Approximation by means of Convergent Fractions.

By A. C. Dixon.

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This is a note on the theory of continued fractions,* in which the chief feature is the use made of the successive remainders or divisors which occur in the reduction of any given ratio to a continued fraction.

The treatment of the Pellian equation also differs from that which is generally given.

1. Let $A = A_0$, $B = A_1$ be two quantities to whose ratio we wish to approximate, and suppose $A > B$, both being positive.

Let a_2B be the greatest integral multiple of B contained in A, and let

$$A_2 = A - a_2B = A_0 - a_2A_1;$$

let a_3A_2 be the greatest integral multiple of A_2 contained in A_1 and

$$A_3 = A_1 - a_3A_2, \text{ and so on,}$$

a_mA_{m-1} being the greatest integral multiple of A_{m-1} contained in A_{m-2}, and $A_m = A_{m-2} - a_mA_{m-1}$.

By successive substitution

$$A_m = (-1)^m\{q_mA - p_mB\}$$

where p_m, q_m are formed by the law

$$p_m = a_mp_{m-1} + p_{m-2},$$

$$q_m = a_mq_{m-1} + q_{m-2},$$

and

$$p_0 = 0,\ q_0 = 1,\ p_1 = 1,\ q_1 = 0,$$

so that

$$p_1, p_2, p_3, \dots$$

$$q_2, q_3, q_4, \dots$$

are two increasing series of positive integers, since $a_2, a_3 \dots$ are positive integers.

* For the theory, see, for instance, Chrystal's *Algebra*, chapters 32, 33.

Also A_0, A_1, A_2,... form a decreasing series of positive quantities.

Thus A_m/q_mB diminishes continually as m increases, that is, the difference between A/B and p_m/q_m diminishes continually as m increases and it is clearly an excess and defect alternately.

Also $p_m q_{m-1} - p_{m-1} q = -(p_{m-1}q_{m-2} - p_{m-2}q_{m-1}) = (-1)^{m-1}$, from which it follows as usual that p_m/q_m is nearer to A/B than any other fraction whose denominator $\not> q_m$.

Again, if we approximate by the same method to p_m/q_m, the quotients a_2, a_3, ...a_m are the same as for A/B: for let

$$A_0' = p_m,\ A_1' = q_m,$$
$$A_r' = (-1)^r\{q_r A' - p_r B'\}.$$

Then when $r < m$, $\frac{p_r}{q_r} - \frac{p_m}{q_m}$ is of the same sign as $\frac{p_r}{q_r} - \frac{A}{B}$, and therefore A_r' is always positive. Moreover

$$A'_{r-2} = a_r A'_{r-1} + A_r'$$
$$> A'_{r-1},$$

so that A_0', A_1', A_2'... form a decreasing series of positive quantities, and $a_r A'_{r-1}$ is the greatest multiple of A'_{r-1} contained in A'_{r-2}.

Thus the convergents to p_m/q_m are, so far, the same as to A/B,

that is, $$p_m/q_m = a_2 + \frac{1}{a_3+}\,\frac{1}{a_4+}\dots+\frac{1}{a_m}.$$

Also $$p_m/p_{m-1} = a_m + p_{m-2}/p_{m-1} = a_m + \frac{1}{a_{m-1}+}\,\frac{1}{a_{m-2}+}\dots+\frac{1}{a_2}$$

and $$q_m/q_{m-1} = a_m + \frac{1}{a_{m-1}}+\dots+\frac{1}{a_3}$$ similarly.

2. If the ratio A/B is a simple quadratic surd, say

$$A_0 = \sqrt{N},\ A_1 = 1$$

where N is a positive integer, then a_{m+1} is the integer next below

$$(q_{m-1}\sqrt{N} - p_{m-1}) \div (p_m - q_m\sqrt{N})$$

which $$= (Nq_m q_{m-1} - p_m p_{m-1} - (-1)^m\sqrt{N}) \div (p_m^2 - q_m^2 N)$$

or $$(q_{m-1}^2 N - p_{m-1}^2) \div (Nq_m q_{m-1} - p_m p_{m-1} + (-1)^m\sqrt{N}).$$

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This fraction is positive and >1, and so is

$$(p_m + q_m \sqrt{N}) \div (q_{m-1}\sqrt{N} + p_{m-1}),$$

and so therefore is the product of the two, namely,

$$(Nq_m q_{m-1} - p_m p_{m-1} - (-1)^m \sqrt{N}) \div (-Nq_m q_{m-1} + p_m p_{m-1} - (-1)^m \sqrt{N}).$$

Hence $(-1)^m\{p_m p_{m-1} - Nq_m q_{m-1}\}$ is positive but $<\sqrt{N}$, and it follows that $(-1)^m(q_m{}^2 - Np_m{}^2)$ is positive but $<2\sqrt{N}$.

