15
Weak interactions

In the 1960s and early 1970s a theory was developed by Glashow [1], Weinberg [2], Salam [3], 't Hooft [4], and others that unified the weak and electromagnetic interactions. This theory is presently in accord with all experimental information. It is not our purpose here to go into a detailed exposition of the model or the history of weak interaction physics. Rather, we want to show that the spontaneously broken gauge symmetry that is the cornerstone of the theory can be restored in a phase transition at a critical temperature of order 100 GeV. The existence and order of this transition depend on details that we shall discuss in this chapter.

15.1 Glashow–Weinberg–Salam model

We begin with a theory involving bosons only. The essence of the model can be found without the inclusion of fermions: they will be added later. The Lagrangian is

$$\begin{aligned}\mathcal{L} = (D_\mu\Phi)^\dagger (D^\mu\Phi) + c^2\Phi^\dagger\Phi - \lambda\left(\Phi^\dagger\Phi\right)^2 \\ - \tfrac{1}{4}g^{\mu\nu}g_{\mu\nu} - \tfrac{1}{4}f_a^{\mu\nu}f^a_{\mu\nu}\end{aligned} \tag{15.1}$$

This Lagrangian has an SU(2) × U(1) symmetry. There is an SU(2) gauge field A^a_μ and a U(1) gauge field B_μ. The field strengths are

$$f^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - g\epsilon^{abc}A^b_\mu A^c_\nu \tag{15.2}$$

$$g_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu \tag{15.3}$$

There is a covariant derivative

$$D_\mu = \partial_\mu + \tfrac{1}{2}igA^a_\mu\tau^a + \tfrac{1}{2}ig'B_\mu \tag{15.4}$$

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which acts on a complex SU(2) field

$$\Phi = \frac{1}{\sqrt{2}} \begin{pmatrix} \phi_1 + i\phi_2 \\ \phi_3 + i\phi_4 \end{pmatrix} \tag{15.5}$$

Note that, according to (15.1), A^a_μ and B_μ are massless spin-1 bosons and, if $c^2 > 0$, Φ is a tachyon. Thus we should expect spontaneous symmetry breaking. Altogether, there are apparently 12 spin degrees of freedom.

Owing to the gauge symmetry we may choose, without loss of generality, the vacuum expectation value

$$\langle \Phi \rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v \end{pmatrix} \tag{15.6}$$

where v is a real constant. Then, for arbitrary Φ, we write

$$\Phi = \frac{1}{\sqrt{2}} U^{-1}(\boldsymbol{\zeta}) \begin{pmatrix} 0 \\ v + \eta \end{pmatrix} \tag{15.7}$$

where $\boldsymbol{\zeta}(\mathbf{x}, t)$ and $\eta(\mathbf{x}, t)$ are the independent fields and

$$U(\boldsymbol{\zeta}) = \exp\left(\frac{-i\boldsymbol{\zeta} \cdot \boldsymbol{\tau}}{2v} \right) \tag{15.8}$$

This is the so-called unitary, or U, gauge. It is a useful gauge since it makes the particle content of the theory manifest.

Now let

$$\Phi \to \Phi' = U(\boldsymbol{\zeta})\Phi = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v + \eta \end{pmatrix} \tag{15.9}$$

This is just a particular SU(2) gauge transformation, such that

$$\begin{aligned} B_\mu &\to B_\mu \ , \\ \boldsymbol{\tau} \cdot \boldsymbol{A}_\mu \to \boldsymbol{\tau} \cdot \boldsymbol{A}'_\mu &= U(\boldsymbol{\zeta}) \left(\boldsymbol{\tau} \cdot \boldsymbol{A}_\mu - \frac{i}{g} U^{-1}(\boldsymbol{\zeta}) \partial_\mu U(\boldsymbol{\zeta}) \right) U^{-1}(\boldsymbol{\zeta}) \end{aligned} \tag{15.10}$$

After some algebra, the Lagrangian is expressed in terms of the independent fields as

$$\begin{aligned} \mathcal{L} = {} & \tfrac{1}{2}(\partial_\mu \eta)(\partial^\mu \eta) + \tfrac{1}{2} c^2 (v + \eta)^2 - \tfrac{1}{4} \lambda (v + \eta)^4 \\ & + \tfrac{1}{4} \Phi'^\dagger (g' B_\mu + g \boldsymbol{\tau} \cdot \boldsymbol{A}_\mu)(g' B^\mu + g \boldsymbol{\tau} \cdot \boldsymbol{A}^\mu) \Phi' \\ & - \tfrac{1}{4} g^{\mu\nu} g_{\mu\nu} - \tfrac{1}{4} f^{\mu\nu}_a f^a_{\mu\nu} \end{aligned} \tag{15.11}$$

This can be written as the sum of a classical part, $\mathcal{L}_{\text{cl}}$, a part quadratic in the fields, $\mathcal{L}_{\text{quad}}$, and a part giving rise to interactions that is cubic and

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quartic in the fields, $\mathcal{L}_{\rm I}$;

$$\mathcal{L}_{\rm cl} = \tfrac{1}{2}c^2v^2 - \tfrac{1}{4}\lambda v^4 \tag{15.12}$$

$$\begin{aligned}\mathcal{L}_{\rm quad} = &\tfrac{1}{2}(\partial_\mu\eta)^2 - \tfrac{1}{2}(3\lambda v^2 - c^2)\eta^2 \\ &- \tfrac{1}{4}(\partial_\mu B_\nu - \partial_\nu B_\mu)^2 - \tfrac{1}{4}(\partial_\mu A^c_\nu - \partial_\nu A^c_\mu)^2 \\ &+ \tfrac{1}{8}v^2\left[(g'B_\mu - gA^3_\mu)^2 + g^2(A^1_\mu)^2 + g^2(A^2_\mu)^2\right]\end{aligned} \tag{15.13}$$

We define new fields

$$\begin{aligned} W^\pm_\mu &= \left(A^1_\mu \pm iA^2_\mu\right)/\sqrt{2} \\ Z_\mu &= \left(g'B_\mu - gA^3_\mu\right)/\sqrt{g^2+g'^2} \\ A_\mu &= \left(gB_\mu + g'A^3_\mu\right)/\sqrt{g^2+g'^2}\end{aligned} \tag{15.14}$$

The masses are

$$\begin{aligned} m^2_\eta &= 3\lambda v^2 - c^2 \\ m_A &= 0 \\ m_W &= \tfrac{1}{2}gv \\ m_Z &= \tfrac{1}{2}\sqrt{g^2+g'^2}\,v\end{aligned} \tag{15.15}$$

The tachyon is avoided as long as $v^2 \geq c^2/3\lambda$. In fact, from (15.12) we see that the classical minimum occurs at $v^2 = v_0^2 = c^2/\lambda$, so that indeed the model shows spontaneous symmetry breaking.

