Bull. Aust. Math. Soc. **100** (2019), 281–289
doi:10.1017/S0004972719000054

A NILPOTENCY CRITERION FOR SOME VERBAL SUBGROUPS

CARMINE MONETTA[✉] **and ANTONIO TORTORA**

(Received 5 December 2018; accepted 14 December 2018; first published online 27 February 2019)

Abstract

The word $w = [x_{i_1}, x_{i_2}, \dots, x_{i_k}]$ is a simple commutator word if $k \geq 2$, $i_1 \neq i_2$ and $i_j \in \{1, \dots, m\}$ for some $m > 1$. For a finite group G, we prove that if $i_1 \neq i_j$ for every $j \neq 1$, then the verbal subgroup corresponding to w is nilpotent if and only if $|ab| = |a||b|$ for any w-values $a, b \in G$ of coprime orders. We also extend the result to a residually finite group G, provided that the set of all w-values in G is finite.

2010 *Mathematics subject classification*: primary 20F18; secondary 20F45.

Keywords and phrases: nilpotent, Engel word, verbal subgroup.

1. Introduction

Let F be the free group on free generators $x_1, \dots, x_m$ for some $m > 1$. A group word is any nontrivial element of F, that is, a product of finitely many x_i and their inverses. The elements of the commutator subgroup of F are called commutator words. We say that the commutator word

$$[x_{i_1}, x_{i_2}, \dots, x_{i_k}] = [\dots[[x_{i_1}, x_{i_2}], x_{i_3}], \dots, x_{i_k}]$$

is a *simple commutator word* if $k \geq 2$, $i_1 \neq i_2$ and $i_j \in \{1, \dots, m\}$ for every $j \in \{1, \dots, k\}$. Examples of simple commutator words are the lower central words and the n-Engel word

$$[x, {}_n y] = [x, \underbrace{y, \dots, y}_{n}].$$

Let $w = w(x_1, \dots, x_k)$ be a group word in the variables $x_1, \dots, x_k$. For any group G and arbitrary $g_1, \dots, g_k \in G$, the elements of the form $w(g_1, \dots, g_k)$ are called the w-values in G. We denote by G_w the set of all w-values in G. The verbal subgroup of G corresponding to w is the (normal) subgroup $w(G)$ of G generated by G_w. If $w(G) = 1$, then w is said to be a law in G.

Recently the following question has been considered in [2] (see also [3]).

The authors are members of National Group for Algebraic and Geometric Structures and their Applications (GNSAGA–INdAM).

QUESTION 1.1. Let w be a commutator word and let G be a finite group with the property that $|ab| = |a||b|$ for any $a, b \in G_w$ of coprime orders. Is the verbal subgroup $w(G)$ nilpotent? (Here $|x|$ stands for the order of the element $x \in G$.)

As remarked in [2], by a result of Kassabov and Nikolov [9], this is not true in general (see Example 4.3). Two easier counterexamples are given in Section 4. On the other hand, the answer to the above question is positive when w is a lower central word [1, 2]. Motivated by this, we prove the following nilpotency criterion for $w(G)$, where w is a simple commutator word without any repetition of the first variable.

THEOREM 1.2. *Let $w = [x_{i_1}, \ldots, x_{i_k}]$ be a simple commutator word with $i_1 \neq i_j$ for every $j \in \{2, \ldots, k\}$ and let G be a finite group. Then $w(G)$ is nilpotent if and only if $|ab| = |a||b|$ for any $a, b \in G_w$ of coprime orders.*

We also extend Theorem 1.2 to a residually finite group G, provided that the set of all w-values in G is finite.

COROLLARY 1.3. *Let $w = [x_{i_1}, \ldots, x_{i_k}]$ be a simple commutator word with $i_1 \neq i_j$ for every $j \in \{2, \ldots, k\}$ and let G be a residually finite group in which G_w is finite. Then $w(G)$ is finite and it is nilpotent if and only if $|ab| = |a||b|$ for any $a, b \in G_w$ of coprime orders.*

Recall that a group is residually finite if the intersection of its subgroups of finite index is trivial. Notice also that in Corollary 1.3 the finiteness of G_w depends on the conciseness of the word w in the class of residually finite groups (see Section 3). This is related to a question of P. Hall on words assuming only finitely many values in a group (see [10, Part 1, page 119]).

