If there were infinitely terms to deal with, then we might need a linear functional like an invariant mean which vanishes at all singleton sets in the group. Also, although a net converging weak* to invariance would have given approximate invariance here, some argument like the one with the configuration equations is needed in order to get exact invariance.

It is also clear from the solution of the configuration equations how to get weak* invariant nets which are not strongly invariant.

Theorem 2.9 *If G is an infinite discrete amenable group, then there exists a net (f_α) in $P(G)$ converging* weak* *to invariance, such that for every $x \in G$, $x \neq e$, eventually $\|f_\alpha - {}_xf_\alpha\|_1 = 2$.*

Proof Fix $\{E_i\}$ and $\{g_j\}$ as above. Observe first that if (f_α) is a net in $P(G)$ which is converging weak* to invariance, then the associated limiting solution to the configuration equations derived from $\{E_i\}$ and $\{g_j\}$ must have f_C equal 0 for any configuration C which has only finitely many occurrences in the Cayley graph $(G, \{g_j\})$ because G is infinite. Also, because G is infinite and there are only a finite number of possible configurations with respect to a fixed choice of $\{E_i\}$ and $\{g_j\}$, there must be configurations with infinitely many incidences in the Cayley graph. Take the normalized solution (f_C) to the configuration equations for all configurations with infinite incidence. Choose distinct base points x_0^C for each C with infinite incidence. Let $f \in P(G)$ be defined as in the proof of Proposition 2.4, with $f(x_0^C) = f_C$ for every C, and $f = 0$ otherwise. In addition, because C has infinite incidence in the Cayley graph, we can also arrange for these points x_0^C to be arbitrarily widely separated. That is, given any finite set $K \subset G \setminus \{e\}$, the choices of x_0^C can be made so that for all $k \in K$, we have f and ${}_k f$ disjointly supported and so $\|f - {}_k f\|_1 = 2$. Now letting the choices of $\{E_i\}$, $\{g_j\}$ and K vary in the usual directed fashion, the functions constructed here form a net (f_α) as needed to finish the proof. ■

Remark 2.10 a) The construction shows that for any fixed finite $K_0 \subset G \setminus \{e\}$, we can arrange for $\|f_\alpha - {}_k f_\alpha\|_1 = 2$ for all $k \in K_0$ and all α. This is reflected in Example 2.3 by the fact that we can put the weights $c_1 = \frac{1}{2}$ and $c_2 = \frac{1}{2}$ anywhere in the sets E_1 and E_2 respectively, and still have a function $f \in l_1(\mathbb{Z})$ which is a solution of the configuration equations in that instance.

b) In the examples that we have examined, where the configuration equations have no normalized solution, there is also a paradoxical decomposition of the group which can be described in terms of the configuration set. There is even a paradoxical decomposition of the form $G = 2G$. See Example 2.11 for a basic instance of this. These examples suggest, as is conjectured in Example 2.11, that it may be possible using just the configuration equations themselves to prove Tarski's Theorem: G is amenable if and only if there are no paradoxical decompositions in G. See Wagon [8] for discussion of paradoxical decompositions in general and a proof of Tarski's Theorem.

c) We think that it is likely that combinatorial properties of configurations can be used to characterize various kinds of behavior of groups (like the group being abelian or the group containing a non-abelian free subgroup). This possibility needs further investigation especially since it may help understand more clearly the way in which amenability is related to other properties of groups.

Example 2.11 Let $\mathbb{F}_2 = \langle g_1, g_2 \rangle$ be the free group with generators g_1 and g_2. Then each element $w \neq e$ of $\mathbb{F}_2$ may be written as a reduced word $w = w_1 w_2 \cdots w_l$ where each w_i is equal to g_1, g_1^{-1}, g_2 or g_2^{-1}. Define subsets E_i, $i = 1, 2, 3$ of $\mathbb{F}_2$ by $E_1 = \{w \in \mathbb{F}_2 : w_1 = g_1\}$, $E_2 = \{w \in \mathbb{F}_2 : w_1 = g_2\}$ and $E_3 = \{w \in \mathbb{F}_2 : w = e,\ w_1 = g_1^{-1} \text{ or } w_1 = g_2^{-1}\}$. Then $\mathcal{E} = \{E_1, E_2, E_3\}$ is a partition of $\mathbb{F}_2$. There are seven configurations corresponding to the partition, namely: $C^{(1)} = (1, 1, 2)$, $C^{(2)} = (2, 1, 2)$, $C^{(3)} = (3, 1, 2)$, $C^{(4)} = (3, 3, 2)$, $C^{(5)} = (3, 2, 2)$, $C^{(6)} = (3, 1, 3)$ and $C^{(7)} = (3, 1, 1)$. These configurations are shown in Figure 3.