After addition of the fermions, it becomes possible to identify the fields and parameters described above: A_μ is the photon, $W^\pm$ and Z are the weak interaction bosons, and η is the as yet unobserved Higgs boson. Since all these are massive except for the photon, the total number of spin degrees of freedom is 12, the same as before, since the $W^\pm$ and Z each acquirie one degree of freedom from the Φ field. The electric charge is

$$e = \frac{gg'}{\sqrt{g^2+g'^2}} \tag{15.16}$$

and the Weinberg angle is defined by

$$\tan\theta_{\rm W} = \frac{g'}{g} \tag{15.17}$$

Experimentally, it is found that $e = 0.3028\ldots$ and $\sin^2\theta_{\rm W} = 0.226 \pm 0.004$. This leads to $g = 0.637$ and $g' = 0.344$. It turns out that the vacuum field v_0 is related directly to the Fermi constant: $v_0^2 = (\sqrt{2}G_{\rm F})^{-1} = (246\,{\rm GeV})^2$. The predicted masses of the gauge bosons in the tree

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approximation are then $m_W = 78.4$ GeV and $m_Z = 89.0$ GeV. (Radiative corrections increase these by several GeV.) These are consistent with observation. Only the combination c^2/λ is known, so there remains one undetermined parameter. It may be taken to be the Higgs mass. The theoretical bounds are currently $130 < m_\eta < 190$ GeV [5]. The lower bound comes from the requirement that the standard model vacuum be stable. The upper bound comes from the requirement that λ be small enough that perturbation theory can be used. It should remain valid up to a supposed grand unification scale $\Lambda_{\text{GUT}} \sim 10^{16}$ GeV.

The fermions are included according to the following scheme. The quark mass eigenstates are not eigenstates of the weak interactions. The matrix connecting the different sets of eigenstates is the Cabibbo–Kobayashi–Maskawa (CKM) matrix. By convention the charge 2/3 quarks (u, c, t) are unmixed. The CKM matrix, U_{CKM}, is unitary and relates the eigenstates of the charge $-1/3$ quarks (d, s, b) as

$$\begin{pmatrix} d' \\ s' \\ b' \end{pmatrix} = U_{\text{CKM}} \begin{pmatrix} d \\ s \\ b \end{pmatrix} \tag{15.18}$$

The elements of U_{CKM} will not be needed in the subsequent discussion.

The fermions are then grouped into left-handed SU(2) doublets and right-handed SU(2) singlets. For example, the electron and its neutrino form the doublet

$$L = \begin{pmatrix} \nu_e \\ e^- \end{pmatrix}_L \tag{15.19}$$

where $e_L^- = \frac{1}{2}(1-\gamma_5)e^-$, and a singlet $R = \frac{1}{2}(1+\gamma_5)e^-$. These are coupled to the gauge bosons via

$$\bar{R}\left(i\partial\!\!\!/ - g' B\!\!\!/\right)R + \bar{L}\left(i\partial\!\!\!/ - \tfrac{1}{2}g' B\!\!\!/ + \tfrac{1}{2}g A\!\!\!/^a \tau^a\right)L \tag{15.20}$$

The other leptons and quarks are included in an analogous way. The coupling to $\gamma, W^\pm$, and Z can be written compactly as

$$\begin{aligned} e\bar{\psi}\gamma^\mu \Big\{ QA_\mu &+ \frac{1}{2^{3/2}\sin\theta_{\text{W}}}(1-\gamma_5)\left(T^+W_\mu^+ + T^-W_\mu^-\right) \\ &+ \frac{1}{\sin\theta_{\text{W}}\cos\theta_{\text{W}}}\left[\tfrac{1}{2}(1-\gamma_5)T_3 - Q\sin^2\theta_{\text{W}}\right]Z_\mu \Big\}\psi \end{aligned} \tag{15.21}$$

where ψ is one of the following doublets,

$$\begin{pmatrix} u \\ d' \end{pmatrix} \quad \begin{pmatrix} c \\ s' \end{pmatrix} \quad \begin{pmatrix} t \\ b' \end{pmatrix} \quad \begin{pmatrix} \nu^e \\ e^- \end{pmatrix} \quad \begin{pmatrix} \nu^\mu \\ \mu^- \end{pmatrix} \quad \begin{pmatrix} \nu^\tau \\ \tau^- \end{pmatrix} \tag{15.22}$$

and where Q is the electric charge operator, T_3 is the third component of the weak SU(2) spin (with eigenvalue 1/2 for ν_e, ν_μ, ν_τ, u, c, t, and eigenvalue $-1/2$ for $e^-, \mu^-, \tau^-, d, s, b$), and $T^\pm$ is the raising or lowering operator, which acts on the left-handed particles. The weak hypercharge Y is determined by $Q = T_3 + \frac{1}{2}Y$.

In order to retain the left-handed SU(2) symmetry for the fermions it is not possible to add a mass term in the usual form $-\bar{\psi}M\psi$. The allowable term for the electron, for example, is of the Yukawa form

$$-f_e \left(\bar{R}\Phi^\dagger L + \bar{L}\Phi R \right) \tag{15.23}$$

After using (15.9) we obtain the electron mass as $m_e = \frac{1}{2} f_e v^0$ and zero neutrino mass. A similar situation prevails for the other fermions. Thus all quarks and leptons receive their masses on account of spontaneous symmetry breaking. In the vacuum, a quark or lepton mass is therefore $\sim f_i G_F^{-1/2}$ where f_i is a dimensionless coupling constant. For all but the t quark the f_i are very small since $G_F^{-1/2} = 293$ GeV.

15.2 Symmetry restoration in mean field approximation

The existence of phase transitions in the early universe has been a question that has preoccupied a generation of cosmologists. Early on, Kirzhnits [6] found that the symmetry between the weak and electromagnetic interactions would be restored at high temperatures. This result was soon complemented by similar works by Weinberg [7] Dolan and Jackiw [8], and Kirzhnits and Linde [9]. Some consequences of this phase transition will be discussed in Chapter 16. In the sections that follow, the stage will be set for the theoretical investigation of the electroweak phase transition, its existence, and its order.