2. Proof of Theorem 1.2

The aim of this section is to prove the 'if' part of Theorem 1.2, since the 'only if' part is clear. We split the proof into two cases, depending on whether the finite group G is soluble or not. In particular, we will show that only the soluble case can occur.

2.1. The soluble case. If G is a finite soluble group, the Fitting height of G is the least integer h such that $F_h(G) = G$, where $F_0(G) = 1$ and $F_i(G)/F_{i-1}(G) = F(G/F_{i-1}(G))$ is the Fitting subgroup of $G/F_{i-1}(G)$ for every $i \geq 1$. A finite soluble group with Fitting height at most 2 is said to be metanilpotent.

The following lemma is well known (see [1, Lemma 3] for a proof).

LEMMA 2.1. *Let G be a finite metanilpotent group with Fitting subgroup $F(G)$. For p a prime, denote by $O_{p'}(G)$ the maximal normal subgroup of G of order coprime to p. If $x \in G$ is a p-element such that $[O_{p'}(F(G)), x] = 1$, then $x \in F(G)$.*

A subgroup H of a finite group G is called a tower of height h if H can be written as a product $H = P_1 \cdots P_h$, where:

(1) P_i is a p_i-group (p_i a prime) for $i = 1, \ldots, h$;

https://doi.org/10.1017/S0004972719000054 Published online by Cambridge University Press

(2) P_i normalises P_j for $i < j$;
(3) $[P_i, P_{i-1}] = P_i$ for $i = 2, \dots, h$.

It follows from (3) that $p_i \neq p_{i+1}$ for $i = 1, \dots, h-1$.

The next lemma is taken from [14, Lemma 1.9].

Lemma 2.2. *A finite soluble group G has Fitting height at least h if and only if G has a tower of height h.*

Given two nonempty subsets X and Y of a group G, let

$$[X, {}_nY] = [[X, {}_{n-1}Y], Y],$$

where $n \geq 2$ and $[X, Y]$ is the commutator subgroup of X and Y. We write $[X, {}_ny]$ when $Y = \{y\}$. Then, assuming that X is normalised by y, it is easy to see that

$$[X, {}_ny] = [X, {}_n\langle y\rangle]$$

for every $n \geq 1$.

The next lemma is a straightforward corollary of [6, Theorem 5.3.6].

Lemma 2.3. *For p a prime, let P be a p-subgroup of a finite group G. Suppose that P is normalised by an element $x \in G$ of p'-order. Then*

$$[P, x] = [P, x, x].$$

Lemma 2.4. *Let $w = [x_{i_1}, \dots, x_{i_k}]$ be a simple commutator word with $i_1 \neq i_j$ for every $j \in \{2, \dots, k\}$ and let G be a finite group in which $|ab| = |a||b|$ for any w-values $a, b \in G$ of coprime orders. For p a prime, let P be a p-subgroup of G normalised by a w-value $x \in G$ of p'-order. Then $[P, x] = 1$.*

Proof. By Lemma 2.3,

$$[P, x^{-1}] = [P, {}_{k-1}x^{-1}];$$

thus, the result will follow once it is shown that $N = [P, {}_{k-1}x^{-1}] = 1$.

Let $[g, {}_{k-1}x^{-1}] \in N$ for some $g \in P$. Of course, the orders of the w-values x and $[g, {}_{k-1}x^{-1}]$ are coprime. Then, by hypothesis,

$$|[g, {}_{k-1}x^{-1}]x| = |[g, {}_{k-1}x^{-1}]|\,|x|.$$

However,

$$[g, {}_{k-1}x^{-1}]x = [g, {}_{k-2}x^{-1}]^{-1}x[g, {}_{k-2}x^{-1}]$$

is a conjugate of x. So, $|[g, {}_{k-1}x^{-1}]x| = |x|$ and consequently $[g, {}_{k-1}x^{-1}] = 1$. □

Lemma 2.5. *Let $w = [x_{i_1}, \dots, x_{i_k}]$ be a simple commutator word and let $G = A \times B$ be an arbitrary group. Then $w(G) = w(A) \times w(B)$.*

Proof. By induction on n,

$$[a_{i_1}b_{i_1}, \dots, a_{i_k}b_{i_k}] = [a_{i_1}, \dots, a_{i_k}][b_{i_1}, \dots, b_{i_k}]$$

for every $a_{i_1}, \dots, a_{i_k} \in A$ and every $b_{i_1}, \dots, b_{i_k} \in B$. □

https://doi.org/10.1017/S0004972719000054 Published online by Cambridge University Press

We are now able to prove the announced result for soluble groups.