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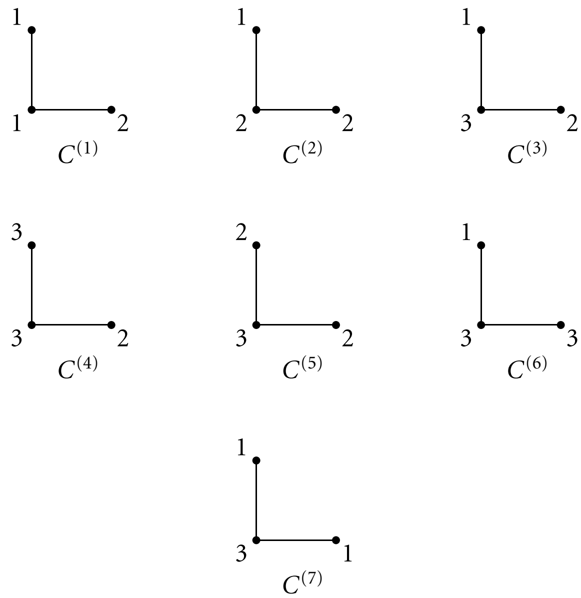

Figure 3: The configuration set corresponding to the partition $\mathcal{E}$.

Letting c_i the value in the configuration equations which is corresponding to the configuration $C^{(i)}$ in this example gives these configuration equations:

From g_1:	From g_2:
$c_1 = c_1 + c_2 + c_3 + c_6 + c_7$	$c_1 = c_7$
$c_2 = c_5$	$c_2 = c_1 + c_2 + c_3 + c_4 + c_5$
$c_3 + c_4 + c_5 + c_6 + c_7 = c_4$	$c_3 + c_4 + c_5 + c_6 + c_7 = c_6$

These equations have no positive solution because the first equation in the first column implies that $c_2+c_3+c_6+c_7 = 0$ while the second equation in the second column implies that $c_1+c_3+c_4+c_5 = 0$ and adding implies that $c_1+c_2+2c_3+c_4+c_5+c_6+c_7 = 0$. Thus we have yet another proof that $\mathbb{F}_2$ is not amenable.

There are many nonzero solutions however because we have only six equations in seven unknowns. In fact, the equations on the last line are dependent on the others, as is always the case with the configuration equations, and so there are only four independent equations.

The configurations determine a refinement, $\mathcal{E}'$ of $\mathcal{E}$. Define, for each $3 \leq i \leq 7$, $E_{3i} = \{w \in \mathbb{F}_2 : w \text{ is the basepoint in } C^{(i)}\}$. Then $\{E_{33}, \ldots, E_{37}\}$ is a partition of E_3 and so $\mathcal{E}' = \{E_1, E_2, E_{33}, \ldots, E_{37}\}$ refines $\mathcal{E}$. Now when the two equations used above to show that the configuration equations have no positive solution are written in terms of sets, they become $g_1E_1 = E_1 \cup E_2 \cup E_{33} \cup E_{36} \cup E_{37}$ and $g_2E_2 = E_1 \cup E_2 \cup E_{33} \cup$

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$E_{34} \cup E_{35}$. Hence we have a paradoxical decomposition for $\mathbb{F}_2$: $\mathbb{F}_2 = g_1E_1 \cup E_{34} \cup E_{35}$ and $\mathbb{F}_2 = g_2E_2 \cup E_{36} \cup E_{37}$. We do not use E_{33} on the right hand side of the equations so this is not an exact paradoxical decomposition of the form $\mathbb{F}_2 = 2\mathbb{F}_2$.