The Glashow–Weinberg–Salam model is relatively easy to study at finite temperatures in the mean field approximation. At high temperature, $T > 50$ GeV, the fermion masses can be ignored except for that of the top quark. For simplicity, we shall ignore that for the moment as well. First, we shall use the U-gauge and show that it leads to an erroneous result, at least in the mean field approximation. This can be corrected in a covariant gauge.

The U-gauge has the advantage of displaying immediately the physical degrees of freedom. From (15.12) and (15.15) we can write the pressure as

$$\begin{aligned} P_{\rm MF} = & -\tfrac{1}{4}c^4/\lambda + \tfrac{1}{2}c^2 v^2 - \tfrac{1}{4}\lambda v^4 \\ & + 6P_0(m_{\rm W}) + 3P_0(m_Z) + 2P_0(0) + P_0(m_\eta) + \tfrac{7}{8}\pi^2 T^4 \end{aligned} \tag{15.24}$$

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The last term is the contribution from three generations of massless quarks and leptons. The previous four terms are the boson contributions, with

$$P_0(m) = \int \frac{d^3k}{(2\pi)^3} \frac{k^2}{3\omega} \frac{1}{e^{\beta\omega} - 1} \sim \frac{\pi^2}{90} T^4 - \frac{m^2}{24} T^2 \tag{15.25}$$

where the high-temperature limit is given. Using this limit we obtain

$$P_{\rm MF} = \left(\tfrac{7}{8} + \tfrac{2}{15}\right) \pi^2 T^4 + \tfrac{1}{2} v^2 \left[c^2 - \tfrac{1}{4} T^2 \left(\lambda + \tfrac{3}{4} g^2 + \tfrac{1}{4} g'^2\right)\right] \\ - \tfrac{1}{4} \lambda v^4 - \tfrac{1}{4} c^4/\lambda \tag{15.26}$$

Maximizing the pressure with respect to the mean field v gives the temperature dependence $v^2(T) = \left[c^2 - \tfrac{1}{4}T^2\left(\lambda + \tfrac{3}{4}g^2 + \tfrac{1}{4}g'^2\right)\right]/\lambda$ if $T^2 < 4c^2/\left(\lambda + \tfrac{3}{4}g^2 + \tfrac{1}{4}g'^2\right)$ and $v(T) = 0$ otherwise. This would indicate restoration of the gauge symmetry that was spontaneously broken at $T = 0$.

However, the result (15.26) is wrong. The reason can be traced to the U-gauge itself. Although it makes the physical particle content of the theory manifest, it is not, in practice, a renormalizable gauge. This follows from the poor ultraviolet behavior of the massive vector meson propagators, which is $p^\mu p^\nu/m^2 p^2$ instead of $1/p^2$. The implication for finite temperature is serious since T effectively acts as a physical high-momentum cutoff. Another way to see the difficulty is to consider the transformation (15.8) in the high-temperature phase, where v is supposed to vanish.

A more appropriate gauge for our purpose is the R-gauge, suitably generalized from its first application to the Abelian Higgs model, given in Section 7.4. Now we take as the independent fields η and $\boldsymbol{\zeta}$, defined by

$$\Phi = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v + \eta \end{pmatrix} + \frac{i\boldsymbol{\zeta} \cdot \boldsymbol{\tau}}{\sqrt{2}v} \begin{pmatrix} 0 \\ v \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \zeta_2 + i\zeta_1 \\ v + \eta - i\zeta_3 \end{pmatrix} \tag{15.27}$$

which is suggested by (15.7) and (15.8). We choose the SU(2) gauge-fixing function to be

$$F^a = \partial^\mu A^a_\mu - \tfrac{1}{2} \rho g v \zeta^a - f^a(\mathbf{x}, \tau) \tag{15.28}$$

and the U(1) gauge-fixing function to be

$$F = \partial^\mu B_\mu + \tfrac{1}{2} \rho g' v \zeta^3 - f(\mathbf{x}, \tau) \tag{15.29}$$

The gauge-fixing delta functions $\delta(F)$ and $\delta(F^a)$ in the functional integral expression for Z are multiplied by

$$\exp\left\{-\frac{1}{2\rho} \int d^3x \, d\tau (f_a^2 + f^2)\right\}$$

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and integration over $f^a(\mathbf{x},\tau)$ and $f(\mathbf{x},\tau)$ is carried out. The result is to add to the Lagrangian the gauge-fixing terms

$$-\frac{1}{2\rho}\left(\partial^\mu A^a_\mu - \tfrac{1}{2}\rho g v \zeta^a\right)^2 - \frac{1}{2\rho}\left(\partial^\mu B_\mu + \tfrac{1}{2}\rho g' v \zeta^3\right)^2 \tag{15.30}$$

The cross terms in (15.30) between the gauge fields and ζ are ρ-independent. They combine with the cross terms from $(D_\mu\Phi)^\dagger(D^\mu\Phi)$ to produce total divergences that integrate to zero. Thus one advantage of using (15.28) and (15.29) is that there is no mixing between the fields. The terms $-(\partial^\mu A^a_\mu)^2/2\rho - (\partial^\mu B_\mu)^2/2\rho$ are familiar from the covariant gauge. The last terms in (15.30), when combined with the quadratic terms in $c^2\Phi^\dagger\Phi - \lambda(\Phi^\dagger\Phi)^2$, yield the masses

$$\begin{aligned} m^2_\eta &= 3\lambda v^2 - c^2 \\ m^2_{\zeta^1} = m^2_{\zeta^2} &= \lambda v^2 - c^2 + \tfrac{1}{4}\rho g v^2 \\ m^2_{\zeta^3} &= \lambda v^2 - c^2 + \tfrac{1}{4}\rho(g^2 + g'^2)v^2 \end{aligned} \tag{15.31}$$

The fact that the $\boldsymbol{\zeta}$ masses are gauge or ρ-dependent suggests that these do not represent physical particles.

