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LINEAR OPERATORS PRESERVING THE REAL SYMPLECTIC GROUP

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1. Introduction and statement of results. Let F be a field and $M_n(F)$ the $n \times n$ matrices over F. Set $GL(n, F)$ to be the units in $M_n(F)$. Suppose H is a subgroup of $GL(n, F)$ and L is an F-linear operator on $M_n(F)$ mapping H into itself. Can L be neatly characterized?

Marcus [**2**] answered this question when F is the complex numbers $\mathbf{C}$ and H the unitary group. The set of such L is a group and L has the form $L(A) = UAV$ or $L(A) = UA'V$ where U, V are fixed unitary matrices and A' is the transpose of A. This result is also valid if F is a subfield of $\mathbf{C}$ having at least three roots of unity.

Marcus and Purves [**4**] got a similar answer when $H = GL(n, \mathbf{C})$, namely, for fixed U, V in H, $L(A) = UAV$ or $L(A) = UA'V$. If, however, F is the real field R and H the orthogonal group, then L need not be invertible if $n = 2$, 4 or 8 [**5**]. If L is invertible, one gets the expected answer. There are similar difficulties for $H = GL(n, F)$ and F not algebraically closed.

For a discussion of work done when L is required to preserve other invariants (e.g., elementary summetric functions) see [**3**].

We are going to do the real symplectic group $Sp(2n, R)$. The fact that we are over the real field is used, but perhaps the proof can be modified to take care of other fields.

Dieudonne [**1**] determined the automorphisms of many of the classical groups. It seems possible that any invertible linear operator mapping the identity to the identity and preserving an irreducible classical group must be a combination of a group automorphism and the transpose map. All known results confirm this, but no connection between the two types of problems has been found. We do have occasion to use the Fundamental Theorem of Projective Geometry in this paper, but it does not play a central role; we use it mostly to lessen the computation.

We set $H = Sp(2n, R)$, and unless otherwise noted, $2n \times 2n$ matrices are written $\begin{bmatrix} A & B \\ C & D \end{bmatrix}$, with A, B, C, D $n \times n$.

MAIN THEOREM. *Let $L : M_{2n}(R) \to M_{2n}(R)$ be linear and let $L(H) \subset H$. Assume $L(I_{2n}) = I_{2n}$. Then the set of all such L is a group and this group is*

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generated by the transpose map, similarity by $(I_n \dotplus -I_n)$ *and the inner automorphisms of* H.

2. Preliminary remarks and notation. Let $J = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}$. Then $M = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in H$ if and only if $M'JM = J$ or, equivalently, if $AD' - BC' = I = A'D - C'B$ and AB', CD', $A'C$, $B'D$ are all symmetric. Let E_{ij} be the matrix with 1 in the (i, j) position and 0 elsewhere. Note that $U \dotplus U$ and $\begin{bmatrix} 0 & U \\ -U & 0 \end{bmatrix}$ are in H if U is orthogonal.

3. Preliminary results. The lemmas in this section will be used in the proof of the main theorem.

LEMMA 1. *Let T be a linear map on the space of real $n \times n$ symmetric matrices. Suppose $T(I_n) = I_n$ and T preserves invertibility. Then there is a fixed orthogonal U such that $T(A) = U'AU$ for all symmetric A.*

Proof. Let A be symmetric and $\lambda_1, \ldots, \lambda_n$ the roots of A. Then all λ_i are real and $T(xI_n - A)$ is invertible unless x is one of the λ_i. It follows that idempotents are mapped to idempotents. Since $E_{ii} + E_{jj}$ is idempotent if $i \neq j$, we have $TE_{ii}TE_{jj} = TE_{jj}TE_{ii} = 0$. Thus, to within orthogonal similarity, we may assume all the TE_{ii} are diagonal.

Suppose $TE_{ii} = 0$, $i = r + 1, \ldots, n$ and $TE_{ii} \neq 0$ if $i \leqq r$. Then $\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$ is mapped to I_n. Let A be $n - r \times n - r$ symmetric invertible. Then if $x \neq 0$, $T\begin{bmatrix} xI_r & 0 \\ 0 & A \end{bmatrix}$ is invertible. Thus

$$\begin{bmatrix} 0 & 0 \\ 0 & A \end{bmatrix} \in \ker T \text{ and hence } \begin{bmatrix} 0 & 0 \\ 0 & A \end{bmatrix} \in \ker T$$

for all $n - r \times n - r$ symmetric A.

