A CHARACTERIZATION OF THE FINITE SIMPLE GROUP $\mathrm{PSp}_4(3)$

ZVONIMIR JANKO

The aim of this paper is to characterize the finite simple group $\mathrm{PSp}_4(3)$ by the structure of the centralizer of an involution contained in the centre of its Sylow 2-subgroup. More precisely, we shall prove the following result.

THEOREM. *Let t_0 be an involution contained in the centre of a Sylow 2-subgroup of* $\mathrm{PSp}_4(3)$. *Denote by H_0 the centralizer of t_0 in* $\mathrm{PSp}_4(3)$.

Let G be a finite group of even order with the following two properties:

(a) *G has no subgroup of index* 2, *and*

(b) *G has an involution t such that the centralizer $C_G(t)$ of t in G is isomorphic to H_0.*

Then G is isomorphic to $\mathrm{PSp}_4(3)$.

Remark. $\mathrm{PSp}_4(3)$ is the subgroup of index 2 of the group of the equation for the 27 lines on a general cubic surface.

The main difficulty in proving our theorem is to show that a group G with properties (a) and (b) possesses two conjugate classes of involutions and to determine the structure of the centralizer of an involution of G which is not conjugate to an involution in the centre of a Sylow 2-subgroup of G. From the knowledge of the structure of such a centralizer the 3-structure of G can be deduced. The identification of G with $\mathrm{PSp}_4(3)$ is then accomplished by using a theorem of J. G. Thompson **(7)**.

1. A preparatory lemma. For the determination of the centralizers of involutions in a group with properties (a) and (b) the following proposition will be used:

PROPOSITION. *Let G be a finite group of even order with the following two properties:*

(1) *The centralizer $C_G(t)$ of an involution t contained in the centre of a Sylow 2-subgroup of G is equal to $\langle t\rangle \times F$, where F is isomorphic to S_4 (the symmetric group in four letters).*

(2) *If S is a Sylow 2-subgroup of G, then $C_G(S') = S$, where S' denotes the commutator group of S.*

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Then if G is soluble, $G = C_G(t)$. If, however, G is not soluble, then G is isomorphic to S_6 (the symmetric group in six letters).

Proof. Let G be a finite group of even order satisfying the conditions (1) and (2). Put $F = V\cdot\langle\rho\rangle\cdot\langle\tau\rangle$, where $V = \langle\tau_1, \tau_2\rangle$ is a four-group, $V\cdot\langle\rho\rangle \cong A_4$, τ inverts ρ and centralizes τ_1, $\rho^{-1}\tau_1\rho = \tau_2$, $\rho^{-1}\tau_2\rho = \tau_1\tau_2$, and $\tau\tau_2\tau = \tau_1\tau_2$. Obviously $S = (V\cdot\langle\tau\rangle) \times \langle t\rangle$ is a Sylow 2-subgroup of $C(t)$ and $V\cdot\langle\tau\rangle$ is a dihedral group of order 8 with the element $a = \tau\tau_2$ of order 4. Also we have $\langle\tau_1\rangle = S'$ and so $C_G(\tau_1) = S$. The four-group $\langle t, \tau_1\rangle$ is equal to the centre $Z(S)$ of S.

The involutions t, τ_1, and $t\tau_1$ lie in three different conjugate classes of G. In fact, suppose that any two of these three involutions are conjugate in G. Then by a theorem of Burnside, they are conjugate in $N_G(S)$ and hence in $N_G(Z(S))$. But $C_G(Z(S)) = S$ and so $N_G(Z(S)) \supset S$. It follows that all three involutions t, τ_1, and $t\tau_1$ would lie in the same conjugate class in G. This is impossible since $|C_G(\tau_1)| = 16$ and $|C_G(t)| = 16\cdot 3$. The intersections of the conjugate classes of $C(t)$ with S are $\{1\}$, $\{\tau_1, \tau_2, \tau_1\tau_2\}$, $\{t\tau_1, t\tau_2, t\tau_1\tau_2\}$, $\{\tau, \tau\tau_1\}$, $\{t\tau, t\tau\tau_1\}$, $\{a, a^{-1}\}$, $\{ta, ta^{-1}\}$, $\{t\}$.

The group G has precisely two conjugate classes of elements of order 4. Suppose that a and ta are conjugate in G. Then there is an element $x \in G$ such that $x^{-1}ax = ta$. Since $a^2 = (ta)^2 = \tau_1$, we get $x^{-1}\tau_1 x = \tau_1$ and so $x \in S$. This is a contradiction since a and ta lie in two different conjugate classes of S.