The determinants must be analyzed. They are $\det(\partial F/\partial\alpha)$ and $\det(\partial F^a/\partial\alpha^b)$, where the infinitesimal gauge transformations are parametrized by $\alpha(\mathbf{x},\tau)$ and $\alpha^b(\mathbf{x},\tau)$. With the help of (8.11), we find

$$\begin{aligned} \frac{\partial F^a}{\partial \alpha^b} &= -\partial^2\delta^{ab} - \tfrac{1}{4}\rho g^2 v^2 \delta^{ab} + \text{linear terms} \\ \frac{\partial F}{\partial \alpha} &= -\partial^2 - \tfrac{1}{4}\rho g'^2 v^2 + \text{linear terms} \end{aligned} \tag{15.32}$$

where "linear terms" indicates terms that are linear in A^a_μ, ζ, and/or η. The determinants can be written as functional integrals over the ghost fields C_a and C. The ghost masses can be read off directly from (15.32):

$$\begin{aligned} m^2_{C_a} &= \tfrac{1}{4}\rho g^2 v^2 \\ m^2_C &= \tfrac{1}{4}\rho g'^2 v^2 \end{aligned} \tag{15.33}$$

The propagators for the W and Z fields are

$$D^{\mu\nu} = \frac{g^{\mu\nu} - p^\mu p^\nu/m^2}{p^2 - m^2} + \frac{p^\mu p^\nu/m^2}{p^2 - \rho m^2} \tag{15.34}$$

where $m^2 = m^2_Z$ or m^2_W. The first term is the usual propagator for a massive vector boson. The second term looks like the propagator for an unphysical longitudinally propagating particle.

Now we are ready to put together this strange zoo of real and fictitious particles. Again, in the mean field approximation at high temperature we have

$$P_{\rm MF} = -\tfrac{1}{4}c^4/\lambda + \tfrac{1}{2}c^2v^2 - \tfrac{1}{4}\lambda v^4 + \left(\tfrac{7}{8}+\tfrac{2}{15}\right)\pi^2 T^4 - \tfrac{1}{24}T^2\Big[3m_Z^2 + 6m_W^2 + m_\eta^2 + \Big(m_{\zeta^1}^2 + m_{\zeta^2}^2 + m_{\zeta^3}^2 + \rho m_Z^2 + 2\rho m_W^2 - 6m_{C_a}^2 - 2m_C^2\Big)\Big] \tag{15.35}$$

The quantity in the second parentheses is all that distinguishes the U-gauge from the R-gauge. These mass-squared terms add up to $3(\lambda v^2 - c^2)$. The pressure is thus

$$P_{\rm MF} = \tfrac{121}{120}\pi^2T^4 + \tfrac{1}{2}v^2\left[c^2 - \tfrac{1}{4}T^2\left(2\lambda + \tfrac{3}{4}g^2 + \tfrac{1}{4}g'^2\right)\right] - \tfrac{1}{4}\lambda v^4 - \tfrac{1}{4}c^4/\lambda + \tfrac{1}{6}c^2T^2 \tag{15.36}$$

This should be compared with (15.26). Note that all ρ-dependence has vanished: a nice check on the calculation.

Minimizing $P_{\rm MF}$ with respect to v we obtain

$$v^2(T) = \begin{cases} (c^2/\lambda)\left(1 - T^2/T_{\rm c}^2\right) & T \le T_{\rm c} \\ 0 & T \ge T_{\rm c} \end{cases} \tag{15.37}$$

$$P_{\rm MF} = \begin{cases} \tfrac{121}{120}\pi^2T^4 + \tfrac{1}{4}(c^4/\lambda)\left(1 - T^2/T_{\rm c}^2\right)^2 + \tfrac{1}{6}c^2T^2 - \tfrac{1}{4}c^4/\lambda & T \le T_{\rm c} \\ \tfrac{121}{120}\pi^2T^4 + \tfrac{1}{6}c^2T^2 - \tfrac{1}{4}c^4/\lambda & T \ge T_{\rm c} \end{cases} \tag{15.38}$$

and

$$T_{\rm c}^2 = \frac{4c^2}{2\lambda + \frac{3}{4}g^2 + \frac{1}{4}g'^2} \tag{15.39}$$

This yields a second-order symmetry-restoring phase transition at $T_{\rm c}$ since $\partial P/\partial T$ is continuous but $\partial^2 P/\partial^2 T$ is not. If we take the zero-temperature Higgs mass to be 120 GeV then $c = 84.9$ GeV and $\lambda = 0.119$. The critical temperature is $T_{\rm c} = 225$ GeV. The effective potential is plotted in Figure 15.1 as a function of v for several values of the temperature, including the critical value. Here the effective potential is $\Omega_{\rm MF}^{\rm eff}(v) \equiv P_{\rm MF}(0,T) - P_{\rm MF}(v,T)$; in the literature it is also written as $V_{\rm eff}$. Minimizing the effective potential is equivalent to maximizing the pressure.

All the previously discussed difficulties associated with spontaneous symmetry breaking and nonabelian gauge theories at finite temperature arise in the Glashow–Weinberg–Salam model as well. For example, at sufficiently high temperature the Higgs-mass-squared of (15.15) becomes

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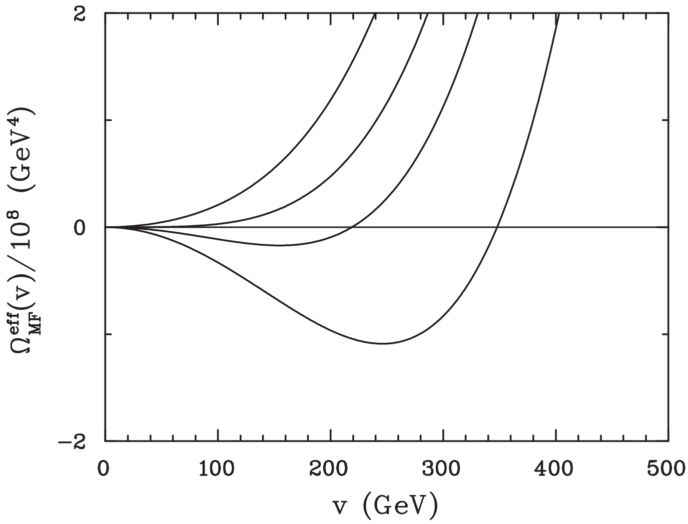

Fig. 15.1. The effective potential in the mean field approximation, as described in the text. The curves shown correspond to potentials calculated at $T = 0$, $T = 175$ GeV, $T = T_c = 225$ GeV, and $T = 275$ GeV, from bottom to top, respectively.

negative, and loop self-energy corrections are necessary to cure it. In the high-temperature phase the mean field masses of the gauge fields are zero. Thus the same infrared problems will arise as in QCD. The contributions of exchange and ring diagrams to the pressure may be computed.