Consider a matrix of the form $A = \begin{bmatrix} 0 & B \\ B' & 0 \end{bmatrix}$ where B is $r \times n - r$, has one 1 and all other entries 0. The eigenvalues of this matrix are $0, 1, -1$. Thus $0, 1, -1$ contain the eigenvalues of TA. Let $A_1 = \begin{bmatrix} 0 & B \\ B' & E \end{bmatrix}$ where E has one 1 and all other entries 0 and the 1 is in the same column as the 1 in B. The eigenvalues of A_1 are 0, $(1 \pm \sqrt{5})/2$. But $TA_1 = TA$. Thus $TA = 0$, and it follows that any matrix of the form $\begin{bmatrix} 0 & B \\ B' & 0 \end{bmatrix} \in \ker T$.

Consider the matrix diag $(1, 2, \ldots, r, 0, \ldots, 0)$. Its image, say S, has roots from among 1, 2, . . . , r. However

$$T\left[\begin{array}{ccc|ccc} 1 & & & 1 & & \\ & & & & & \\ & & r & & & 0 \\ \hline 1 & & & & & \\ & & & & & \\ & & 0 & 0 & & \end{array}\right] = S$$

and hence 1 is eliminated as a root of S. Similarly, we can eliminate 2, 3, . . . , r as eigenvalues of S. But S is invertible and hence we have a contradiction.

Thus we may assume that rank one idempotents are mapped to rank one idempotents, and, in particular, that $TE_{ii} = E_{ii}$, $i = 1, \ldots, n$. If $\langle v \rangle$ is any line of the underlying space V, then there is exactly one rank one symmetric idempotent fixing $\langle v \rangle$. Thus T sets up a correspondence between the lines of V. Let P be a plane in V and $\langle v \rangle$, $\langle w \rangle$ be two orthogonal lines in P. Let E_v, E_w be the corresponding idempotents. Then TE_v, TE_w are still rank one independent idempotents fixing orthogonal lines. It follows that this correspondence between lines is 1-1 and preserves orthogonality. Thus, if $n \geqq 3$, we can use the Fundamental Theorem of Projective Geometry to see that this correspondence between lines is induced by a linear map in projective space. Since we assume to within orthogonal similarity that the E_{ii} are fixed, this map is the identity and the lemma is proved. (If $n = 2$, the computations are easy.)

LEMMA 2. *Let T be a linear operator mapping the $n \times n$ real symmetric matrices into $M_m(R)$. Assume $T(I_n) = I_m$, T preserves invertibility and T maps idempotents to idempotents. Then $m \geqq n$.*

Proof. Use induction on n. If $n = 2$, it is easy, so assume $n \geqq 3$. We may assume that $T(E_{11}) \neq 0$ and since the problem is unaltered under change of basis, assume $T(E_{11}) = I_r \dotplus 0$. Let A be any $n - 1 \times n - 1$ symmetric idempotent and set $X = 0 \dotplus A$. Since E_{11}, X, and $E_{11} + X$ are idempotent, $T(X) = 0 \dotplus B$, where B is $m - r \times m - r$ idempotent. Symmetric idempotents span the symmetric matrices, and hence any matrix of the form $0 \dotplus A$, where A is $n - 1 \times n - 1$ is mapped to $0 \dotplus B$, where B is $m - r \times m - r$. The linear correspondence $A \rightarrow B$ satisfies the inductive hypothesis and we are done, unless $T(E_{11}) = I_m$.

If $T(E_{11}) = I_m$, let $A^2 = A$ where A is $n - 1 \times n - 1$ symmetric. Set $X = 0 \dotplus A$. Then $T(E_{11} + X) = I_m + T(X)$ is idempotent. Thus $T(X) = 0$. Thus $0 \dotplus A \in \ker T$, where A is $n - 1 \times n - 1$ symmetric.

