1

Motivations from Equivariant Topology

Introduction

In this chapter we describe context from equivariant topology and the theory of stable model categories that motivates our further study of multicategorically enriched categories, enriched diagrams, enriched Mackey functors, and change of enrichment.

Convention 1.0.1. Assume throughout this chapter that G is a finite group. See Remarks 1.1.1 and 1.2.8 for further comments on this convention. $\diamond$

Connection with Main Content

The purpose of this chapter is to indicate the role that categorical diagrams – particularly Mackey functors – play in equivariant homotopy theory. None of the mathematics in this present work depends on the content of this chapter, but the attendant applications are a key motivation.

For example, the Burnside 2-category $G\mathcal{E}$ (Definition 1.3.5) is enriched in the multicategory of permutative categories, $\mathsf{PermCat}^{\mathsf{su}}$ (Section 2.4). We give a treatment of

- categories enriched in closed multicategories, in Chapter 7,
- change of enrichment, in Chapter 8,
- the closed multicategory structure of $\mathsf{PermCat}^{\mathsf{su}}$ in Chapter 9, and
- self-enrichment for closed multicategories in Chapter 10.

In the Guillou–May Theorem 1.3.9, the domain of spectral Mackey functors, $(G\mathcal{E}_{\mathbb{K}})^{\mathsf{op}}$, is given by a change of enrichment $(-)_{\mathbb{K}}$ and requires a distinction between enriched diagrams, with domain $G\mathcal{E}_{\mathbb{K}}$, and enriched Mackey functors, with domain $(G\mathcal{E}_{\mathbb{K}})^{\mathsf{op}}$. We describe the relevant subtleties further in Remarks 1.3.7 and 11.5.5.

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Similarly, but in a more abstract context, the spectral presheaves in the Schwede–Shipley Characterization Theorem 1.4.3 have domain $\mathcal{E}(P)^{\mathsf{op}}$. The input $\mathcal{E}(P)$ is the spectral endomorphism category of a set of compact generators P for a simplicial, cofibrantly generated, proper, and stable model category M.

We give a general treatment of enriched diagrams and enriched Mackey functors, including interactions with change of enrichment, in Chapter 11. We develop techniques and applications for the corresponding homotopy theory in Chapters 12 and 13.

Chapter Summary

A substantive treatment of equivariant homotopy theory is well beyond our current scope. At the end of this introduction we give a list of key references. The remaining content in this chapter is restricted to those definitions and results that provide motivating context for our work in what follows.

Section 1.1 concerns equivariant spaces.

- The orbit category of G is denoted $\mathcal{O}_G$; see Definition 1.1.3.
- Elmendorf's Theorem 1.1.9 shows that the homotopy theory of G-spaces is equivalent to that of topological presheaves on $\mathcal{O}_G$.

Section 1.2 concerns Abelian Mackey functors.

- The Burnside ring of G is denoted $G\mathcal{A}$. Its elements are isomorphism classes of finite G-sets with disjoint union and Cartesian product; see Definition 1.2.5.
- The Burnside category of G is denoted $G\mathcal{B}$. Its morphisms are isomorphism classes of spans between finite G-sets. Disjoint union provides an enrichment over Abelian groups; see Definition 1.2.5.
- Abelian Mackey functors are enriched presheaves on the Burnside category; see Definition 1.2.9.

Sections 1.3 and 1.4 concern spectral Mackey functors.

- The Burnside 2-category of G is denoted $G\mathcal{E}$. Its 1- and 2-cells are categories of spans between finite G-sets. Disjoint union, together with a choice of pullbacks and whiskering by a strict unit, provides an enrichment over permutative categories; see Definition 1.3.5.
- Spectral Mackey functors are enriched presheaves on a spectral enrichment of the Burnside 2-category; see Definition 1.3.8.
- The Guillou–May Theorem 1.3.9 shows that the homotopy theory of G-spectra is equivalent to that of spectral Mackey functors.

- The Schwede–Shipley Characterization Theorem 1.4.3 shows that the homotopy theory of a simplicial, cofibrantly generated, proper, and stable model category is equivalent to that of spectral presheaves on an endomorphism category of generating objects.

