J. Austral. Math. Soc. 19 (Series A) (1975), 309–311.

AN EXAMPLE FOR HOMOTOPY COMMUTATIVITY OF H-SPACES

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(Received 7 September 1973)

Communicated by G. Versik

In this note we give an example of a homotopy commutative H-space X that is not dominated by any homotopy associative, homotopy commutative H-space. In particular, X is not dominated by $\Omega^2 S^2 X$.

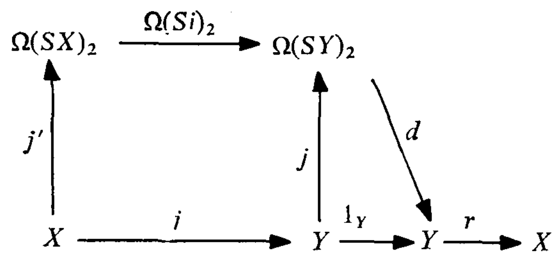
We recall that a space X is *dominated* by a space Y provided that there exists maps $i: X \to Y$ and $r: Y \to X$ such that $r \circ i$ is homotopic to the identity map of X (written $r \circ i \sim 1_X$). We let Ω and S denote the loop space and suspension functors, respectively.

If P denotes a homotopy property, then we call a functor T a *universal example* for P provided that a space X possesses property P if and only if X is dominated by $T(X)$. For example, X is an H-space if and only if it is dominated by ΩSX; hence ΩS is a universal example for the property of being an H-space. For another example, let X be an H-space. The *projective plane* of $X, P_2(X)$, is defined to be the mapping cone of the Hopf construction $X * X \to SX$. Then X is homotopy associative if and only if it is dominated by $\Omega P_2(X)$, since in that case the Hopf construction extends to a fibration over $P_2(X)$, Stasheff (1963). It is conjectured in Stasheff (1970) that an H-space X is homotopy commutative if and only if it is dominated by $\Omega^2 S^2 X$. We shall show that this conjecture is false by showing that there is *no* functor to the category of homotopy commutative, homotopy associative H-spaces that serves as universal example for homotopy commutativity. Our example is based on the following dfact.

THEOREM. *Suppose that a space X is dominated by a homotopy associative, homotopy commutative H-space Y. Then X is dominated by $\Omega(SX)_2$.* (Here $(SX)_2$ denotes $SX \times SX$ modulo the relation $\sim$ given by $(*, w) \sim (w, *)$, $w \in SX$, $*$ the basepoint of SX.)

Supported in part by a National Science Foundation Grant.

Proof. If Y is as hypothesized, then it was shown in Williams (1971) that Y is dominated by $\Omega(SY)_2$. Let $i: X \to Y$, $r: Y \to X$ be such that $r \circ i \sim 1_X$ and j: $Y \to \Omega(SY)_2$, $d: \Omega(SY)_2 \to Y$ be such that $d \circ j \sim 1_Y$. Let $j': X \to \Omega(SX)_2$ denote the inclusion. Then from the diagram



We see that

$$(r \circ d \circ \Omega(Si)_2) \circ j' \sim r \circ d \circ j \circ i \sim r \circ i \sim 1_X.$$

Remark 1. It was shown in Williams (1971) that $\Omega(SY)_2$ *does* serve as uni· versal example for homotopy commutativity for homotopy associative H-spaces (This fact also follows from Husseini (1963) and from Stasheff (1961).)

Remark 2. For a space X to be a homotopy commutative H-space, it i: sufficient that it be dominated by $\Omega(SX)_2$, Williams (1971).

Example. Our example is based on ideas in Adams (1961). Let $m: S^3 \times S^3 \to S^{\cdot}$ be an H-space multiplication whose separation element for homotopy commuta tivity is of order 2 or 4 in $\pi_6(S^3)$, James (1955). (Such a multiplication is no homotopy associative.) Let X be the space obtained by attaching cells to S^3 t(kill the 2-component of $\pi_k(X)$ in dimensions $\geqq 6$. The multiplication on S^3 ex tends to a homotopy commutative multiplication on X, since the obstruction: lie in vanishing cohomology groups. Observe that $\pi_9(X) = \pi_9(S^3) \approx Z_3$. Further the 3-component of $\pi_9(\Omega^2 S^2 X)$ is isomorphic to the 3-component of $\pi_9(\Omega^2 S^5$ which is trivial, since $\pi_9(\Omega^2 S^5) = \pi_{11}(S^5) = Z_2$. Since $\Omega^2 S^2 X = \Omega(\Omega S(SX)$ and $\Omega S(SX)$ is obtained from $(SX)_2$ by attaching cells in dimensions greater thaı eleven, James (1955), $\pi_9(\Omega SX)_2) \approx \pi_9(\Omega^2 S^2 X)$. Since Z_3 cannot be a subgrou] of a trivial group, X cannot be dominated by $\Omega(SX)_2$.

In conclusion, we remark that X provides an example similar to that o: Adams (1961) of a homotopy vommutative, non-homotopy associative H-space

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https://doi.org/10.1017/S1446788700031517 Published online by Cambridge University Press

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