Let S be any symmetric matrix with only λ, μ for eigenvalues, and with $\lambda \neq \mu$. Then $(S - \mu I)/(\lambda - \mu)$ is idempotent and hence $T(S)$ has λ, μ as its only roots. Put

$$X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \dotplus I_{n-2}, \quad Y = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \dotplus \theta I_{n-2}, \quad \text{where } \theta = (1 + \sqrt{5})/2.$$

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Then $T(X)$ has ± 1 for its only roots and $T(Y)$ has $(1 \pm \sqrt{5})/2$ for its only roots. Since $T(X) = T(Y)$, we have proved Lemma 2.

LEMMA 3. *Let T be a linear map of the $n \times n$ symmetric matrices into $M_n(R)$. Assume T preserves invertibility, $T(I_n) = I_n$ and T maps idempotents to idempotents. Then T is a similarity.*

Proof. Use induction on n. We do $n = 2$ later. Since idempotents are mapped to idempotents, we may assume that $T(E_{ii})$ are all diagonal. If $T(E_{nn}) = 0$, then some $T(E_{ii})$ has rank larger than 1. Assume $T(E_{11}) = I_r \dot{+} 0$, $r \geqq 2$. Suppose A is $n - 1 \times n - 1$ invertible symmetric. We can show as in Lemma 2 that $T(0 \dot{+} A)$ has the form $0 \dot{+} B$, where B is $n - r \times n - r$ symmetric invertible. This is a contradiction by Lemma 2.

Thus, to within conjugation, $T(E_{11}) = E_{11}$. If A is $n - 1 \times n - 1$ symmetric, and $A^2 = A$, then $T(0 \dot{+} A) = 0 \dot{+} B$, with B $n - 1 \times n - 1$, and $B^2 = B$. The correspondence $A \rightarrow B$ satisfies the inductive hypothesis and hence we assume $0 \dot{+} A$ is fixed by T, when A is $n - 1 \times n - 1$. Hence, $T(E_{ii}) = E_{ii}$ for all i, and thus all diagonal matrices are fixed.

Let A be any $n - 1 \times n - 1$ symmetric matrix. Then $T(A \dot{+} 0) = B \dot{+} 0$, with B $n - 1 \times n - 1$. Moreover, since E_{nn} is fixed, there is an $n - 1 \times n - 1$ S such that $S^{-1}BS = A$ for all symmetric A. But T fixes all diagonal matrices and hence S is diagonal. Write $S = \text{diag}\ (s_1, \ldots, s_n)$. Clearly $\text{diag}\ (s_2, \ldots, s_{n-1})$ commutes with all symmetric matrices and hence $s_2 = \ldots = s_{n-1}$. Replacing E_{11}, E_{nn} with E_{ii}, E_{jj}, we have $s_k = s_l$ for $k, l \neq i, j$. Thus if $n \geqq 4$, S is scalar and the lemma is proved.

If $n = 2$, $T\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ is an involution. Write it as $\begin{bmatrix} x & y \\ z & -x \end{bmatrix}$. Since $\begin{bmatrix} x + k & y \\ z - x & \end{bmatrix}$ is invertible for all k, $x = 0$ and

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 0 & y \\ y^{-1} & 0 \end{bmatrix}.$$

A similarity with $\begin{bmatrix} 1 & 0 \\ 0 & y \end{bmatrix}$ finishes the proof.

If $n = 3$, let B be the image of $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \dot{+} 0$. Since $(B \pm E_{33})^2 = I_3$, $b_{3i} = b_{j3} = 0$. Since $B + kE_{11} + E_{33}$ is invertible for all k, $b_{11} = b_{22} = 0$. Thus

$$B = \begin{bmatrix} 0 & a \\ a^{-1} & 0 \end{bmatrix} \dot{+} 0.$$

A similarity with $\text{diag}\ (a, 1, 1)$ allows us to assume $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \dot{+} 0$ is fixed. Also,

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 0 & 0 & b \\ 0 & 0 & 0 \\ b^{-1} & 0 & 0 \end{bmatrix},$$

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and we are done if we show $b = 1$. Let J_3 be the matrix with all entries 1. Let $X = T(J_3)$. Then $X^3 = 3X$ and since

$$X = \begin{bmatrix} 1 & 1 & b \\ 1 & 1 & 1 \\ b^{-1} & 1 & 1 \end{bmatrix},$$

we must have $b = 1$. Lemma 3 is proved.