15.3 Symmetry restoration in perturbation theory

The applicability of finite-temperature perturbation expansions in the electroweak theory will now be more closely examined. Consider a scalar field theory, $\lambda\phi^4$, like that discussed elsewhere in this book. At each order in a loop expansion there will be terms of the form

$$T \sum_n \int \frac{d^3p}{(2\pi)^3} f(\omega_n, \mathbf{p}) \tag{15.40}$$

where $f(\omega_n, \mathbf{p})$ is a functional of propagators and vertices. The tadpole diagram is a simple example, namely

$$T \sum_n \int \frac{d^3p}{(2\pi)^3} \frac{1}{\omega_n^2 + \omega^2} \tag{15.41}$$

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with $\omega = \sqrt{\mathbf{p}^2 + m^2}$. Clearly this will be dominated by the Matsubara zero mode. Omitting the integral over momentum, this fact is expressed as

$$T \sum_n \frac{1}{\omega_n^2 + \omega^2} \sim \frac{T}{\omega^2} \tag{15.42}$$

What might constitute a dimensionless loop-expansion parameter? The argument above suggests that when the vertices contribute an overall constant λ then the expansion parameter that controls the convergence is $\lambda T/\omega \sim \lambda T/m_{\text{eff}}$ for bosons (where m_{eff} is some soft scale in our theory) and $\lambda T/T = \lambda$ for fermions. For bosons the perturbation expansion could be ill defined if $m_{\text{eff}} < \lambda T$, and then non-perturbative techniques would be required. What happens in the standard model is more complicated because of the inclusion of the gauge bosons. In what follows we study the electroweak theory with the inclusion of the ring diagrams, that are known to be important for long wavelengths.

In the Glashow–Weinberg–Salam model, the gauge boson mass term is of the form

$$(A_\mu^a, B_\mu)\, M^2 \begin{pmatrix} A_a^\mu \\ B^\mu \end{pmatrix}$$

with a non-diagonal mass matrix

$$M^2(v) = \frac{v^2}{4} \begin{pmatrix} g^2 & 0 & 0 & 0 \\ 0 & g^2 & 0 & 0 \\ 0 & 0 & g^2 & -gg' \\ 0 & 0 & -gg' & g'^2 \end{pmatrix} \tag{15.43}$$

The customary procedure is to define the physical fields $W_\mu^\pm$, Z_μ, and A_μ as linear combinations of the A_μ^a and B_μ fields, such that the physical masses are $m_W^2(v) = g^2 v^2/4$, $m_Z^2(v) = (g^2 + g'^2)v^2/4$, and $m_A^2(v) = 0$. For this application, we now assume that only the top quark Yukawa coupling, f_t, is nonzero. The shift in the Higgs field generates a mass through the term $\mathcal{L}_{\text{Yukawa}} = f_t t\bar{t} v/\sqrt{2}$.

The one-loop contribution to the thermodynamic potential is split into zero-temperature and finite-temperature contributions. Following Carrington [10], one may write the contribution from the Higgs boson (ϕ), the gauge boson (gb), and the top quark (ψ) loops as

$$\Omega_1(v) = \Omega_1^{\text{vac}}(v) + \Omega_1^{\text{mat}}(v) \tag{15.44}$$

where

$$\begin{aligned} \Omega_1^{\text{vac}}(v) &= \Omega_{1,\phi}^{\text{vac}}(v) + \Omega_{1,gb}^{\text{vac}}(v) + \Omega_{1,\psi}^{\text{vac}}(v) \\ \Omega_1^{\text{mat}}(v) &= \Omega_{1,\phi}^{\text{mat}}(v) + \Omega_{1,gb}^{\text{mat}}(v) + \Omega_{1,\psi}^{\text{mat}}(v) \end{aligned} \tag{15.45}$$

The zero-temperature loops are regularized using a cut-off. Their contribution may be obtained from Section 7.3; for an arbitrary mass $m_x(v)$ it is

$$\frac{\Lambda_c^2}{32\pi^2}m_x^2(v) + \frac{m_x^4(v)}{64\pi^2}\left[\ln\left(\frac{m_x^2(v)}{\Lambda_c^2}\right) - \frac{1}{2}\right] \tag{15.46}$$

This procedure generates a correction to the tree-level zero-temperature effective potential, so that

$$\Omega^{\rm vac}(v) = \Omega^{\rm tree}(v) + \Omega_1^{\rm vac}(v) \tag{15.47}$$

with

$$\Omega^{\rm tree} = -\frac{1}{2}c^2v^2 + \frac{1}{4}v^4$$

and

$$\begin{aligned}\Omega_1^{\rm vac}&(v)\\ &= \frac{3}{32\pi^2}\lambda c^2v^2 - \frac{v^4}{64\pi^2}\left(6\lambda^2 + \frac{3}{16}g^4 + \frac{3}{32}(g^2+g'^2)^2 - \frac{3}{2}f^4\right)\\ &\quad + \frac{1}{64\pi^2}\left[6m_W^4(v)\ln\left(\frac{\lambda v^2}{c^2}\right) + 3m_Z^4(v)\ln\left(\frac{\lambda v^2}{c^2}\right) - 12m_t^4(v)\ln\left(\frac{\lambda v^2}{c^2}\right)\right.\\ &\qquad\left. + m_1^4(v)\ln\left(\frac{m_1^2(v)}{2c^2}\right) + 3m_2^4(v)\ln\left(\frac{m_2^2(v)}{2c^2}\right)\right]\end{aligned} \tag{15.48}$$

The one-loop finite-temperature thermodynamic potential for bosons and fermions is just the negative of the pressure for the free particle of mass $m_x(v)$. There will be contributions to the ring diagrams from both gauge and Higgs bosons. The finite-temperature part of the one-loop potential will combine with the ring contribution to define a potential in terms of the shifted mass-squared. Therefore we need to evaluate the gauge boson and Higgs boson self-energies in the leading infrared limit. For the ith Higgs field,

$$\Pi_i(0) = \Pi_\phi^{(A_\mu^a)}(0) + \Pi_\phi^{(B_\mu)}(0) + \Pi_\phi^{(\phi)}(0) + \Pi_\phi^{(\psi)}(0) \tag{15.49}$$

where the individual contributions are

$$\begin{aligned}\Pi_\phi^{(A_\mu^a)}(0) &= \frac{1}{8}g^2T^2 \qquad \Pi_\phi^{(B_\mu)}(0) = \frac{1}{16}\left(g^2+g^2\right)T^2\\ \Pi_\phi^{(\phi)}(0) &= \frac{1}{2}\lambda T^2 \qquad \Pi_\phi^{(\psi)}(0) = \frac{1}{4}f_tT^2\end{aligned} \tag{15.50}$$