The focal group S^* of S in G contains V. This is obvious, since $\rho^{-1}\tau_1\rho = \tau_2$ and $\rho^{-1}\tau_2\rho = \tau_1\tau_2$. (For the concept of a focal group see D. G. Higman **(5)**.)

If $S^* = V$, then $G = C_G(t) = \langle t\rangle \times F$. We have in this case a normal subgroup M of G such that $M \cap S = V$ and $[G : M] = 4$. Because $\rho \in M$ and $V\langle\rho\rangle \cong A_4$, all involutions are conjugate in M and a Sylow 2-subgroup of M is a four-group. Also we have $C_M(\tau_1) = V$. By a result of Suzuki **(8)** we have either $V \lhd M$ (and then $M = V\langle\rho\rangle$, $G = S\cdot M$, $G = C_G(t) = \langle t\rangle \times F$) or $M \cong A_5$. We shall show that the second case is impossible. Because the automorphism group of A_5 is S_5, it follows that $C_G(M) \neq \langle 1\rangle$ and

$$C_G(M) \cap M = \langle 1\rangle.$$

The condition $C_G(\tau_1) = S$ gives $C_G(M) \subseteq S$. Since $C_S(V) = \langle t\rangle \times V$, it follows that $C_G(M) \subseteq \langle t\rangle \times V$ and so $C_G(M) = \langle z\rangle$, where z is an involution contained in $(\langle t\rangle \times V)\backslash V$. It follows that $t = z\cdot v$, where $v \in V$. Both t and z centralize ρ. Hence v commutes with ρ. By the structure of A_4, $v = 1$. We get $C_G(M) = \langle t\rangle$, which contradicts our assumption (1).

The case $S^* = S$ is not possible. Hence G must have a normal subgroup N of index 2, *and t cannot be an element of S^*.* By way of contradiction, suppose that $t \in S^*$. Then at least one of the involutions τ or $t\tau$ must be conjugate in G to an involution in $Z(S)$. Replacing τ by $t\tau$, if necessary, we may suppose that τ is conjugate in G to an involution in $Z(S)$. Put $U = \langle Z(S), \tau\rangle$. Then

$$C(\tau) \cap C(t) = U$$

and a Sylow 2-subgroup of $C_G(\tau)$ has order 16. It follows that

$$N_G(U) \cap C(t) = S$$

and $N_G(U) \not\subseteq C(t)$. Also $C_G(U) = U$ and so $N_G(U)/U$ is isomorphic to a subgroup of GL(3, 2). Obviously 7 cannot divide $|N_G(U)|$ (because all involutions in U do not lie in the same conjugate class in G) and so 3 must divide $|N_G(U)|$. Let ζ be an element of order 3 contained in $N(U)$. We want to determine the orbits of ζ in $U \backslash \langle 1 \rangle$. Since t, τ_1, and $t\tau_1$ lie in three different conjugate classes in G, it follows that t, τ_1, and $t\tau_1$ must lie in three different orbits under the action of ζ. In particular, ζ must fix one of these three involutions and since $\zeta \notin C(t)$ and $C_G(\tau_1) = S$, it follows that $\zeta^{-1} \cdot t\tau_1 \cdot \zeta = t\tau_1$. The other two orbits are either $\{t, \tau, \tau\tau_1\}$, $\{\tau_1, \tau t, \tau\tau_1 t\}$ or $\{t, \tau t, \tau\tau_1 t\}$, $\{\tau_1, \tau, \tau\tau_1\}$. In the first case we get $S^* = \langle V, t\tau \rangle$ and in the second case $S^* = \langle V, \tau \rangle$. Hence in any case $t \notin S^*$. It follows that G has a normal subgroup N such that $G = \langle t \rangle \cdot N$ and replacing τ by $t\tau$, if necessary, we may suppose that $\tau \in N$ and so $F \subseteq N$, $N \cap C(t) = F$.