The configurations argument used here to obtain a paradoxical decomposition applies to any group having the configuration set in Figure 3, not just the free group. It may be that the free group is the only one having this particular configuration set, see [1] Proposition 20, but in general there are many groups having a given configuration set. Should the configuration equations for such a set of configurations have no non-zero positive solution then none of these groups is amenable. We conjecture that, given a configuration set having no positive solutions, there is always a paradoxical decomposition associated with it as in the previous paragraph. The configuration set would then contain enough information to show that there is a paradoxical decomposition.

It is much easier to check whether the configuration equations of a given partition of a group have a non-zero positive solution than it is to find a paradoxical decomposition and so it seems that the configurations method may be a convenient way to determine whether a group is amenable. However, it may in practice turn out to be just as difficult to determine just what configurations occur in a given partition, and hence what the configuration equations are, as it is to decide whether the partition gives a paradoxical decomposition.

In the case of this configuration set, there is a paradoxical decomposition of the exact form $G = 2G$ but to exhibit it we need to consider the g_2-cosets in G. The configuration sets imply that there are at most four types of g_2-coset and these are shown in Figure 4. Observe from Figure 3 that if w and g_2w both belong to E_3, then w is in E_{36}. The vertices in E_{36} are ringed in Figure 4. The cosets of type III come in three subtypes, depending on whether the single vertex in E_3 but not in E_{36} belongs to E_{33}, E_{34} or E_{35}; call these subtypes III(3), III(4) and III(5). Now partition E_{36} in to sets $A = \{w \in E_{36} : w \text{ lies on a coset of type III(3)}\}$ and $B = E_{36} \setminus A$. Define a new partition $\mathcal{F} = \{F_1, F_2, F_3, F_4\}$ of G by $F_1 = E_1$, $F_2 = E_{34} \cup E_{35}$, $F_3 = E_2 \cup E_{33} \cup A$ and $F_4 = E_{37} \cup B$. Then $G = F_1 \cup g_1F_2$ and $G = F_3 \cup g_2F_4$. This construction of a partition associated with $\mathcal{E}$ is not finite in the sense that $\mathcal{F}$ is not formed by intersecting finitely many translates of the sets in $\mathcal{E}$. The definition of the sets A and B seems to require an infinite number of steps from the original data and is thus the analogue of the Schröder-Bernstein argument from Tarski's proof.

3 Configurations in Non-discrete Groups

The same ideas that were used for discrete groups can be used in the case of general locally compact Hausdorff groups with suitable modifications in the construction. A configuration $(C_0, C_1, \dots, C_n)$ is assumed to be just as before except that the finite choice of colors is determined by a λ-measurable partition $(E_1, \dots, E_m)$ of G. For $f \in L_1(G)$, we let $f_C = \int_{x_0(C)} f \, d\lambda$. There again are configuration equations which play the same role as for discrete groups. Assume that the λ-measurable partition $(E_1, \dots, E_m)$ and the elements $(g_1, \dots, g_n)$ have been fixed.

Proposition 3.1 *Given $f \in L_1(G)$, we have for all $i = 1, \dots, m$ and $j = 1, \dots, n$,*

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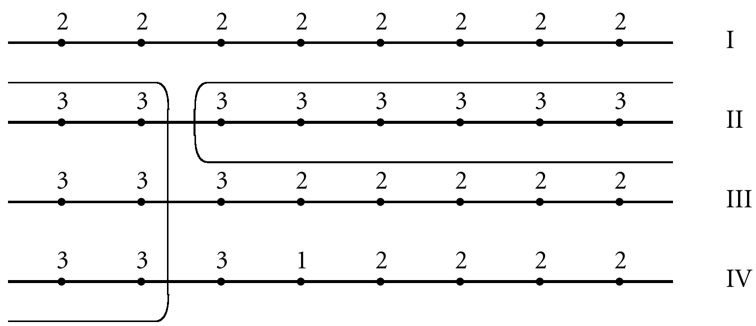

Figure 4: The four types of g_2-cosets. Ringed vertices belong to E_{36}.

$$\langle f - {}_{g_j}f, 1_{E_i}\rangle = 0, \tag{3}$$

if and only if for all $i = 1, \ldots, m$ *and* $j = 1 \ldots, n$

$$\sum\{f_C : x_0(C) \subset E_i\} = \sum\{f_C : x_j(C) \subset E_i\}. \tag{4}$$

Proof We see easily that $\langle f, 1_{E_i}\rangle = \int_{E_i} f\, d\lambda = \sum\{f_C : x_0(C) \subset E_i\}$. But also as in Proposition 2.1, $\langle {}_{g_j}f, 1_{E_i}\rangle = \int_{E_i} f(g_j^{-1}x)\, d\lambda(x) = \sum\{f_C : x_j(C) \subset E_i\}$. ■