The ring contribution for the Higgs field is

$$\Omega_{\rm ring}^{\rm mat}(v) = -\frac{1}{2}T\sum_n\int\frac{d^3q}{(2\pi)^3}\sum_{\ell=1}^{\infty}\frac{1}{\ell}\left(-\frac{1}{\omega_n^2+\mathbf{q}^2+m_i^2(v)}\Pi_i(0)\right)^\ell \tag{15.51}$$

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which, combined with the finite-temperature part of the one-loop potential, gives

$$\Omega_\phi^{\text{mat}}(v) = -P_0(\widetilde{m}_1) - 3P_0(\widetilde{m}_2) \tag{15.52}$$

where $P_0(m)$ is as in Section 15.2, $\widetilde{m}_i^2 = m_i^2(v) + \Pi_i(0)$, and the factor 3 is a degeneracy factor. For the gauge boson polarization tensors, as used in Section 5.4,

$$\Pi_{\mu\nu}(0) = \Pi_{\text{T}}(0)P_{\text{T}\,\mu\nu} + \Pi_{\text{L}}(0)P_{\text{L}\,\mu\nu} \tag{15.53}$$

In the static infrared limit $\Pi_{\mu\nu}^{AB}(0) = \Pi_{00}^{AB}(0)P_{\text{L}\,\mu\nu}$, and Π_{00}^{AB} is approximately diagonal if the ratio of the gauge boson masses and the temperature is small:

$$\Pi_{00}(0) = \begin{bmatrix} \Pi_{00}^{(2)}(0) & 0 & 0 & 0 \\ 0 & \Pi_{00}^{(2)}(0) & 0 & 0 \\ 0 & 0 & \Pi_{00}^{(2)}(0) & 0 \\ 0 & 0 & 0 & \Pi_{00}^{(1)}(0) \end{bmatrix} \tag{15.54}$$

Here the superscripts (1) and (2) refer to the U(1) and SU(2) gauge bosons, respectively. One defines as $\Pi_{gb}^{(2)}(0)$, $\Pi_\phi^{(2)}(0)$, and $\Pi_\psi^{(2)}(0)$, the contribution to the SU(2) gauge boson polarization tensor from the gauge boson, Higgs boson, and t quark loops. One may use a similar notation for the polarization of the U(1) gauge boson. Then

$$\begin{aligned} \Pi_{00}^{(1)}(0) &= \Pi_\phi^{(1)}(0) + \Pi_\psi^{(1)}(0) \\ \Pi_{00}^{(2)}(0) &= \Pi_{gb}^{(2)}(0) + \Pi_\phi^{(2)}(0) + \Pi_\psi^{(2)}(0) \end{aligned} \tag{15.55}$$

where

$$\begin{aligned} &\Pi_\phi^{(1)}(0) = \frac{1}{6}g'^2T^2 \qquad \Pi_\psi^{(1)}(0) = \frac{5}{3}g'^2T^2 \\ &\Pi_{\text{gb}}^{(2)}(0) = \frac{2}{3}g^2T^2 \qquad \Pi_\phi^{(2)}(0) = \frac{1}{6}g^2T^2 \qquad \Pi_\psi^{(2)}(0) = g^2T^2 \end{aligned} \tag{15.56}$$

The rest of the calculation for the ring contribution to the gauge boson effective potential proceeds as in the case of the Higgs particle. In terms of the mass and self-energy matrices, it may be written as

$$\Omega_{\text{ring}}^{gb}(v) = -\frac{T}{12\pi}\,\text{Tr}\left\{[M^2(v) + \Pi_{00}(0)]^{3/2} - M^3(v)\right\} \tag{15.57}$$

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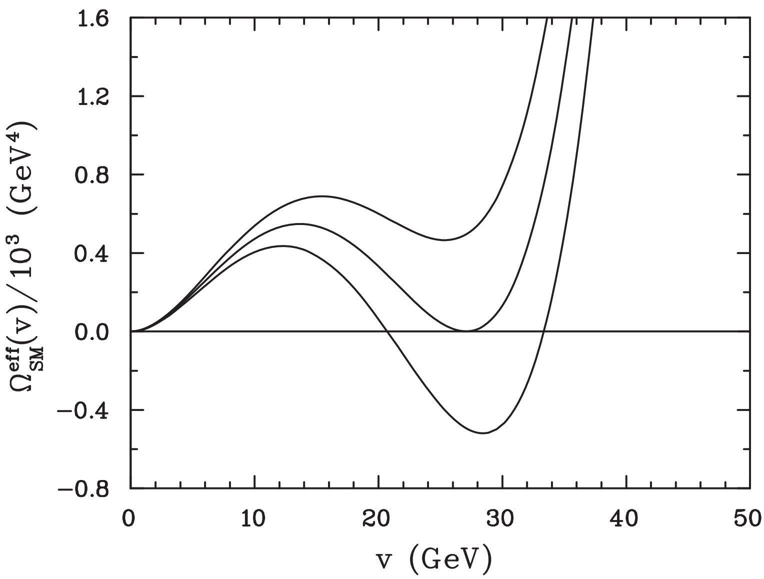

Fig. 15.2. The ring-improved effective potential that is relevant for the electroweak phase transition in the standard model. The physical parameters that enter this calculation are the masses of the Higgs particle and the top quark. The values here are $m_H = 120$ GeV, and $m_t = 175$ GeV. The curve at the critical temperature $T_{\rm c} = 140.42$ GeV is enclosed by curves at $T = 140.40$ (lower) and 140.43 GeV (upper).

One may show that

$$\mathrm{Tr}[M^2(v) + \Pi_{00}(0)]^{3/2} = 2a^{3/2} + \frac{1}{2\sqrt{2}}\left[(a+c) - \sqrt{(a-c)^2 + 4b^2}\right]^{3/2} + \left[(a+c) + \sqrt{(a-c)^2 + 4b^2}\right]^{3/2} \quad (15.58)$$

where $a = g^2 v^4/4 + \Pi^{(2)}_{00}(0)$, $b = -gg'v^2/4$, and $c = g'^2 v^2/4 + \Pi^{(1)}_{00}(0)$.

The final expression for the effective potential is obtained by adding to the zero-temperature parts the ring-improved finite-temperature expressions [10]. We remark that two-loop topologies have also been considered, along with their contribution (with resummations) to the effective potential [11, 12, 13].

