108.44 Expressing the area of a circle in terms of line segments of perpendicular chords

Claim: Let two perpendicular chords of a circle be cut into segments of length a, b, c and d as shown.

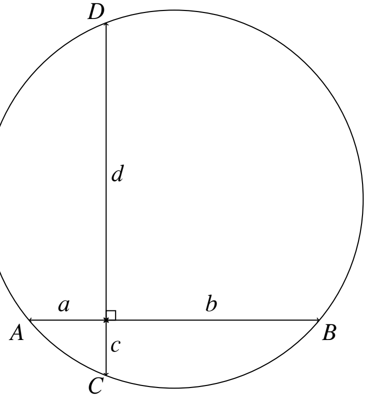

FIGURE 1

Then the area of the circle is $\frac{\pi}{4}\left(a^2 + b^2 + c^2 + d^2\right)$.

Proof: Construct a chord AX parallel to CD.

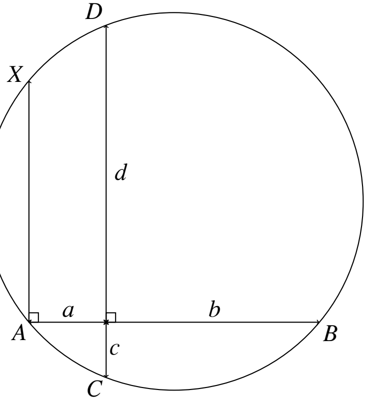

FIGURE 2

The radius of the circle perpendicular to both AX and CD bisects them.

https://doi.org/10.1017/mag.2024.128 Published online by Cambridge University Press

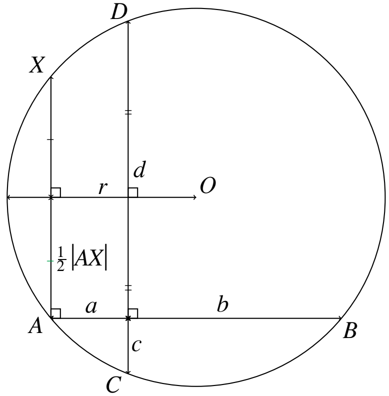

FIGURE 3

Therefore, $\frac{1}{2}|AX| + c = d - \frac{1}{2}|AX|$ and hence $|AX| = d - c$.

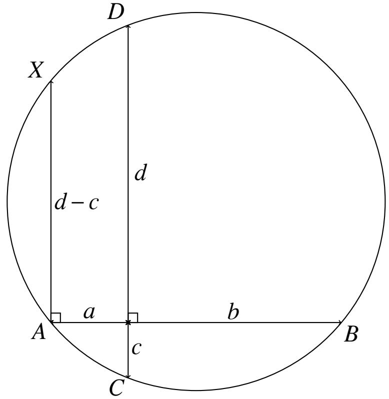

FIGURE 4

By the converse of Thales' theorem, BX is a diameter of the circle.

By Pythagoras' theorem,

$$|BX| = \sqrt{(a + b)^2 + (d - c)^2} = \sqrt{a^2 + 2ab + b^2 + d^2 - 2cd + c^2}.$$

By the intersecting chords theorem, $ab = cd$ implying

$$|BX| = \sqrt{a^2 + b^2 + c^2 + d^2}.$$

Therefore, the area of the circle is

$$\pi\left(\frac{\sqrt{a^2 + b^2 + c^2 + d^2}}{2}\right)^2 = \frac{\pi}{4}\left(a^2 + b^2 + c^2 + d^2\right).$$

Acknowledgement

The author would like to thank the anonymous referee for their useful comments in improving this proof.

10.1017/mag.2024.128 © The Authors, 2024
Published by Cambridge University Press
on behalf of The Mathematical Association

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108.45 The golden section from three congruent semicircles

Let R be a positive real number and let A_1B_1 be a line segment with length $2R$. Two rays ℓ, ℓ' with origins at A_1, B_1, respectively, are perpendicular to A_1B_1. We show how to obtain the following configuration where $A_2B_2 = A_3B_3 = 2R$, points A_3, B_3 are on ℓ, B_2 is on ℓ', and the semicircles $\omega_1, \omega_2, \omega_3$ with respective diameters A_1B_1, A_2B_2, A_3B_3 satisfy:

- A_2B_2 is tangent to ω_1 at A_2
- ω_2 is tangent to ω_3.

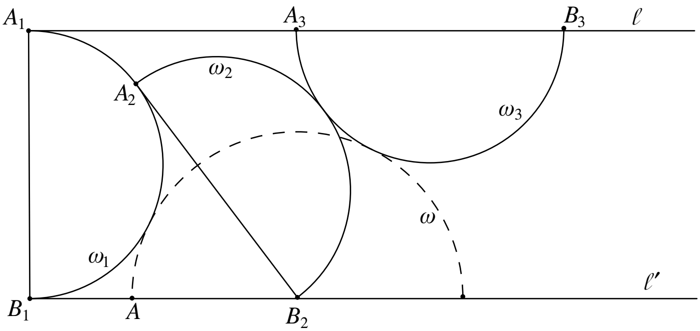

FIGURE 1

As a by-product, the construction will provide the following proposition:

Proposition 1: The semicircle ω with centre B_2 externally tangent to ω_1 is also tangent to ω_3. In addition, if it intersects the line segment B_1B_2 in A, then $\frac{AB_2}{AB_1} = \phi$, the golden ratio $\left(\phi = \frac{1}{2}(\sqrt{5} + 1)\right)$.

Constructing Figure 1

The construction of ω_2 is easy: since the tangents to ω_1 from B_2 are of equal length, we must have $B_2B_1 = B_2A_2 = 2R$. Thus, we first locate B_2 on ℓ such that $B_1B_2 = 2R$, then draw the tangent B_2A_2 to ω_1 and ω_2 follows.