References

Main references for equivariant homotopy theory include, at least, the following. We include further specialized references at relevant points in the discussion.

- The text by tom Dieck [tD79] lays the foundations for equivariant homotopy theory of spaces, including equivariant (co)homology theories known as Bredon cohomology.
- The monograph [LMS86], by Lewis, May, and Steinberger, gives the foundational treatment of equivariant stable homotopy theory, particularly equivariant spectra.
- The CBMS Alaska conference proceedings [May96] refines and significantly extends the preceding theory, including more development of the closed monoidal structure for equivariant spectra.
- The recent textbook account by Hill, Hopkins, and Ravenel [HHR21] provides a more modern perspective, with thorough treatment of norm operations and the slice filtration that are essential in their solution of the Kervaire invariant problem [HHR16].

1.1 Equivariant Spaces and Presheaves on the Orbit Category

Recall Convention 1.0.1 that G is assumed to be a finite group.

Remark 1.1.1. Many, but not all, of the following concepts extend to more general cases of interest, such as G being a compact Lie group or a general topological group. The most important exception is our definition of the Burnside category in Definitions 1.2.5 and 1.3.5, which depends on finiteness of G. See Remark 1.2.8 for further comments and references regarding that point. ◇

Definition 1.1.2. Suppose C and M are categories, with C small. A *diagram of shape* C, or C*-diagram* in M is a functor

$$\mathsf{C} \longrightarrow \mathsf{M}.$$

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A *presheaf on* C or C-*presheaf* in M is a diagram of shape C^{op} in M, where C^{op} is the opposite category of C. That is, a presheaf on C is a functor

$$\mathsf{C}^{\mathsf{op}} \longrightarrow \mathsf{M}.$$

The phrase "in M" is often omitted when M is clear from context. Morphisms between diagrams and presheaves are natural transformations, and so

$$\mathsf{Cat}(\mathsf{C},\mathsf{M}) \quad \text{and} \quad \mathsf{Cat}(\mathsf{C}^{\mathsf{op}},\mathsf{M})$$

are the respective categories of diagrams and presheaves on C. If M is a symmetric monoidal closed category (Definition A.1.19) or, more generally, a closed multicategory (Definition 9.1.1), then there are corresponding enriched variants described in Definition 11.1.1. ◇

Definition 1.1.3. The *orbit category* of a group G, denoted $\mathcal{O}_G$, consists of the following. Its objects are the G-orbits G/H, where H is a subgroup of G, and its morphisms are the G-equivariant morphisms. ◇

Remark 1.1.4. Note that each G-equivariant map

$$f\colon G/H \longrightarrow G/K \quad \text{in} \quad \mathcal{O}_G$$

determines and is determined by an element $g \in G$, where $f(eH) = gK$, such that $g^{-1}Hg \subset K$. Thus, the morphisms in $\mathcal{O}_G$ are given by subconjugacy relations. ◇

Definition 1.1.5 (G-Spaces)**.** A G-*space* is a topological space on which G acts continuously. Morphisms of G-spaces are continuous functions that commute with the G-action. The category of G-spaces and their morphisms is denoted Top^G. ◇

Definition 1.1.6 (Fixed Points)**.** For each G-space X, and for each subgroup H in G, the H-*fixed point space*, denoted X^H, consists of the subspace of points on which H acts trivially. As a G-space, X^H can be defined equivalently as the space of G-equivariant morphisms

$$\mathsf{Top}^G(G/H, X),$$

where G/H has the discrete topology. The assignment

$$G/H \longmapsto X^H$$

determines a presheaf of spaces on the orbit category $\mathcal{O}_G$,

$$\Phi X\colon \mathcal{O}_G^{\mathsf{op}} \longrightarrow \mathsf{Top}^G, \tag{1.1.7}$$

called the *fixed point functor*. ◇

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Discussion of equivariant homotopy and (co)homology is beyond our current scope, but the following gives an indication of the role that the orbit category plays in equivariant topology.