Proposition 2.6. *Let* $w = [x_{i_1}, \dots, x_{i_k}]$ *be a simple commutator word with* $i_1 \neq i_j$ *for every* $j \in \{2, \dots, k\}$ *and let* G *be a finite soluble group in which* $|ab| = |a||b|$ *for any* $a, b \in G_w$ *of coprime orders. Then* $w(G)$ *is nilpotent.*

Proof. Let h be the Fitting height of G. Firstly, we show that $h \leq 2$. Suppose by way of contradiction that $h \geq 3$. Then, by Lemma 2.2, there exists a tower

$$P_1 P_2 P_3 \cdots P_h$$

of height h in G. Since $P_2 = [P_2, P_1]$ and $P_3 = [P_3, P_2]$,

$$P_3 = [P_3, [P_2, P_1]].$$

Furthermore, by Lemma 2.3,

$$[P_2, x] = [P_2, {}_{k-1}x]$$

for every $x \in P_1$. Hence, $[P_2, x]$ is generated by w-values of p_2-orders. Applying Lemma 2.4, we deduce that P_3 commutes with $[P_2, x]$. Thus,

$$[P_3, [P_2, P_1]] = 1,$$

which is impossible.

Now let $h = 2$, the case $h = 1$ being obvious. Denote by F the Fitting subgroup of G. If $w(G) \leq F$, we are done. Suppose that $w(G)$ is not contained in F. Since G/F is nilpotent, by Lemma 2.5, there exists a Sylow p-subgroup P of G such that $w(P/F) = w(P)F/F$ is nontrivial. Let $x \in w(P)$ be a w-value which does not belong to F. Then $[O_{p'}(F), x] = 1$, by Lemma 2.4, from which it follows that $x \in F$, by Lemma 2.1, which is a contradiction. □

2.2. The general case. The following lemma is a well-known consequence of the Baer–Suzuki theorem (see, for instance, [8, Theorem 2.13]).

Lemma 2.7. *Let* G *be a finite nonabelian simple group. If* x *is an element of* G *of order* 2*, then there exists* $g \in G$ *such that* $[x, g]$ *has odd prime order.*

We will require a property of finite simple groups whose proper subgroups are soluble. These groups have been classified by Thompson in [13] and they are known as finite minimal simple groups.

Proposition 2.8. *Let* G *be a finite minimal simple group. Then* G *contains a subgroup* $H = A \rtimes T$*, where* A *is an elementary abelian* 2*-group and* T *is a subgroup of odd order such that* $C_A(T) = 1$*. Further,* $A = [A, T]$*.*

Proof. According to Thompson's classification [13, Corollary 1], the group G is isomorphic to one of the following groups:

(1) $\mathrm{PSL}(2, 2^p)$, where p is any prime;
(2) $\mathrm{PSL}(2, 3^p)$, where p is any odd prime;

https://doi.org/10.1017/S0004972719000054 Published online by Cambridge University Press

(3) $\mathrm{PSL}(2, p)$, where $p > 3$ is any prime such that $p^2 + 1 \equiv 0 \pmod 5$;
(4) $\mathrm{PSL}(3, 3)$;
(5) $\mathrm{Sz}(2^p)$, where p is any odd prime.

Since the groups in (2), (3) and (4) have a subgroup isomorphic to the alternating group of degree 4 (see, for instance, [12, Theorem 6.26] and [4, Theorem 7.1(2)]), we may consider the other two cases.

If G is isomorphic to $\mathrm{PSL}(2, 2^p)$, then [12, Theorem 6.25] shows that G contains a Frobenius group $H = A \rtimes T$, where A is an elementary abelian 2-group of order q and T is a cyclic group of order $q - 1$.

If G is isomorphic to the Suzuki group $\mathrm{Sz}(2^p)$, then G contains a Frobenius group $F = Q \rtimes T$, where Q is a Sylow 2-subgroup of G of order 2^{2p} and T is a cyclic subgroup of order $2^p - 1$ (see [11, Theorem 9]). Thus, taking A to be a minimal normal subgroup of F contained in Q, the subgroup $H = A \rtimes T$ is as required.