4. Proof of the main theorem. We do the proof in a series of propositions.

PROPOSITION 1. *Let* $X = \begin{bmatrix} 0 & S \\ 0 & 0 \end{bmatrix}$ *where S is $n \times n$ symmetric invertible. Then LX is symplectically similar to a matrix of the form* $\begin{bmatrix} 0 & D \\ 0 & 0 \end{bmatrix}$, *where* $D = I_r \dotplus - I_{n-r}$.

Proof. Clearly $I_{2n} + kX \in H$ for all real k. Thus LX has the same property. Set $Y = LX$. Then from $(I + kY)J(I + kY') = J$ for all k, we obtain $YJ + JY' = YJY' = Y^2 = 0$. Thus

$$(1) \quad Y = \begin{bmatrix} A & B \\ C & -A' \end{bmatrix}$$

where AB, B, C, CA are symmetric and $A^2 + BC = 0$. Thus rank $Y \leqq n$. Now $Y_1 = L\begin{bmatrix} 0 & 0 \\ -S' & 0 \end{bmatrix}$ has the form (1), and $Y + Y_1 \in H$. Thus, rank $Y +$ rank $Y_1 = 2n$. The elementary divisors of $I_{2n} + Y$ are $(x - 1)^2$ n times.

Consider all members of H with $(x - 1)^2$ n times for elementary divisors. Williamson [**6**, p. 613] has a result to tell us when two of these are conjugate in H. To each elementary divisor is associated (in a prescribed way) a sign, $+$ or $-$, and two such members are conjugate if and only if they both have the same number of $+$ signs. Thus members of H with $(x - 1)^2$ for elementary divisors break up into at most $n + 1$ different conjugacy classes in H. Let $D_r = I_r \dotplus -I_{n-r}$, $r = 0, \ldots, n$. Then no two of $\begin{bmatrix} 0 & D_r \\ 0 & 0 \end{bmatrix}$ are conjugate in H. If they were, then

$$\begin{bmatrix} P & Q \\ R & S \end{bmatrix}\begin{bmatrix} 0 & D_r \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & D_s \\ 0 & 0 \end{bmatrix}\begin{bmatrix} P & Q \\ R & S \end{bmatrix} \quad \text{for} \begin{bmatrix} P & Q \\ R & S \end{bmatrix} \in H.$$

This forces $R = 0$ and $S = P'^{-1}$, and hence $PD_rP' = D_s$, a contradiction if $r \neq s$. This proves Proposition 1.

Thus we may now assume that

$$L\begin{bmatrix} 0 & I \\ 0 & 0 \end{bmatrix} = Y = \begin{bmatrix} 0 & D_r \\ 0 & 0 \end{bmatrix}.$$

Do a similarity with $I_n \dot{+} -I_n$ if necessary so that we can assume $r \geqq n - r$. At this point, we had better have $D_r = I_n$; if not, the Main Theorem is false.

PROPOSITION 2. *In fact,* $D_r = I_n$.

Proof. Let S be invertible symmetric. Then

$$L\begin{bmatrix} 0 & S \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} A & B \\ C & -A' \end{bmatrix}$$

as in (1). Now $\begin{bmatrix} A & B + D_r \\ C & -A' \end{bmatrix}$ also has the form (1) and hence $C = 0$ and $A^2 = 0$. It is also clear that for large enough real k, $\begin{bmatrix} A & B + kD_r \\ 0 & -A' \end{bmatrix}$ has rank n. Let W_1, W_2 be matrices which transform A, A' respectively to Jordan form:

$$W_1^{-1}AW_1 = \left[\dot{\sum_{i=1}^{z}} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right] \dot{+}\, 0 \quad \text{and } W_2^{-1} - A'W_2 =$$

$$\left[\dot{\sum_{i=1}^{z}} \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\right] \dot{+}\, 0.$$

Then $W_1^{-1}(B + kD_r)W_2$ still has full rank for all large k. Thus

$$\begin{bmatrix} W_1^{-1}AW_1 & W_1^{-1}(B + kD_r)W_2 \\ 0 & W_2^{-1}AW_2 \end{bmatrix} \tag{2}$$

has rank n for all large k. But some judicious row and column operations will easily show that the rank of (2) can be larger than n unless $A = 0$. Thus $L\begin{bmatrix} 0 & S \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & S_1 \\ 0 & 0 \end{bmatrix}$ for all symmetric S and the correspondence preserves invertibility. Call this linear map ϕ_1.