Explanation 1.1.8. The coefficient systems for Bredon cohomology of G-spaces are given by presheaves

$$A\colon \mathcal{O}_G^{\mathsf{op}} \longrightarrow \mathsf{Ab},$$

where Ab is the category of Abelian groups and group homomorphisms. In particular, for a G-space X, the composite with π_n for $n \geq 2$ yields a coefficient system

$$\mathcal{O}_G^{\mathsf{op}} \xrightarrow{\Phi X} \mathsf{Top} \xrightarrow{\pi_n} \mathsf{Ab}. \qquad \diamond$$

The following result due to Elmendorf [Elm83] gives a different indication of the importance of presheaves on the orbit category.

Theorem 1.1.9 ([Elm83, GMR19]) *The fixed point functor, Φ, induces a Quillen equivalence*

$$\Phi\colon \mathsf{Top}^G \xrightarrow{\simeq_Q} (\mathsf{Top}^G\text{-}\mathsf{Cat})(\mathcal{O}_G^{\mathsf{op}}, \mathsf{Top})$$

between the category of G-equivariant topological spaces and the category of topological presheaves on $\mathcal{O}_G$.

As we outline in the following section, presheaves on the Burnside (2-)category, which are known as Mackey functors, fill an analogous role in the generalization to stable equivariant homotopy.

1.2 The Burnside Category and Abelian G-Mackey Functors

The Burnside category (Definition 1.2.5) extends the orbit category of G using spans of finite G-sets. The key motivation for this expansion of $\mathcal{O}_G$ is to account for the restriction, induction, and transfer morphisms on finite G-sets. Further explanation and examples of this perspective can be found in [Web00] and [HHR21, Sections 8.1 and 8.2].

Definition 1.2.1 (Finite G-Sets)**.** Let $\mathcal{N}_G$ denote the following skeleton of the category of finite G-sets. The objects of $\mathcal{N}_G$ are pairs $(\overline{n}, \alpha)$, where n is a natural number, $\overline{n} = \{1, \ldots, n\}$, and

$$\alpha\colon G \longrightarrow \Sigma_n$$

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is a group homomorphism. We regard $X = (\overline{n}, \alpha)$ as a G-set with the action

$$g \cdot i = \alpha(g)(i)$$

for $g \in G$ and $i \in \overline{n}$. The morphisms $f \colon (\overline{n}, \alpha) \longrightarrow (\overline{p}, \beta)$ in $\mathcal{N}_G$ are G-equivariant morphisms. That is, f is a map of sets $\overline{n} \longrightarrow \overline{p}$ such that

$$\beta(g)(f(i)) = f\big(\alpha(g)(i)\big) \quad \text{for} \quad g \in G \quad \text{and} \quad i \in \overline{n}.$$

We call n the *cardinality* of $X = (\overline{n}, \alpha)$ and write $|X| = n$. Additionally, we define the following.

(1) The disjoint union of finite G-sets, $\coprod$, defines a permutative structure with unit given by the empty G-set. We write $(\overline{0}, !)$ for the empty finite set and the unique action homomorphism $G \longrightarrow \Sigma_0$.

(2) The Cartesian product, together with the lexicographic ordering

$$\overline{n} \times \overline{p} \cong \overline{np} \quad \text{via} \quad (i,j) \longmapsto p(i-1) + j, \tag{1.2.2}$$

defines a second permutative structure on $\mathcal{N}_G$. Its unit is the terminal G-set $(\overline{1}, !)$, consisting of the terminal set and the unique action homomorphism $G \longrightarrow \Sigma_1$. $\diamond$

Definition 1.2.3 (Bicategory of Spans)**.** Suppose C is a small category with pullbacks, equipped with a choice of pullbacks for each pair of morphisms having a common codomain. The *bicategory of spans in* C is denoted $\mathsf{Span}(\mathsf{C})$ and consists of the following.

0-Cells: The 0-cells are objects $X \in \mathsf{C}$.

1-Cells: The 1-cells with domain X and codomain Y are triples (A, f, g) that are spans

$$X \xleftarrow{f} A \xrightarrow{g} Y \quad \text{in} \quad \mathsf{C}. \tag{1.2.4}$$

Since the object A is determined by the two morphisms, a span is sometimes denoted by its pair of morphisms, (f, g).