If G has no normal subgroup of index 4, then $G \cong S_6$. In this case we have $G = \langle t \rangle \cdot N$, $N \lhd G$, $N \cap C(t) = F$, and $S^* = \langle V, \tau \rangle$. N has no normal subgroup of index 2, $C_N(t) = F$, and $C_N(\tau_1) = \langle V, \tau \rangle$. A Sylow 2-subgroup of N is dihedral of order 8 and since N has no normal subgroup of index 2, all involutions in N are conjugate in N. Considering the action of V on $O(N)$ (and using the fact that the centralizer of any involution in N has order 8), it follows that $O(N) = \langle 1 \rangle$. N has no non-trivial normal subgroup of odd order. Using a result of Gorenstein and Walter **(3)**, it follows that $N \cong \mathrm{PSL}(2, q)$, q odd, or $N \cong A_7$. However, the second case cannot happen since the order of the centralizer of an involution in A_7 is divisible by 3. Since the order of the centralizer of an involution in $\mathrm{PSL}(2, q)$, q odd, is $q + \epsilon$ ($\epsilon = \pm 1$), it follows that $N \cong \mathrm{PSL}(2, 7)$ or $N \cong \mathrm{PSL}(2, 9) \cong A_6$. It is easy to see that the first case cannot happen. Suppose that $N \cong \mathrm{PSL}(2, 7)$. The case $C_G(N) = \langle 1 \rangle$ gives $G \cong \mathrm{Aut}(\mathrm{PSL}(2, 7)) = \mathrm{PGL}(2, 7)$. We know that a Sylow 2-subgroup of PGL(2, 7) is dihedral of order 16. This is a contradiction, since G has no elements of order 8. Hence $C_G(N) \neq \langle 1 \rangle$ and so $G = N \times C_G(N)$, $C_G(N) = \langle z \rangle$, where z is an involution contained in $(\langle t \rangle \times V) \backslash V$. It follows that $t = zv$ with $v \in V$. Both t and z centralize F and so v centralizes $F \cong S_4$. However, S_4 has no non-trivial centre and so $v = 1$. It follows that t centralizes N, a contradiction.

We have proved that $N \cong \mathrm{PSL}(2, 9) \cong A_6$. The automorphism group $\mathfrak{A}$ of A_6 has the property that $\mathfrak{A}/A_6$ is elementary abelian of order 4. Certainly $C_G(N) = \langle 1 \rangle$ and so G is a subgroup of $\mathfrak{A}$ containing $N \cong A_6$. Also G is not isomorphic to PGL(2, 9) because a Sylow 2-subgroup of PGL(2, 9) is dihedral of order 16.

Now, $\mathfrak{A}$ is the extension of PGL(2, 9) by the field automorphism f of order 2. PGL(2, 9) is the group of all 2×2 matrices

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix},$$

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where $a_{ij} \in GF(9)$ considered modulo the group of all scalar matrices

$$\begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}, \qquad k \in GF(9),$$

and we have

$$f \cdot \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \cdot f = \begin{bmatrix} a_{11}^3 & a_{12}^3 \\ a_{21}^3 & a_{22}^3 \end{bmatrix}.$$

$PSL(2, 9)$ is the subgroup of $PGL(2, 9)$ consisting of all matrices whose determinant is square in $GF(9)$. Let ζ be a generator of the multiplicative group of $GF(9)$. Then $\zeta^4 = -1$. Put

$$\alpha = \begin{bmatrix} \zeta & 0 \\ 0 & \zeta^{-1} \end{bmatrix}, \qquad \beta = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \qquad \delta = \begin{bmatrix} 1 & 0 \\ 0 & \zeta \end{bmatrix} \cdot f$$

and verify that $\alpha^4 = 1$, $\beta^2 = 1$, $\beta\alpha\beta = \alpha^{-1}$, $\delta^{-1}\alpha\delta = \alpha^{-1}$, $\delta^{-1}\beta\delta = \alpha^{-1}\beta$, $\delta^2 = \alpha^2$. Since $\langle \alpha, \beta \rangle$ is the dihedral Sylow 2-subgroup of $PSL(2, 9)$, it follow that $\langle \alpha, \beta, \delta \rangle$ is a Sylow 2-subgroup of $\langle PSL(2, 9), \delta \rangle$. Note that

$$\begin{bmatrix} 1 & 0 \\ 0 & \zeta \end{bmatrix}$$

is an element of $PGL(2, 9) \backslash PSL(2, 9)$. However,

$$(\delta\beta)^2 = \delta^2\delta^{-1}\beta\delta\beta = \alpha^2 \cdot \alpha^{-1}\beta \cdot \beta = \alpha$$

and so $\delta\beta$ is an element of order 8. Hence G cannot be isomorphic to $\langle PSL(2, 9), \delta \rangle$. It follows that G is isomorphic to $\langle PSL(2, 9), f \rangle$. Because $PSL(2, 9)$ has a subgroup isomorphic to A_5, we have $PSL(2, 9) \cong A_6$. Hence S_6 is a subgroup of $\mathrm{Aut}(PSL(2, 9))$ containing A_6. Since S_6 has no elements of order 8, it follows that $S_6 \cong \langle PSL(2, 9), f \rangle$ and so $G \cong S_6$. The proposition is completely proved.