Next, let

$$L\begin{bmatrix} 0 & 0 \\ -S^{-1} & 0 \end{bmatrix} = \begin{bmatrix} A & B \\ C & -A' \end{bmatrix}$$

when S is $n \times n$ symmetric invertible. If $\phi_1(S) = S$, then $\begin{bmatrix} A & B + S_1 \\ C & -A' \end{bmatrix} \in H$, and since $A^2 + BC = 0$, $C = -S_1^{-1}$. Call the linear correspondence induced by $S^{-1} \to -S_1^{-1}$, ϕ_2. Invertibility is preserved by ϕ_1, ϕ_2. Define linear maps ψ_1, ψ_2 on the $n \times n$ symmetric matrices by $\psi_1(S) = \phi_1(S)D_r$ and $\psi_2(S) = D_r\phi_2(S)$. Then $\psi_1(I_n) = \psi_2(I_n) = I_n$ and ψ_1, ψ_2 preserve invertibility. Furthermore, if S is invertible, and $\psi_1(S) = S_1$, then $\psi_2(S^{-1}) = S_1^{-1}$.

Let S be any symmetric matrix whose only eigenvalues are $(1 \pm \sqrt{5})/2$. Then $S^2 - S - I_n = 0$, i.e., $S - S^{-1} = I_n$. Let $\psi_1(S) = S_1$. Then $\psi_2(S^{-1}) = S_1^{-1}$. Since $(S - I)(S^{-1} + I) = I$, we have

$$\psi_1(S - I)\psi_2(S^{-1} + I) = I, \quad \text{or } (S_1 - I)(S_1^{-1} + I) = I.$$

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Thus, $S_1^2 - S_1 - I = 0$ and $(1 \pm \sqrt{5})/2$ are the only possible eigenvalues of S_1. It follows that ψ_1 maps idempotents to idempotents, and hence we may use Lemma 3 to conclude that for some $X \in M_n(R)$, $\psi_1(S) = X^{-1}SX$ for all symmetric S. Thus $\phi_1(S)D_r = X^{-1}SX$.

Now ϕ_1 maps the symmetric matrices into themselves, and thus ψ_1 maps the symmetric matrices into the space

$$\mathscr{U} = \begin{bmatrix} P & Q \\ -Q' & R \end{bmatrix},$$

where P is $r \times r$ symmetric, and R is $n - r \times n - r$ symmetric. Since ψ_1 is invertible, $\mathscr{U}$ = range ψ_1. But ψ_1 is a similarity and hence $\mathscr{U}$ contains no non-zero skew-symmetric matrices. Thus, $r = n$ and $D_r = I_n$.

PROPOSITION 3. *By conjugation in H, we may assume* $\begin{bmatrix} 0 & S \\ S_1 & 0 \end{bmatrix}$ *is fixed by L, where S, S_1 are arbitrary symmetric matrices.*

Proof. If S is $n \times n$ symmetric, then $\psi_1(S)$ is symmetric. By Lemma 1, $\psi_1(S) = U'SU$ for some orthogonal U. Since $U \dotplus U \in H$, we assume ψ_1 is the identity.

Suppose S is symmetric invertible. Let

$$L\begin{bmatrix} 0 & 0 \\ S & 0 \end{bmatrix} = \begin{bmatrix} A & B \\ C & -A' \end{bmatrix}.$$

Then $\begin{bmatrix} A & B \\ C + I & -A' \end{bmatrix}$ has the form (1). Thus, A is symmetric, and $B = 0$. Now $A^2 = 0$, so $A = 0$ and $\begin{bmatrix} 0 & -S^{-1} \\ C & 0 \end{bmatrix} \in H$. Hence $C = S$ and Proposition 3 is proved.