2-Cells: The 2-cells $(A, f, g) \longrightarrow (A', f', g')$ are morphisms $w \colon A \longrightarrow A'$ in C that make the following diagram commute in C.

$$\begin{tikzcd} & A \arrow[dl, "f"'] \arrow[dr, "g"] \arrow[d, "w"] & \\ X & A' \arrow[l, "f'"] \arrow[r, "g'"'] & Y \end{tikzcd}$$

Identities: The identity 1-cell on a 0-cell X is the triple $\Delta_X = (X, 1_X, 1_X)$ given by the identity morphisms in C. The identity 2-cell on a 1-cell (A, f, g) is the identity morphism 1_A in C.

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Composition: For objects X, Y, and Z in C, the composition functor

$$\mathsf{Span}(\mathsf{C})(Y,Z) \times \mathsf{Span}(\mathsf{C})(X,Y) \longrightarrow \mathsf{Span}(\mathsf{C})(X,Z)$$

sends a composable pair to the span given by their chosen pullback, as shown below.

$$\begin{tikzcd}[column sep=small]
& & A \times_Y B \arrow[dl] \arrow[dr] & & \\
& A \arrow[dl, "f"'] \arrow[dr, "g"] & & B \arrow[dl, "h"'] \arrow[dr, "k"] & \\
X & & Y & & Z
\end{tikzcd}$$

Having a chosen pullback for each pair of morphisms with a common codomain makes the composition of 1-cells well defined. Universality of pullbacks makes it associative and unital up to isomorphisms that satisfy the axioms of bicategorical composition. See [JY21, Example 2.1.22] for further details of this construction. $\diamond$

Now we use Definitions 1.2.1 and 1.2.3 to define the Burnside category and its specialization, the Burnside ring. In Definition 1.3.5 we generalize further to a Burnside 2-category.

Definition 1.2.5 (Burnside Category and Burnside Ring)**.** The *Burnside category* of a finite group G, denoted $G\mathcal{B}$, is an Ab-enriched category defined as follows. The objects of $G\mathcal{B}$ are the finite G-sets $X \in \mathcal{N}_G$. The Abelian group $G\mathcal{B}(X,Y)$ for $X, Y \in \mathcal{N}_G$ is the Grothendieck group of isomorphism classes of spans

$$X \longleftarrow A \longrightarrow Y \quad \text{in} \quad \mathcal{N}_G.$$

Thus, $G\mathcal{B}$ is the category obtained from $\mathsf{Span}(\mathcal{N}_G)$ by taking isomorphism classes of 1-cells and then group-completing each set of morphisms with respect to the Abelian monoid structure given by disjoint union.

The *Burnside ring* of G, denoted $G\mathcal{A}$, is obtained by taking isomorphism classes of objects in $G\mathcal{B}$. Equivalently, the additive group of elements is given by the Grothendieck group of isomorphism classes of finite G-sets, with addition given by disjoint union. Its multiplication is induced by Cartesian product. $\diamond$

Lemma 1.2.6 (Self-Duality of $G\mathcal{B}$) *There is an isomorphism of* Ab*-enriched categories*

$$G\mathcal{B} \xrightarrow{\cong} G\mathcal{B}^{\mathsf{op}} \tag{1.2.7}$$

that is the identity on objects and is induced on hom Abelian groups by the isomorphism

$$\mathsf{Span}(\mathcal{N}_G)(X,Y) \xrightarrow{\cong} \mathsf{Span}(\mathcal{N}_G)(Y,X)$$

that sends a span (f,g) *to its reverse,* (g,f).

Proof. Functoriality of the indicated isomorphism follows from universality of the pullbacks defining composition. □

We warn the reader that the 2-categorical analog of the self-duality (1.2.7) does *not* hold for the Burnside 2-category $G\mathcal{E}$ in Definition 1.3.5 below. Sending (f,g) to its reverse (g,f) does not define a 2-functor in that context; see Remark 1.3.7.

Remark 1.2.8 (Self-Duality and Stable Orbit Spectra)**.** Self-duality of the Burnside category $G\mathcal{B}$ (1.2.7) is nearly transparent in its simplicity, but it is an algebraic artifact of a much deeper topological phenomenon. Each orbit G/H has an equivariant suspension spectrum, $\Sigma^\infty G/H_+$, and there is an equivalent definition of $G\mathcal{B}$ with morphisms given by stable equivariant morphisms $\Sigma^\infty G/H_+ \longrightarrow \Sigma^\infty G/K_+$; see [May96, Section XIX.3]. The stable orbit spectra $\Sigma^\infty G/H_+$ satisfy an equivariant self-duality ([May96, Section XVI.7] or [HHR21, Section 8.0C]) that implies that of Lemma 1.2.6.