This leads to a generalization of Proposition 2.4. The only extra point worth observing first is that a configuration needs to be not only realized in the group in order to be relevant now, but it should also be *essential i.e.* it should be that $\lambda\big(x_0(C)\big) > 0$. Indeed, it is clear that in the configuration equations (4) as derived in Proposition 3.1, the terms f_C are all zero whenever the configuration is not essential. We therefore to avoid trivialities assume that the configuration equations in the case of a general group exclude any term f_C where C is not essential; this revised set of equations is called the *essential configuration equations*.

Proposition 3.2 *There is a normalized solution of every possible instance of the essential configuration equations if and only if G is amenable.*

Proof Suppose that G is amenable. Let (f_α) be a net in $P(G)$ converging weak* to invariance. For a fixed choice of $\{E_i\}$ and $\{g_j\}$, the values $\big((f_\alpha)_C\big)$ are normalized. Because of the weak* invariance, the values $\big((f_\alpha)_C\big)$ give approximate solutions of the configuration equations, with the error in the approximation tending to 0 as $\alpha \to \infty$. But the configuration equations form a finite linear system. Therefore, taking the limit as $\alpha \to \infty$, we get a normalized solution of the configuration equations. On the other hand, suppose there is a normalized solution for every instance of the essential configuration equations. Given fixed $\{E_i\}$ and $\{g_j\}$ as above, let (f_C) be such a solution. Because we are assuming that every C in the configuration equations

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is essential, we can choose a compact set $x_0^C \subset x_0(C)$ with $\lambda(x_0^C) > 0$ for each configuration C. Let $f \in P(G)$ be the function such that $f = f_C/\lambda(x_0^C)$ on x_0^C for every C, and $f = 0$ otherwise. By Proposition 3.1, for every i and j, we have $\langle f -_{g_j} f, 1_{E_i}\rangle = 0$. Since the choice of $\{E_i\}$ and $\{g_j\}$ is arbitrary, this shows that there is a net in $P(G)$ converging weak* to invariance, and hence that G is amenable. ■

Now one can see using configurations that generally weak* convergence to invariance does not imply strong convergence to invariance.

Theorem 3.3 *If G is a non-discrete amenable locally compact Hausdorff group, then there exists a net (f_α) in $P(G)$ converging* weak* *to invariance, such that for every $x \in G$, $x \neq e$, eventually $\|f_\alpha - {}_x f_\alpha\|_1 = 2$.*

Proof Fix $\{E_i\}$ and $\{g_j\}$ as above. Let $f \in P(G)$ be defined as in the proof of Proposition 3.2, with $f = f_C/\lambda(x_0^C)$ on x_0^C for each of the C such that $\lambda\big(x_0(C)\big) > 0$, and $f = 0$ otherwise. There is nothing restricting us in this choice from also keeping x_0^C to be within a very small neighborhood of some particular point in $x_0(C)$. Since G is non-discrete, this fact shows that given any finite set $K \subset G \setminus \{e\}$, the choices of x_0^C can be made inductively so that for all $k \in K$, we have f and ${}_k f$ disjointly supported and so $\|f - {}_k f\|_1 = 2$. Now letting the choices of $\{E_i\}$, $\{g_j\}$ and K vary in the usual directed fashion, the functions constructed here form a net (f_α) as needed to finish the proof. ■

Remark 3.4 It is possible that another argument involving extreme points can be used for constructing nets in $P(G)$ which are weak* converging to invariance, but not converging strongly to invariance. C. Chou has suggested that it is possible that any net (f_α) in $P(G)$ which is strongly converging to invariance cannot be converging in the weak* topology to an extreme point of the set of left-invariant means. If so, by the Krein-Milman Theorem, one could choose a left-invariant mean ϕ that is an extreme point of the set of left-invariant means, and take a net (f_α) in $P(G)$ converging in the weak* topology to ϕ, and thus obtain a net which is weak* converging to invariance, but is not strongly converging to invariance. This argument is of course not going to be constructive.

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