Let X be any matrix of the form (1). We now know that $L(X)$ has the same rank as X. Let $\mathscr{R}$ be those X of the form (1) such that $I_{2n} + X$ is a transvection. Then X has rank 1. Suppose $I + X$ is a transvection in the direction of the vector v. Let $v, v_2, \ldots, v_{2n}$ be a basis of R^{2n}. We can make this change of basis by a member of H. Then

$$X = \begin{bmatrix} A & B \\ C & -A' \end{bmatrix} = \left[\begin{array}{c|c} 0 & bE_{11} \\ \hline 0 & 0 \end{array}\right],$$

where $b \in R$. It follows that there is a 1-1 correspondence between these special transvections in projective space and the lines of R^{2n}.

PROPOSITION 4. *Consider the space* $\begin{bmatrix} A & B \\ C & -A' \end{bmatrix}$, *$B$, C symmetric, A arbitrary.*

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There are two possibilities; either L fixes this space or for all members

$$L\begin{bmatrix} A & B \\ C & -A' \end{bmatrix} = \begin{bmatrix} -A & B \\ C & A' \end{bmatrix}.$$

Proof. Put $A = E_{ij}$, $B = E_{ii}$, $C = -E_{jj}$. Then

$$L(A \dotplus -A') \pm \begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix} \in \mathscr{R},$$

and this forces $L(A \dotplus -A') = \pm(A \dotplus -A')$. Thus, $L(A \dotplus -A') = X \dotplus Y$ for any A.

Let $\langle v \rangle$ be a line in $\langle v_1, \ldots, v_n \rangle$, where $v_1, \ldots, v_{2n}$ is our symplectic basis. Consider the set of all rank one matrices A fixing $\langle v \rangle$. The set of all these such that symmetric B and C exist such that $\begin{bmatrix} A & B \\ C & -A' \end{bmatrix} \in \mathscr{R}$ is a one-dimensional space. Now

$$L\begin{bmatrix} A & B \\ C & -A' \end{bmatrix} = \begin{bmatrix} X & B \\ C & -X' \end{bmatrix} \in \mathscr{R};$$

thus $X = \alpha A$. Hence, the correspondence $A \rightarrow X$ is scalar, and this proves Proposition 4.

If $L\begin{bmatrix} A & B \\ C & -A' \end{bmatrix}\begin{bmatrix} -A & B \\ C & A' \end{bmatrix}$, do a similarity with $\begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}$ and then the transpose map. Thus, we may assume the above space is fixed by L.

PROPOSITION 5. *If S is $n \times n$ symmetric, then $S \dotplus S$ is fixed by L.*

Proof. Let $S = -1 \dotplus I_{n-1}$. Then

$$S \dotplus S, \qquad \begin{bmatrix} S & I \\ 0 & S \end{bmatrix}, \qquad \begin{bmatrix} S & 0 \\ I & S \end{bmatrix}$$

are in H and hence $L(S \dotplus S) = S_1 \dotplus S_2$ with S_1, S_2 symmetric. If A is symmetric and SA is symmetric, then S_1A and AS_2 are symmetric. If follows that $S_1 = S_{11} \dotplus kI_{n-1}$, and $S_2 = S_{11}^{-1} \dotplus k^{-1}I_{n-1}$. Now $S \dotplus -S$ is fixed and $\begin{bmatrix} 2S & I \\ -I & 0 \end{bmatrix} \in H$. Since $L(2S \dotplus 0) = (s_{11} - 1) \dotplus (k+1)I_{n-1} \dotplus (s_{11}^{-1} + 1) \dotplus (k-1)I_{n-1}$, we have $s_{11} = \pm 1$, $k = \pm 1$. Thus $S_1 = \pm I$, or $\pm S$. In any case, $S_2 = S_1$. If $S_1 = \pm I$, then

$$\begin{bmatrix} 2S & 0 \\ 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} S \pm I & 0 \\ 0 & -S \pm 1 \end{bmatrix}$$

and thus

$$\begin{bmatrix} S \pm I & I \\ -I & -S \pm I \end{bmatrix} \in H,$$

a contradiction.