The definition of the Burnside category in terms of stable orbit spectra is the more general one, with origins in work of tom Dieck [tD79]; see [May96, Section XVII.2]. The proofs that this definition can be given equivalently by spans of finite G-sets, as in Definition 1.2.5, depend on the assumption that G is finite. In more general cases, the definition of the Burnside category in terms of stable orbit spectra is necessary. ◇

Definition 1.2.9. An *Abelian G-Mackey functor* is an Ab-enriched presheaf

$$G\mathcal{B}^{\mathsf{op}} \longrightarrow \mathsf{Ab}.$$

Because $G\mathcal{B}$ is isomorphic to $G\mathcal{B}^{\mathsf{op}}$ (Lemma 1.2.6), an Abelian G-Mackey functor is equivalently defined as a functor $G\mathcal{B} \longrightarrow \mathsf{Ab}$. ◇

Remark 1.2.10. Each Abelian G-Mackey functor M has an associated Eilenberg–Mac Lane G-spectrum, HM. See [May96, Section V.4] or [HHR21, Theorem 8.8.4] for constructions via Elmendorf's Theorem 1.1.9. Such Mackey functors M, and their associated G-spectra HM, are the coefficient systems for Bredon cohomology of G-spectra. ◇

Explanation 1.2.11. An Abelian G-Mackey functor M can be defined equivalently as a pair of functors

$$M_*\colon \mathcal{N}_G \longrightarrow \mathsf{Ab} \quad \text{and} \quad M^*\colon \mathcal{N}_G^{\mathsf{op}} \longrightarrow \mathsf{Ab}$$

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that agree on objects and are subject to the following two axioms, where

$$MX = M_*X = M^*X$$

denotes the common value on objects.

(1) For each pair of objects X and Y in $\mathcal{N}_G$, applying M_* to the structure morphisms of the coproduct

$$X \longrightarrow X \coprod Y \longleftarrow Y$$

induces a universal morphism with domain $MX \oplus MY$ that is an isomorphism

$$MX \oplus MY \xrightarrow{\cong} M(X \coprod Y).$$

(2) For each pullback diagram in $\mathcal{N}_G$,

$$\begin{tikzcd} W \arrow[r, "p"] \arrow[d, "q"'] & X \arrow[d, "f"] \\ Y \arrow[r, "g"] & Z \end{tikzcd}$$

the following equality of composite morphisms holds in Ab:

$$(M^*f)(M_*g) = (M_*p)(M^*q).$$

See [Web00, Section 2] and [HHR21, Definition 8.2.3] for further discussion of this perspective, explanation of the equivalence with Definition 1.2.5, and several compelling examples. ◇

1.3 Equivariant Spectra and Presheaves on the Burnside 2-Category

For the category $\mathsf{C} = \mathcal{N}_G$, there is a choice of pullbacks that makes $\mathsf{Span}(\mathcal{N}_G)$ nearly a 2-category. Following Guillou–May [GM22, Remark 1.8 and Definition 6.2], the following will be used in the definition of the Burnside 2-category (Definition 1.3.5). A more general approach to such strictifications can be found in [Gui10].

Explanation 1.3.1 (Choices of Pullbacks in $\mathcal{N}_G$)**.** Recall the lexicographic ordering of products from (1.2.2). We use this to determine choices of pullbacks in $\mathcal{N}_G$, as follows. Suppose given the following composable pair of spans in $\mathcal{N}_G$,

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$$\begin{tikzcd}[column sep=small]
 & A \arrow[dl, "f"'] \arrow[dr, "g"] & & B \arrow[dl, "h"'] \arrow[dr, "k"] & \\
X & & Y & & Z,
\end{tikzcd}$$

where

$$X = (\overline{n}_X, \alpha_X), \quad A = (\overline{n}_A, \alpha_A), \quad Y = (\overline{n}_Y, \alpha_Y),$$
$$B = (\overline{n}_B, \alpha_B), \text{ and } Z = (\overline{n}_Z, \alpha_Z).$$