Finally, notice that in both cases we have $A = [A, T]$ by [6, Theorem 5.2.3]. □

Lemma 2.9. *Let $w = [x_{i_1}, \ldots, x_{i_k}]$ be a simple commutator word with $i_1 \neq i_j$ for every $j \in \{2, \ldots, k\}$ and let G be a finite group such that $G = G'$. If $q \in \pi(G)$, then G is generated by w-values of p-power order for primes $p \neq q$.*

Proof. For each prime $p \in \pi(G) \setminus \{q\}$, denote by N_p the subgroup of G generated by all w-values of p-power order. Let us show that each Sylow p-subgroup of G is contained in N_p. Suppose that this is false and choose p such that a Sylow p-subgroup of G is not contained in N_p. Of course, N_p is a normal subgroup of G. We may pass to the quotient G/N_p and assume that $N_p = 1$. Since $G = G'$, it is clear that G does not possess a normal p-complement. Thus, the Frobenius theorem (see [6, Theorem 7.4.5]) implies that G has a p-subgroup H and a p'-element $a \in N_G(H)$ such that $[H, a] \neq 1$. By Lemma 2.3,

$$1 \neq [H, a] = [H, {}_{k-1}a] \leq N_p,$$

which is a contradiction. Hence, N_p contains the Sylow p-subgroups of G. Let T be the product of all subgroups N_p, with $p \neq q$. Then G/T is a q-group and, since $G = G'$, we conclude that $G = T$. It follows that G can be generated by w-values of p-power order for $p \neq q$. □

In order to complete the proof of Theorem 1.2, we recall that if a simple commutator word is a law in a finite group G, then G is nilpotent [7].

Proposition 2.10. *Let $w = [x_{i_1}, \ldots, x_{i_k}]$ be a simple commutator word with $i_1 \neq i_j$ for every $j \in \{2, \ldots, k\}$ and let G be a finite group in which $|ab| = |a||b|$ for any $a, b \in G_w$ of coprime orders. Then G is soluble and $w(G)$ is nilpotent.*

Proof. By Proposition 2.6, it is enough to show that G is soluble. Suppose that G is not soluble. Of course, we may assume that G is a counterexample of minimal order. Then every proper subgroup K of G is soluble: indeed, $K/w(K)$ is nilpotent [7, Satz 6.1] and so is $w(K)$ by Proposition 2.6. It follows that $G = G'$.

https://doi.org/10.1017/S0004972719000054 Published online by Cambridge University Press

Let R be the soluble radical of G, that is, the subgroup of G generated by all normal soluble subgroups of G. Then G/R is a nonabelian simple group and, by Proposition 2.6, $w(R)$ is nilpotent. We claim that $R = Z(G)$. Choose $q \in \pi(F(G))$ and let Q be the Sylow q-subgroup of $F(G)$. According to Lemma 2.9, the group G is generated by w-values of p-power order for primes $p \neq q$. Also, by Lemma 2.4, $[Q, x] = 1$ for every w-value x of q'-order. Thus, $Q \leq Z(G)$. This happens for each choice of $q \in \pi(F(G))$, so that $F(G) = Z(G)$. Since $w(R) \leq F(G)$, we have $[x_{i_1}, \ldots, x_{i_k}, y] = 1$ for every $x_{i_1}, \ldots, x_{i_k}, y \in R$. Hence, R is nilpotent [7, Satz 6.1] and therefore $R \leq F(G)$. In particular, $R = Z(G)$.

Next, we prove that G contains a w-value x such that x is a 2-element of order 2 modulo $Z(G)$. First, notice that $G/Z(G)$ is a finite minimal simple group. Then, by Proposition 2.8, $G/Z(G)$ has a subgroup

$$H/Z(G) = A/Z(G) \rtimes T/Z(G),$$

where $A/Z(G)$ is an elementary abelian 2-group and $T/Z(G)$ is a group of odd order such that

$$A/Z(G) = [A/Z(G),\ _{k-1}T/Z(G)].$$

Let P be the Sylow 2-subgroup of A. Thus, $x = [a,\ _{k-1}t]$, for some $a \in P$ and $t \in T$, is a w-value, as desired.