2. Properties of H_0. We shall now study the structure of H_0 where H_0 denotes the centralizer in $PSp_4(3)$ of an involution contained in the centre of a Sylow 2-subgroup of $PSp_4(3)$. Let F_3 be the finite field of three elements. Let V be a four-dimensional vector space over F_3 equipped with a non-singular skew-symmetric bilinear form $x \cdot y \in F_3$ $(x, y \in V)$. Then V has a "symplectic basis," i.e. a basis n_1, m_1, n_2, m_2 such that $n_1 \cdot n_2 = m_1 \cdot m_2 = n_1 \cdot m_2 = m_1 \cdot n_2 = 0$ and $n_1 \cdot m_1 = n_2 \cdot m_2 = 1$. The group of all linear transformations σ of V such that $\sigma(x) \cdot \sigma(y) = x \cdot y$ for all x, y in V is called the symplectic group $Sp_4(3)$. This group has the centre of order 2 and the corresponding factor-group is $PSp_4(3)$. See Artin **(1)**.

Obviously a linear transformation σ of V belongs to $Sp_4(3)$ if and only if

$$\sigma(n_1) \cdot \sigma(n_2) = \sigma(m_1) \cdot \sigma(m_2) = \sigma(n_1) \cdot \sigma(m_2) = \sigma(m_1) \cdot \sigma(n_2) = 0,$$
$$\sigma(n_1) \cdot \sigma(m_1) = \sigma(n_2) \cdot \sigma(m_2) = 1.$$

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It follows that a linear transformation σ given by the matrix (α_{ij}) $(i, j = 1, \ldots, 4)$ in terms of the basis n_1, m_1, n_2, m_2, where

$$\sigma(n_1) = \alpha_{11} n_1 + \alpha_{12} m_1 + \alpha_{13} n_2 + \alpha_{14} m_2,$$

etc., belongs to $Sp_4(3)$ if and only if

$$\begin{aligned}
\alpha_{11}\alpha_{32} - \alpha_{12}\alpha_{31} + \alpha_{13}\alpha_{34} - \alpha_{14}\alpha_{33} &= 0,\\
\alpha_{21}\alpha_{42} - \alpha_{22}\alpha_{41} + \alpha_{23}\alpha_{44} - \alpha_{24}\alpha_{43} &= 0,\\
\alpha_{11}\alpha_{42} - \alpha_{12}\alpha_{41} + \alpha_{13}\alpha_{44} - \alpha_{14}\alpha_{43} &= 0,\\
\alpha_{21}\alpha_{32} - \alpha_{22}\alpha_{31} + \alpha_{23}\alpha_{34} - \alpha_{24}\alpha_{33} &= 0,\\
\alpha_{11}\alpha_{22} - \alpha_{12}\alpha_{21} + \alpha_{13}\alpha_{24} - \alpha_{14}\alpha_{23} &= 1,\\
\alpha_{31}\alpha_{42} - \alpha_{32}\alpha_{41} + \alpha_{33}\alpha_{44} - \alpha_{34}\alpha_{43} &= 1.
\end{aligned}$$

Take

$$t'_0 = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & -1 & \\ & & & -1 \end{bmatrix},$$

which is an involution in $Sp_4(3)$. (We identify the linear transformations in $Sp_4(3)$ with the corresponding matrices in terms of the basis n_1, m_1, n_2, m_2.) The centre of $Sp_4(3)$ is generated by the following matrix:

$$c = \begin{bmatrix} -1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{bmatrix}.$$

Then a matrix (α_{ij}) from $Sp_4(3)$ satisfies

$$(\alpha_{ij})\cdot t'_0 = t'_0\cdot(\alpha_{ij})\cdot c^r \qquad (r = 0, 1)$$

if and only if

$$(\alpha_{ij}) = \begin{bmatrix} \alpha_{11} & \alpha_{12} & & \\ \alpha_{21} & \alpha_{22} & & \\ & & \alpha_{33} & \alpha_{34} \\ & & \alpha_{43} & \alpha_{44} \end{bmatrix} \quad \begin{matrix} \text{with } \alpha_{11}\alpha_{22} - \alpha_{12}\alpha_{21} = 1 \\ \text{and } \alpha_{33}\alpha_{44} - \alpha_{34}\alpha_{43} = 1, \end{matrix}$$

or

$$(\alpha_{ij}) = \begin{bmatrix} & & \alpha_{13} & \alpha_{14} \\ & & \alpha_{23} & \alpha_{24} \\ \alpha_{31} & \alpha_{32} & & \\ \alpha_{41} & \alpha_{42} & & \end{bmatrix} \quad \begin{matrix} \text{with } \alpha_{13}\alpha_{24} - \alpha_{14}\alpha_{23} = 1 \\ \text{and } \alpha_{31}\alpha_{42} - \alpha_{32}\alpha_{41} = 1. \end{matrix}$$