Let

$$A \times_Y B = \{(a, b) \in \overline{n}_A \times \overline{n}_B \mid g(a) = h(b)\}$$

denote the pullback of G-sets, with its ordering induced by the lexicographic ordering on $\overline{n}_A \times \overline{n}_B$. This determines a unique order-preserving isomorphism of finite G-sets

$$(\overline{p}, \rho) \xrightarrow{\cong} A \times_Y B \tag{1.3.2}$$

with $(\overline{p}, \rho) \in \mathcal{N}_G$.

We write $A \circ B = (\overline{p}, \rho)$ to denote this choice of pullback in $\mathcal{N}_G$ and let π_A and π_B denote the following composites, where the unlabeled isomorphism is that of (1.3.2).

$$\begin{tikzcd}[column sep=small]
 & & A \circ B \arrow[dl, "\pi_A"'] \arrow[d, "\cong"] \arrow[dr, "\pi_B"] & & \\
 & A \arrow[dl, "f"'] \arrow[dr, "g"] & A \times_Y B \arrow[l] \arrow[r] & B \arrow[dl, "h"'] \arrow[dr, "k"] & \\
X & & Y & & Z
\end{tikzcd} \tag{1.3.3}$$

We note three consequences of these choices via lexicographic ordering.

(1) These choices for pullbacks make composition in $\mathsf{Span}(\mathcal{N}_G)$ strictly associative.
(2) The morphism π_A is always order-preserving.
(3) The morphism π_B is generally not order-preserving.

For each $Y = (\overline{n}_Y, \alpha_Y)$ in $\mathcal{N}_G$, let Δ_Y denote the unit 1-cell for Y in $\mathsf{Span}(\mathcal{N}_G)$:

$$\Delta_Y = \left(Y \xleftarrow{1_Y} Y \xrightarrow{1_Y} Y \right).$$

In (1.3.3), if the span (h, k) is the unit Δ_Y, then $B = Y$ and we have

$$A \circ B = A, \qquad \pi_A = 1_A, \quad \text{and} \quad \pi_B = g.$$

Thus, Δ_Y is a strict right unit.

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Now suppose, instead, that the span (f,g) in (1.3.3) is the unit Δ_Y. Then $A = Y$, but $\pi_B = g$ if and only if h is an order-preserving G-map. In general, π_B is an isomorphism of finite G-sets determined by the reordering of $\overline{n}_B$ that is induced by the fibers of h. $\diamond$

To construct a 2-category from $\mathsf{Span}(\mathcal{N}_G)$, the identity 1-cells Δ_X are augmented by new strict identities via the following construction.

Definition 1.3.4 (Whiskering a Category)**.** Given a small category D with a distinguished object $\Delta \in \mathsf{D}$. Define the *whiskering at* Δ, denoted $\mathsf{D}^\dagger$, as a category whose objects consist of those of D, together with a new object I and an isomorphism

$$I \xrightarrow[\cong]{\zeta_\Delta} \Delta.$$

The morphisms in $\mathsf{D}^\dagger$ are generated by those of D and composition with ζ_Δ and its inverse. Thus, $\mathsf{D}^\dagger$ is the pushout in Cat of the two inclusions

$$\mathsf{D} \longleftarrow \{\Delta\} \longrightarrow \{\zeta_\Delta^{\pm 1}\},$$

where $\{\Delta\}$ denotes the discrete category on Δ and the right-hand side denotes the category generated by the isomorphism ζ_Δ and its inverse. A further elaboration of the whiskering construction is given in [GM22, Definition 6.1]. $\diamond$