Thus there must come a stage when the values of these two integers are repeated, that is, when

$$\frac{q_{m-1}\sqrt{N} - p_{m-1}}{p_m - q_m\sqrt{N}} = \frac{q_{n-1}\sqrt{N} - p_{n-1}}{p_n - q_n\sqrt{N}} (n > m)$$

and the series $a_{n+1}, a_{n+2} \ldots$ is the same as

$$a_{m+1}, a_{m+2}, \ldots.$$

Since the rational and irrational parts must be equal separately in the last equation, we may reverse the radical sign and thus

$$\frac{p_m + q_m\sqrt{N}}{p_{m-1} + q_{m-1}\sqrt{N}} = \frac{p_n + q_n\sqrt{N}}{p_{n-1} + q_{n-1}\sqrt{N}}$$

or

$$a_m + \frac{p_{m-2} + q_{m-2}\sqrt{N}}{p_{m-1} + q_{m-1}\sqrt{N}} = a_n + \frac{p_{n-2} + q_{n-2}\sqrt{N}}{p_{n-1} + q_{n-1}\sqrt{N}}.$$

In this, when $m > 2$, the second term on each side is positive and <1. Thus $a_m = a_n$, and the recurrence begins a step further back, unless $m = 2$; that is, the recurrence begins with the fractional part.

If $m = 2$, $\dfrac{p_2 + q_2\sqrt{N}}{p_1 + q_1\sqrt{N}} = a_2 + \sqrt{N}$, the integral part of which is $2a_2$, and thus $a_n = 2a_2$, a well-known result. Also

$$(q_{n-1}^2 N - p_{n-1}^2)(-1)^n = (q_{m-1}^2 N - p_{m-1}^2)(-1)^m$$
$$= -1 \text{ if } m = 2,$$

and thus

$$p_{n-1}^2 - Nq_{n-1}^2 = (-1)^n,$$

affording a solution of the Pellian equation

$$x^2 - Ny^2 = \pm 1.$$

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3. To prove that the Pellian equation has no other solutions than those thus given, let x, y be a pair of positive integers such that

$$x^2 - Ny^2 = \pm 1.$$

Since this may be written

$$x \cdot x - Ny \cdot y = \pm 1$$

it follows from the known theory of the equation

$$ax - by = \pm 1$$

that x/y is the last convergent when Ny/x is reduced to a continued fraction (in one of the two possible ways).

If we write this fraction $a_2 + \frac{1}{a_3 +} \dots \frac{1}{+ a_{m+1}}$ we have $Ny = p_{m+1}$, $x = q_{m+1} = p_m$, $y = q_m$.

Now the quotients in the continued fractions for p_{m+1}/p_m and p_{m+1}/q_{m+1} are the same in reverse order, and therefore in this case $a_2 = a_{m+1}$, $a_3 = a_m$, $a_4 = a_{m-1}$, ..., since $p_m = q_{m+1}$.

Also, if we add $\sqrt{N}$ to the last quotient the fraction takes the value

$$\frac{(a_{m+1} + \sqrt{N})p_m + p_{m-1}}{(a_{m+1} + \sqrt{N})q_m + q_{m-1}}$$

$$\text{or } \frac{p_{m+1} + \sqrt{N}p_m}{q_{m+1} + \sqrt{N}q_m} \text{ or } \frac{Ny + x\sqrt{N}}{x + y\sqrt{N}} \text{ or } \sqrt{N}.$$

Thus the quotients in the infinite continued fraction representing $\sqrt{N}$ are

$$a_2, a_3, \dots, a_m, a_{m+1} + a_2, a_3, \dots, a_m, a_{m+1} + a_2, \dots$$

which was to be proved, and it has further been shewn that the quotients in any period are the same when read in the reverse order.