Thus the correspondence $S \to S_1$ maps every reflection to itself or its negative. If $n \geqq 3$, the fundamental theorem of projective geometry implies $S = S_1$ for all S. If $n = 2$, and $S \neq S_1$ for all S, then

$$\text{if } S = \begin{bmatrix} a & b \\ b & c \end{bmatrix}, \quad S = \begin{bmatrix} c & \pm b \\ \pm b & a \end{bmatrix} \text{ or } \begin{bmatrix} a & \pm b \\ \pm b & c \end{bmatrix}.$$

In particular,

$$\left[\begin{array}{cc|cc} 2 & 1 & w+x & x \\ 1 & 1 & x & w \\ \hline & & 1 & -1 \\ \multicolumn{2}{c|}{0} & -1 & 2 \end{array}\right] \in H,$$

and this eliminates the possibility $S \neq S_1$. Proposition 5 is proved.

Let us summarize. If A, B, C, D are arbitrary $n \times n$ symmetric matrices, then $\begin{bmatrix} A & B \\ C & D \end{bmatrix}$ is fixed by L.

PROPOSITION 6. *Under the above assumptions, any matrix is fixed by L.*

Proof. We will show that the matrix

$$A = \begin{bmatrix} 2 & 1 \\ -1 & -1 \end{bmatrix} \dotplus I_{n-2} \dotplus \begin{bmatrix} 1 & -1 \\ 1 & -2 \end{bmatrix} \dotplus I_{n-2}$$

is fixed. By Propositions 4, 5, this will imply that

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \dotplus O_{n-2} \dotplus \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \dotplus O_{n-2}$$

is fixed, and since by Proposition 4, $A \dotplus A$ is fixed if A is skew-symmetric, we will have $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} + O_{2n-2}$ fixed by L. Hence $A \dotplus B$ is fixed by L, where A, B are arbitrary. The proof that $\begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}$ is fixed by L is done the same way.

Let

$$A = \begin{bmatrix} 2 & 1 \\ -1 & -1 \end{bmatrix} \dotplus I_{n-2}, \quad \text{and } A_2 = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} \dotplus I_{n-2}.$$

If S is any matrix of the form $\begin{bmatrix} -3b-c & b \\ b & c \end{bmatrix} \dotplus S_1$ with S_1 arbitrary symmetric, then $\begin{bmatrix} A_1 & S \\ 0 & A_2 \end{bmatrix}$, $\begin{bmatrix} A_1 & 0 \\ S & A_2 \end{bmatrix}$ are in H. It follows that $L(A) = B_1 \dotplus B_2$ where

$$B_1 = \begin{bmatrix} 3x+z & x \\ -x & z \end{bmatrix} \dotplus kI_{n-2} \quad \text{and} \quad B_2 = \begin{bmatrix} 3x+z & -x \\ x & z \end{bmatrix}^{-1} \dotplus k^{-1} I_{n-2}.$$

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Now $B_1 \dotplus \text{diag}\ (0, 0, 1, \ldots, 1) \dotplus B_2 + \text{diag}\ (0, 0, -\frac{1}{2}, \ldots, -\frac{1}{2})$ and $B_1 + \text{diag}\ (0, 0, 2, \ldots, 2) \dotplus B_2 + \text{diag}\ (0, 0, -\frac{2}{3}, \ldots, -\frac{2}{3})$ are in H and hence $k = 1$.

Next, note that

$$\left[\begin{bmatrix} 2 & 1 \\ -1 & -1 \end{bmatrix} - I_2\right]\left[\begin{bmatrix} 1 & -1 \\ 1 & -2 \end{bmatrix} + I_2\right]' = I_2.$$

Any matrix A such that $(A - I)(A^{-1} + I) = I$ has characteristic polynomial $x^2 - x - 1 = 0$. Thus $3x + 2z = 1$, and $3xz + z^2 + x^2 = -1$.

Do the same procedure with A'. It's image will have the form

$$\begin{bmatrix} -3x_1 + z & x_1 \\ -x_1 & z_1 \end{bmatrix} \dotplus I_{n-2} \dotplus \begin{bmatrix} -3x_1 + z_1 & -x_1 \\ x_1 & z_1 \end{bmatrix}^{-1} \dotplus I_{n-2}.$$

We know $A + A'$ is fixed by L, so $x + x_1 = 0$ and $z + z_1 = -2$, and hence $3x + z_1 + 3x + z = 6x - 2 = 4$. Thus $x = 1$ and Proposition 6 and the main result are proved.

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