Definition 1.3.5 (The Burnside 2-Category)**.** The *Burnside 2-category* of a finite group G is a $\mathsf{PermCat}^{\mathsf{su}}$-enriched category (Explanation 7.3.2) denoted $G\mathcal{E}$ and defined as follows. Its objects are the finite G-sets $X = (\overline{n}, \alpha)$ of $\mathcal{N}_G$ (Definition 1.2.1). For each pair of objects

$$X = (\overline{n}, \alpha) \quad \text{and} \quad Y = (\overline{p}, \beta) \quad \text{in} \quad \mathcal{N}_G,$$

the category of 1- and 2-cells is given by

$$G\mathcal{E}(X,Y) = \begin{cases} \mathsf{Span}(\mathcal{N}_G)(X,Y) & \text{if} \quad X \neq Y \quad \text{or} \quad |X| \leq 1, \\ \mathsf{Span}(\mathcal{N}_G)(X,X)^\dagger & \text{if} \quad X = Y \quad \text{and} \quad |X| \geq 2 \end{cases} \tag{1.3.6}$$

where $\mathsf{Span}(\mathcal{N}_G)$ is the bicategory of spans (Definition 1.2.3) with the lexicographic choice of pullbacks from Explanation 1.3.1 and $\mathsf{Span}(\mathcal{N}_G)(X,X)^\dagger$ is the whiskering of the category $\mathsf{Span}(\mathcal{N}_G)(X,X)$ as in Definition 1.3.4 at the unit 1-cell Δ_X.

The horizontal composition of $\mathsf{Span}(\mathcal{N}_G)$ extends uniquely to $G\mathcal{E}$ such that the 1-cells $I_{\Delta_X} \in \mathsf{Span}(\mathcal{N}_G)(X,X)^\dagger$ are strictly unital. The permutative structure of each $\mathsf{Span}(\mathcal{N}_G)(X,Y)$ given by disjoint union (Definition 1.2.1 (1)) also extends uniquely such that $(\overline{0}, !)$ remains its unit and for $Y \neq (\overline{0}, !)$ we have

$$I_{\Delta_X} \coprod Y = X \coprod Y \quad \text{and} \quad Y \coprod I_{\Delta_X} = Y \coprod X.$$

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For further explanation of this structure, see [GM22, Definition 6.2], where our $G\mathcal{E}$ is denoted $G\mathcal{E}'$. ◇

Remark 1.3.7 (Non-Self-Duality of $G\mathcal{E}$)**.** Recall that the Burnside 1-category, $G\mathcal{B}$ in Definition 1.2.5 is self-dual (Lemma 1.2.6). However, the assignment that sends a span (f,g) as in (1.2.4) to its reverse (g,f) does not define a 2-functor

$$G\mathcal{E} \longrightarrow G\mathcal{E}^{\mathsf{op}}$$

because it does not preserve composition strictly. It is natural to consider the generalization from 2-functors to pseudofunctors, but the latter structure does not provide a $\mathsf{PermCat}^{\mathsf{su}}$-enriched functor in the sense of Explanation 7.3.12. This subtlety has further implications to be noted in Remark 11.5.5. ◇

The following is a special case of more general enriched Mackey functors introduced in Definition 11.1.1.

Definition 1.3.8. Suppose given a (possibly nonsymmetric) K-theory multifunctor

$$K\colon \mathsf{PermCat}^{\mathsf{su}} \longrightarrow \mathsf{Sp}$$

from permutative categories to spectra, and let $(-)_K$ denote the corresponding change of enrichment (Definition 8.1.1). The category of *spectral G-Mackey functors* for K is the enriched presheaf category

$$\mathsf{Sp}\text{-}\mathsf{Cat}\big((G\mathcal{E}_K)^{\mathsf{op}}, \mathsf{Sp}\big),$$

consisting of Sp-enriched functors and transformations, as in (11.1.3). ◇

Note that if K is a multifunctor in the symmetric sense – for example, if K is the Elmendorf–Mandell K-theory, K^{EM}, in (3.5.8) – then $(G\mathcal{E}_K)^{\mathsf{op}}$ and $(G\mathcal{E}^{\mathsf{op}})_K$ are equal as Sp-categories by Proposition 8.2.1. In such a case, the category of spectral G-Mackey functors is equal to $\mathsf{Sp}\text{-}\mathsf{Cat}\big((G\mathcal{E}^{\mathsf{op}})_K, \mathsf{Sp}\big)$. However, if K is not symmetric, then there is no such identification. See, for example, Theorem 11.5.1 and Remark 11.5.5 for particular uses of these details.