With the methods described here, it has been shown that the standard model has a first-order phase transition, driven by the v^3 term [10, 14]. Using modern values of the physical parameters yields the effective potential shown in Figure 15.2. One observes that the perturbation approach appears to predict a very weak first-order phase transition with a critical

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temperature $T_c = 140.42$ GeV. However, this brings us to the core of the issue. Consider the behavior of the perturbative expansion in the standard model. To make the discussion specific, consider a temperature near T_c. A parameter can be associated with each loop in the expansion of the effective potential. For example, the expansion parameter for vector loops is $g^2T_c/m_W(v_c)$ (generically writing g^2 for a linear combination of g^2 and g'^2), according to the analysis earlier in this section. We may write $g^2T_c/m_W(v_c) \sim gT_c/v_c \sim \lambda/g^2$. This last value is $\sim m_H^2/m_W^2$, evaluated at $T = 0$. The current experimental value of the mass of the W boson is 80.425 ± 0.033 GeV [15], and comes from direct measurements. The Higgs boson, at the time of this writing, is still a hypothetical particle. The bounds on its mass placed by self-consistent arguments have been reviewed earlier. Indirect experimental bounds for the standard model Higgs mass can also be obtained from precision electroweak measurements and from fits to measured top quark and $W^\pm$ masses. The global electroweak fits give a preferred value of 96^{+40}_{-38} GeV [15]. However, a recent high-precision measurement of the top quark mass raised the world average for m_t to 178.0 ± 4.3 GeV [16]. The impact on the best standard-model fit of the Higgs mass is that it is raised from 96 to 117 GeV. In line with arguments presented earlier, those numbers clearly cast doubt on the usefulness of a perturbative loop expansion in theoretical searches for an electroweak phase transition in the standard model. It is therefore important to consider lattice-based nonperturbative numerical approaches.

15.4 Symmetry restoration in lattice theory

As the quartic self-coupling λ becomes large, the accuracy of perturbative calculations decreases. For large enough λ, corresponding to a large Higgs mass, the order of the phase transition, and even its existence, cannot be determined using perturbation theory. As for QCD one might turn to numerical calculations of electroweak theory on a lattice. In general this is a more intensive numerical endeavor than in the QCD case for several reasons: there are two types of gauge field, there is a scalar doublet field, and there are three generations of fermion fields to deal with. Also, surprisingly, the weaker gauge coupling makes the simulations more demanding since it introduces a scale hierarchy that is very difficult to handle numerically.

Significant progress in finite-temperature lattice calculations of electroweak-like gauge theories has been realized in recent years with the help of the technique of dimensional reduction. Provided that we are interested in the computation of static quantities, we may generically write a

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four-dimensional boson or fermion field in terms of its Matsubara modes:

$$\begin{aligned}\phi(\tau,\mathbf{x}) &= \sum_{n=-\infty}^{\infty} \exp(i2n\pi T\tau)\,\phi_n(\mathbf{x})\\ \psi(\tau,\mathbf{x}) &= \sum_{n=-\infty}^{\infty} \exp(i(2n+1)\pi T\tau)\,\psi_n(\mathbf{x})\end{aligned} \tag{15.59}$$

The original four-dimensional theory is formally equivalent to a three-dimensional theory albeit with an infinite number of fields, each corresponding to a mode. The three-dimensional "masses" of bosons are $m_{\rm B} = 2\pi nT$ and those of fermions are $m_{\rm F} = (2n+1)\pi T$. If we are concerned with soft physics below some scale Λ, the heavier fields (on that scale) may be integrated out. This leaves an effective field theory where the parameters of the effective Lagrangian are functions of the temperature and of the scale Λ. This integration over heavy modes might be done perturbatively, and the expansion parameter would be $\Lambda/\pi T$. If the relevant scale is T or smaller then all fermionic modes, and all bosonic modes with $n \neq 0$, will have masses larger than πT and can be integrated out.

The effective three-dimensional action can be written as

$$S_{\rm eff} = bVT^3 + \int d^3x\,\mathcal{L}_{\rm eff} + \sum_n \frac{O_n}{T^n} \tag{15.60}$$

Here $\mathcal{L}_{\rm eff}$ is a three-dimensional effective Lagrangian with temperature-dependent parameters, b is some number that is related to the number of degrees of freedom, V is the volume, and the O_n represent the contribution from operators of dimension n. The latter will be suppressed by powers of the temperature but, in the high-T limit, the three-dimensional couplings contained therein will also be large. A typical way to rewrite the last term in the equation above is $\mathcal{O}(m_i^2(T)/T^2)$, the $m_i(T)$ being relevant mass scales for the problem at hand, such as inverse screening lengths, etc. The condition for omitting the last term in the effective action is tantamount to that controlling the convergence of the zero-temperature perturbative expansion, namely, $g^2 \ll 1$ where g is a dimensionless coupling constant. At first it would appear that little has been gained by formulating the problem in a reduced number of dimensions. However, the expansion parameter is different at zero and finite temperature. At finite T the perturbative expansion should prove useful if $g^2T/\Lambda = g_3^2/\Lambda \ll 1$, where g_3 is the three-dimensional coupling. Therefore, at finite temperature it is entirely possible for the four-dimensional perturbation expansion to be unsuitable but for the dimensionally reduced theory to be applicable. For applications in the vicinity of a critical temperature, it turns out that the criterion of applicability of dimensional reduction is satisfied for

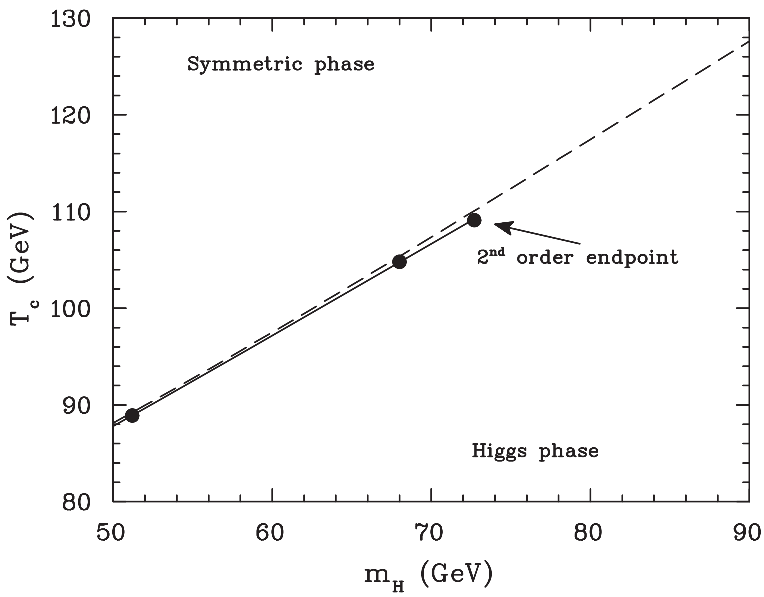

Fig. 15.3. A plot of the phase diagram of the standard model, investigated with lattice Monte Carlo techniques. The broken line is the perturbative (first-order) result; the solid line is a fit to the numerical results and shows a first-order transition with a second-order endpoint. The figure is adapted from Ref. [18].

electroweak theory but not for QCD, since the four-dimensional gauge coupling is not small at T_c.