Now take $x \in G_w$ with the above properties. By Lemma 2.7, there exists an element $g \in G$ such that the order of $[x, g]$ is an odd prime. Since

$$1 = [x^2, g] = [x, g]^x [x, g],$$

x inverts $[x, g]$ and so, by Lemma 2.4, $[\langle [x, g] \rangle, x] = 1$. This gives $[x, g] = 1$, which is a contradiction. □

3. Proof of Corollary 1.3

Following [5], we say that a word w implies virtual nilpotency if every finitely generated metabelian group, where w is a law, has a nilpotent subgroup of finite index. Since finitely generated n-Engel groups are nilpotent (see [10, Part 2, Theorem 7.3.5]), the Engel words imply virtual nilpotency. More generally, this is true for simple commutator words.

Lemma 3.1. *Let $w = [x_{i_1}, \ldots, x_{i_n}]$ be a simple commutator word and let G be a metabelian group such that $w(G) = 1$. Then G is n-Engel.*

Proof. Since G is metabelian, we have $[c, x_{i_j}, x_{i_k}] = [c, x_{i_k}, x_{i_j}]$ for every $c \in G'$. Then, without loss of generality, we may assume that

$$w = [x_{i_1}, \ldots, x_{i_m},\ _{n-m}x_{i_1}],$$

where $1 < m < n$ and $i_1 \neq i_j$ for every $j \in \{2, \ldots, m\}$. Thus, for any $x, y \in G$, taking $x_{i_1} = y[x, y]$ and $x_{i_2} = \cdots = x_{i_m} = y$, we have $[x,\ _n y] = 1$ and therefore G is n-Engel. □

Corollary 3.2. *Every simple commutator word implies virtual nilpotency.*

A word w is said to be boundedly concise in a class of groups C if for every integer m there exists a number $\nu = \nu(C, w, m)$ such that whenever $|G_w| \leq m$ for a group $G \in C$ it always follows that $|w(G)| \leq \nu$. According to [5, Theorem 1.2], words implying virtual nilpotency are boundedly concise in residually finite groups. This, together with Corollary 3.2, yields the following corollary.

Corollary 3.3. *Every simple commutator word is boundedly concise in the class of residually finite groups.*

Proof of Corollary 1.3. Of course, $w(G)$ is finite, by Corollary 3.2. Let us show that $w(G)$ is nilpotent whenever $|ab| = |a||b|$ for any $a, b \in G_w$ of coprime orders. The converse is clear.

Since G is residually finite, there exists a normal subgroup N of G such that $N \cap w(G) = 1$ and G/N is finite. Notice that for any w-value $xN \in G/N$, we have $x \in G_w$ and $|xN| = |x|$. It follows that G/N satisfies the hypotheses of Theorem 1.2 and therefore $w(G/N) \simeq w(G)$ is nilpotent. □

4. Examples

In this section we collect some examples showing that Theorem 1.2 does not hold for an arbitrary commutator word.

Example 4.1. Let $w = [x, y]^3$ and let $G = (S \times S) \rtimes C$, where S is the symmetric group of degree 3, $C = \langle g \rangle$ is the cyclic group of order 2 and the action is given by $(a, b)^g = (b, a)$ for every $(a, b) \in S \times S$. Then every nontrivial w-value has order 2 and $w(G) = S \times S$.

Example 4.2. Let $w = [x, y^{10}, y^{10}, y^{10}]$ and let G be the alternating group of degree 5. Then G_w consists of the identity and all products of two transpositions. In particular, $w(G) = G$.

Proof. Of course, G_w is the set of all commutators $[g, h, h, h]$, where $g, h \in G$ and h is a 3-cycle. Since $[g, h, h] = (h^{-1})^{[g,h]}h$,

$$[g, h, h, h] = [(h^{-1})^{[g,h]}, h]^h = [(h^{-1})^{(h^{-1})^g h}, h]^h = [(h^{-1})^k, h]^{h^{-1}},$$

where $k = (h^{-1})^g$ is a 3-cycle. For any 3-cycles $h, k \in G$, we claim that $[(h^{-1})^k, h]$ is either trivial or a product of two transpositions, from which it follows that so is $[(h^{-1})^k, h]^{h^{-1}}$.