For further development of both theory and applications of spectral Mackey functors in equivariant algebraic K-theory, the reader is referred to [BO15, Bar17, MM19, BGS20, MM22, GM22, GMMO∞]. The key result for our purposes is the following from Guillou and May [GM22], which is a stable analog of Elmendorf's Theorem 1.1.9. Here, $\mathbb{K}$ denotes the nonsymmetric K-theory multifunctor in [GM22, GMMO∞].

Theorem 1.3.9 ([GM22, Theorem 0.1]) *There is a zigzag of Quillen equivalences*

$$G\text{-}\mathsf{Sp} \simeq_Q \mathsf{Sp}\text{-}\mathsf{Cat}\big((G\mathcal{E}_{\mathbb{K}})^{\mathsf{op}}, \mathsf{Sp}\big)$$

where G-Sp is the category of G-spectra.

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Thus, the Guillou–May theorem shows that the homotopy theory of G-spectra is equivalent to that of spectral G-Mackey functors for $\mathbb{K}$.

1.4 Stable Model Categories as Spectral Presheaf Categories

Definition 1.4.1. Suppose given a model category M. We recall the following terms briefly and refer the reader to [Hov99, Hir03] for more detailed descriptions.

(1) We say M is *simplicial* if it is enriched, tensored, and cotensored over simplicial sets, such that the following *pullback powering* condition holds. For each cofibration $i: A \longrightarrow B$ and fibration $p: X \longrightarrow Y$ in M, the universal morphism induced by $\mathsf{M}(i, X)$ and $\mathsf{M}(B, p)$,

$$\mathsf{M}(B,X) \longrightarrow \mathsf{M}(A,X) \times_{\mathsf{M}(A,Y)} \mathsf{M}(B,Y),$$

is a Kan fibration that is acyclic whenever either i or p is acyclic.

(2) We say M is *cofibrantly generated* if it is equipped with two sets of morphisms, $\mathcal{I}$ and $\mathcal{J}$, such that the following three statements hold.
   - Both $\mathcal{I}$ and $\mathcal{J}$ permit the small object argument.
   - A morphism of M is a fibration if and only if it has the right lifting property with respect to every element of $\mathcal{J}$.
   - A morphism of M is an acyclic fibration if and only if it has the right lifting property with respect to every element of $\mathcal{I}$.

(3) We say that M is *proper* if the following two conditions hold.
   - Every pushout of a weak equivalence along a cofibration is a weak equivalence.
   - Every pullback of a weak equivalence along a fibration is a weak equivalence.

(4) We say that M is *stable* if the suspension and loop functors on its homotopy category are inverse equivalences. ◇

For the remainder of this section we suppose that M is a simplicial, cofibrantly generated, proper, and stable model category. The category of symmetric spectra over M [SS03, Definition 3.6.1] is denoted Sp^{M}. The following, from [SS03, Definition 3.7.5], describes an Sp-enriched category generalizing the endomorphism spectrum associated to an object of M.

Definition 1.4.2. Suppose P is a set of cofibrant objects in M. The *spectral endomorphism category* $\mathcal{E}(P)$ is the full Sp-subcategory of Sp^{M} with objects given by the fibrant replacements, relative to the level model structure on Sp^{M}, of the symmetric suspension spectra of the objects in P. ◇

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The following result of Schwede and Shipley gives a characterization of M via Sp-enriched presheaves. In this result, $\mathsf{Sp}\text{-}\mathsf{Cat}\big(\mathcal{E}(P)^{\mathsf{op}}, \mathsf{Sp}\big)$ denotes the $\mathcal{E}(P)$-presheaf category of Sp as in (11.1.3).

Theorem 1.4.3 ([SS03, Theorem 3.3.3]) *Suppose P is a set of compact generators of a simplicial, cofibrantly generated, proper, and stable model category M. Then there is a chain of simplicial Quillen equivalences*

$$\mathsf{M} \simeq_Q \mathsf{Sp}\text{-}\mathsf{Cat}\big(\mathcal{E}(P)^{\mathsf{op}}, \mathsf{Sp}\big).$$

The work of Schwede and Shipley goes on to give a number of applications in (derived) Morita theory and equivariant stable homotopy. In each case, their work characterizes the relevant stable model category as a category of spectral presheaves, also called enriched Mackey functors (see Definition 11.1.1).

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