Finite-temperature electroweak theory has been studied in lattice Monte Carlo simulations for a number of Higgs mass values by Kajantie *et al.* [17]. The effective Lagrangian they use is

$$\mathcal{L}_{\text{eff}} = \tfrac{1}{4} f^a_{ij} f^a_{ij} + (D_i\Phi)^\dagger (D_i\Phi) + m_3^2 \Phi^\dagger\Phi + \lambda_3(\Phi^\dagger\Phi)^2 \tag{15.61}$$

This is electroweak theory in three-dimensions without the U(1) gauge field, without fermions, and where the time component of the SU(2) gauge field has been integrated out. There are three parameters: g_3, that enters via the covariant derivative D_i, m_3, and λ_3. To lowest order (and ignoring Yukawa couplings and g') they are

$$\begin{aligned} g_3^2 &= g^2 T \\ m_3^2 &= \left(\tfrac{3}{16} g^2 + \tfrac{1}{2}\lambda\right) T^2 - c^2 \\ \lambda_3 &= \lambda T \end{aligned} \tag{15.62}$$

where g, c, and λ are all parameters in the fundamental four-dimensional theory (15.1). These parameters have been computed with one-loop

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corrections too. This allows for a precise connection between physically measurable quantities such as the W, Z, and Higgs-boson masses and the thermodynamic properties of the electroweak theory.

The numerical results indicate that the theory has a first-order phase transition for small Higgs masses. The transition gets weaker as m_H grows and terminates around $m_H \sim 80$ GeV at a second-order endpoint. Those results, together with those obtained in perturbation theory, are summarized in Figure 15.3. It might be that these conclusions are modified by physics beyond the standard model; this is a topic still under investigation.

15.5 Exercises

15.1 Find explicitly the "linear terms" in (15.32).

15.2 Verify that (15.34) is the propagator for the W and Z bosons.

15.3 Express T_c in (15.39) in terms of the observable parameters e and θ_W and the zero-temperature Higgs mass. Assuming that perturbation theory is valid and using the quoted bounds on m_H, determine the allowable range for T_c.

15.4 Show that the term in the effective potential that is cubic in the vacuum expectation value of the scalar field is from the Matsubara zero-mode.

15.5 Derive the three-dimensional couplings in (15.62).

15.6 How is the temperature dependence of m_3^2 in (15.62) related to the critical temperature given by (15.39)?

References

1. Glashow, S., *Nucl. Phys.* **22**, 579 (1961).
2. Weinberg, S., *Phys. Rev. Lett.* **19**, 1264 (1967).
3. Salam, A., (1968). *Nobel Symposium on Elementary Particle Theory* (Wiley, New York).
4. 't Hooft, G., *Nucl. Phys.* **B35**, 167 (1971).
5. Cabibbo, N. *et al.*, *Nucl. Phys.* **B158**, 295 (1979); Hambye, T., and Riesselmann, K., *Phys. Rev. D* **55**, 7255 (1997); Isidori, G., Ridolfi, G., and Strumia, A., *Nucl. Phys.* **B609**, 387 (2001).
6. Kirzhnits, D. A., *JETP Lett.* **15**, 374 (1972).
7. Weinberg, S., *Phys. Rev. D* **9**, 3357 (1974).
8. Dolan, L., and Jackiw, R., *Phys. Rev. D* **9**, 3320 (1974).
9. Kirzhnits, D. A., and Linde, A. D., *Ann. Phys. (NY)* **101**, 195 (1976).
10. Carrington, M., *Phys. Rev. D* **45**, 2933 (1992).
11. Arnold, P., and Espinosa, O., *Phys. Rev. D* **47**, 3546 (1993).
12. Bagnasco, J. E., and Dine, M., *Phys. Lett.* **B303**, 308 (1993).
13. Fodor, Z., and Hebecker, A., *Nucl. Phys.* **B432**, 127 (1994).

14. Dine, M., Leigh, R. G., Huet, P., Lindé, A., and Lindé, D., *Phys. Rev. D* **46**, 550 (1992).
15. Eidelman S., *et al.*, *Phys. Lett.* **B592**, 1 (2004).
16. Abazov, V. M., *et al.*, *Nature* **429**, 638 (2004).
17. Kajantie, K., Laine, M., Rummukainen, K., and Shaposhnikov, M., *Phys. Rev. Lett.* **77**, 2887 (1996).
18. Laine, M., and Rummukainen, K., *Nucl. Phys.* **B73** (Proc. Suppl.), 180 (1999).

Bibliography

The standard SU(2) × U(1) electroweak model

Glashow, S., *Nucl. Phys.* **22**, 579 (1961).
Weinberg, S., *Phys. Rev. Lett.* **19**, 1264 (1967).
Salam, A. (1968). *Nobel Symposium on Elementary Particle Theory* (Wiley, New York).
't Hooft, G., *Nucl. Phys.* **B35**, 167 (1971).

Pioneering works on the electroweak phase transition

Kirzhnits, D. A., *JETP Lett.* **15**, 374 (1972).
Dolan, L., and Jackiw, R., *Phys. Rev. D* **9**, 3320 (1974).
Weinberg, S., *Phys. Rev. D* **9**, 3357 (1974).
Kirzhnits, D. A., and Linde, A. D., *Ann. Phys. (NY)* **101**, 195 (1976).

Dimensional reduction and its use in electroweak lattice calculations

Ginsparg, P., *Nucl. Phys.* **B170**, 388 (1980).
Appelquist, T., and Pisarski, R., *Phys. Rev. D* **23**, 2305 (1981).
Landsman, N. P., *Nucl. Phys.* **B322**, 498 (1989).
Farakos, K., Kajantie, K., Rummukainen, K., and Shaposhnikov, M. E., *Nucl. Phys.* **B425**, 67 (1994); **B442**, 317 (1995).
Kajantie, K., Laine, M., Rummukainen, K., and Shaposhnikov M. E., *Nucl. Phys.* **B458**, 90 (1996); **B466**, 189 (1996).

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