Let $h = (a\ b\ c)$ and $k = (d\ e\ f)$. Clearly, we may assume that $a = d$. Furthermore, it is enough to consider the cases:

(1) $b = e$ and $c \neq f$;

(2) $e, f \notin \{b, c\}$.

https://doi.org/10.1017/S0004972719000054 Published online by Cambridge University Press

The other (nontrivial) cases can be deduced by applying the identities

$$[(h^{-1})^k, h]^{-(h^{-1})^k h^{-1} k^{-1}} = [((h^{-1})^k)^{-1}, h^{-1}]^{-k^{-1}} = [(h^{-1})^{k^{-1}}, h],$$

$$[(h^{-1})^k, h]^{h^k k^{-1}} = [(h^k)^{-1}, h]^{h^k k^{-1}} = [h^k, h]^{-k^{-1}} = [h^{k^{-1}}, h].$$

Now, in the first case, h and k belong to the alternating group of degree 4 and therefore $[(h^{-1})^k, h]$ is the product of two transpositions. In the second case,

$$[(h^{-1})^k, h] = [(a\ c\ b)^{(a\ e\ f)}, (a\ b\ c)] = [(b\ e\ c), (a\ b\ c)] = (a\ c)(b\ e).$$

This proves our claim. Also, it implies that G_w contains all products of two transpositions. □

Example 4.3 (see [9, Theorem 1.2]). For every $n \geq 7$, there exists a commutator word v such that, for $w = v^{10}$, the set of w-values of the alternating group of degree n consists of the identity and all 3-cycles.

References

[1] R. Bastos, C. Monetta and P. Shumyatsky, 'A criterion for metanilpotency of a finite group', *J. Group Theory* **21**(4) (2018), 713–718.

[2] R. Bastos and P. Shumyatsky, 'A sufficient condition for nilpotency of the commutator subgroup', *Sib. Math. J.* **57**(5) (2016), 762–763.

[3] B. Baumslag and J. Wiegold, 'A sufficient condition for nilpotency in a finite group', Preprint, 2014, arXiv:1411.2877 [math.GR].

[4] D. M. Bloom, 'The subgroups of PSL$(3, q)$ for odd q', *Trans. Amer. Math. Soc.* **127** (1967), 150–178.

[5] E. Detomi, M. Morigi and P. Shumyatsky, 'Words of Engel type are concise in residually finite groups', *Bull. Math. Sci.* (2018), to appear.

[6] D. Gorenstein, *Finite Groups*, 2nd edn (Chelsea, New York, 1980).

[7] B. Huppert, *Endliche Gruppen I* (Springer, Berlin–New York, 1967).

[8] I. M. Isaacs, *Finite Group Theory*, Graduate Studies in Mathematics, 92 (American Mathematical Society, Providence, RI, 2008).

[9] M. Kassabov and N. Nikolov, 'Words with few values in finite simple groups', *Q. J. Math.* **64**(4) (2013), 1161–1166.

[10] D. J. S. Robinson, *Finiteness Conditions and Generalized Soluble Groups, Parts 1 and 2* (Springer, Berlin, 1972).

[11] M. Suzuki, 'On a class of doubly transitive groups', *Ann. of Math. (2)* **75** (1962), 105–145.

[12] M. Suzuki, *Group Theory I* (Springer, Berlin–New York, 1982).

[13] J. G. Thompson, 'Nonsolvable finite groups all of whose local subgroups are solvable', *Bull. Amer. Math. Soc.* **74** (1968), 383–437.

[14] A. Turull, 'Fitting height of groups and of fixed points', *J. Algebra* **86**(2) (1984), 555–566.

CARMINE MONETTA, Dipartimento di Matematica,
Università di Salerno, Via Giovanni Paolo II, 132,
84084 Fisciano (SA), Italy
e-mail: cmonetta@unisa.it

https://doi.org/10.1017/S0004972719000054 Published online by Cambridge University Press

ANTONIO TORTORA, Dipartimento di Matematica,
Università di Salerno, Via Giovanni Paolo II, 132,
84084 Fisciano (SA), Italy
and
Dipartimento di Matematica e Fisica,
Università della Campania 'Luigi Vanvitelli' Viale Lincoln, 5,
81100 Caserta (CE), Italy
e-mail: antortora@unisa.it, antonio.tortora@unicampania.it