4. Again, if h is a positive integer, and

$$x^2 - Ny^2 = \epsilon h, \text{ where } \epsilon = \pm 1, \text{ and } x \text{ is prime to } y,$$

take p, q positive integers, so that $qx - py = \epsilon$, and $p < x$, $q < y$, that is, p/q is the last convergent to x/y.

Then $$x(x - hq) = y(Ny - hp),$$
$$x - hq = ay, \quad Ny - hp = ax$$
and $$x = ay + hq, \quad Ny = ax + hp, \quad a \text{ being integral.}$$

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Thus $$\frac{Ny}{x} = a_2 + \frac{1}{a_3} + \ldots \frac{1}{a_m} + \frac{h}{a}$$

where $a_2, a_3 \ldots, a_m$ are positive integers, and $\frac{p}{q}, \frac{x}{y}$ are the two last convergents.

It follows that

$$a_2 + \frac{1}{a_3} + \ldots \frac{1}{a_m} + \frac{h}{a + \sqrt{N}} = \frac{(a + \sqrt{N})x + hp}{(a + \sqrt{N})y + hq} = \sqrt{N},$$

and $\sqrt{N}$ is the value of an infinite continued fraction

$$a_2 + \frac{1}{a_3} + \frac{1}{a_4} + \ldots \frac{1}{a_m} + \frac{h}{a + a_2} + \frac{1}{a_3} + \ldots$$
$$\qquad * \qquad\qquad\qquad *$$

recurring from * to *.

Also $$\frac{Ny}{x} = a + \frac{hp}{x} = a + \frac{h}{a_m} + \frac{1}{a_{m-1}} + \ldots + \frac{1}{a_2}$$

$$\frac{x}{y} = a + \frac{h}{a_m} + \frac{1}{a_{m-1}} + \ldots + \frac{1}{a_3}$$

and $$a + \frac{h}{a_m} + \frac{1}{a_{m-1}} + \ldots + \frac{1}{a_2 + \sqrt{N}} = \frac{Ny + x\sqrt{N}}{x + y\sqrt{N}} = \sqrt{N}\dagger$$

so that $$\sqrt{N} = a + \frac{h}{a_m} + \frac{1}{a_{m-1}} + \ldots + \frac{1}{a_2 + a} + \frac{h}{a_m} + \ldots$$
$$\qquad * \qquad\qquad\qquad *$$

recurring from * to *.

In the above work if $h^2 < N$

$$x^2 = Ny^2 + \epsilon h > h^2y^2 - h > (hy - 1)^2$$

so that $x \geqq hy$ and $x - hq$ is positive.

Thus a is positive and $\frac{h}{a + \sqrt{N}}$ is positive but < 1.

Thus x/y is one of the convergents to $\sqrt{N}$, a known theorem.

† Hence $\sqrt{N} - a$ is positive, that is, a must be $< \sqrt{N}$ or else negative: in the former case x/y is a convergent, ordinary or intermediate, to $\sqrt{N}$.

5. To extend the above proof of recurrence (§2) to the case of a positive quantity (>1) of the form $\frac{a+b\sqrt{N}}{c+d\sqrt{N}}$ where a, b, c, d are rational, reduce the fraction to the form $\frac{-a_0+\sqrt{N}}{a_1}$ where a_0, a_1 are integral and positive or negative.

Then take $\epsilon A_0 = -a_0 + \sqrt{N}$, $\epsilon A_1 = a_1$, ϵ being ± 1 and of the same sign as a_1,

$$\epsilon A_m = (-1)^m \{\beta_m \sqrt{N} - a_m\},$$

so that $\beta_0 = 1$, $\beta_1 = 0$ and the law of formation is again

$$a_m = a_m a_{m-1} + a_{m-2},$$

$$\beta_m = a_m \beta_{m-1} + \beta_{m-2}.$$

$\beta_2, \beta_3 \ldots$ are all positive since $\beta_0, \beta_1, a_2, a_3 \ldots$ are so, and since the sequence $A_0, A_1, \ldots A_m \ldots$ diminishes without limit,* a_m must be always positive for values of m exceeding a certain number. Then the reasoning of §2 applies with a, β in the place of p, q, and thus the fraction A_0/A_1 yields a recurring continued fraction, the recurrence beginning where negative values of a_m stop.

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* Since $A_{m-2} = a_m A_{m-1} + A_m$ and $A_{m-1} > A_m$ it follows that $A_m < \frac{1}{2} A_{m-2}$.