Denote by H'_0 the group of all elements (α_{ij}) of $Sp_4(3)$ which "commute projectively" with t'_0, i.e. which satisfy $(\alpha_{ij})\cdot t'_0 = t'_0\cdot(\alpha_{ij})\cdot c^r$ $(r = 0, 1)$ and denote by K' the centralizer $C(t'_0)$ of t'_0 in $Sp_4(3)$.

The matrix

$$\beta' = \begin{bmatrix} & & 1 & 0 \\ & & 0 & 1 \\ 1 & 0 & & \\ 0 & 1 & & \end{bmatrix}$$

belongs to H'_0 and satisfies $\beta'^2 = 1$ and

$$\beta' \cdot \begin{bmatrix} \alpha_{11} & \alpha_{12} & & \\ \alpha_{21} & \alpha_{22} & & \\ & & \alpha_{33} & \alpha_{34} \\ & & \alpha_{43} & \alpha_{44} \end{bmatrix} \cdot \beta' = \begin{bmatrix} \alpha_{33} & \alpha_{34} & & \\ \alpha_{43} & \alpha_{44} & & \\ & & \alpha_{11} & \alpha_{12} \\ & & \alpha_{21} & \alpha_{22} \end{bmatrix}.$$

We have $[H'_0 : K'] = 2$ and $H'_0 = K' \cdot \langle \beta' \rangle$. Let S'_1 be the subgroup of K' consisting of all matrices of the form

$$\begin{bmatrix} 1 & 0 & & \\ 0 & 1 & & \\ & & \alpha_{33} & \alpha_{34} \\ & & \alpha_{43} & \alpha_{44} \end{bmatrix} \quad \text{with } \alpha_{33}\alpha_{44} - \alpha_{34}\alpha_{43} = 1.$$

Then we have $K' = S'_1 \times S'_2$, $t'_0 \in S'_1$, $S'_1 \cong S'_2 \cong SL(2, 3)$ with

$$\beta' \cdot S'_1 \cdot \beta' = S'_2.$$

Also β' commutes projectively with a matrix (α_{ij}) in K' if and only if

$$(\alpha_{ij}) = \begin{bmatrix} A & 0 \\ 0 & \pm A \end{bmatrix},$$

where A is any 2×2 matrix (over F_3) with determinant 1. Now put $H_0 = H'_0/\langle c \rangle$ and in the natural homomorphism from H'_0 onto H_0 let the images of t'_0, β', K', S'_1, S'_2 be t_0, β, K, S_1, S_2 respectively. Then obviously H_0 is the centralizer $C(t_0)$ of the involution t_0 in $PSp_4(3) = Sp_4(3)/\langle c \rangle$. We have $S_1 \cong S_2 \cong S'_1 \cong S'_2 \cong SL(2, 3)$, $H_0 = K \cdot \langle \beta \rangle$, $\beta^2 = 1$, $K = S_1 \cdot S_2$, $[S_1, S_2] = 1$ (which means that S_1 and S_2 commute elementwise), $S_1 \cap S_2 = \langle t_0 \rangle$, and $\beta \cdot S_1 \cdot \beta = S_2$. These relations completely determine the structure of H_0. But of course we have to show that t_0 is in fact an involution contained in the centre of a Sylow 2-subgroup of $PSp_4(3)$.

Let Q be a Sylow 2-subgroup of K. Then $Q = Q_1 \cdot Q_2$, $Q_1 \cap Q_2 = \langle t \rangle$, $[Q_1, Q_2] = 1$, $\beta Q_1 \beta = Q_2$, $Q_1 \cong Q_2$ is the quaternion group (of order 8), where $Q_i = Q \cap S_i$ ($i = 1, 2$). Note that K is 2-closed because S_1 and S_2 are 2-closed. It follows that $\langle \beta, Q \rangle$ is a Sylow 2-subgroup of H_0 and obviously the centre of $\langle \beta, Q \rangle$ is contained in Q. But the centre $Z(Q)$ of Q is equal to $\langle t_0 \rangle$. It follows that $Z(H_0) = Z(\langle \beta, Q \rangle) = Z(Q) = \langle t_0 \rangle$ and so $\langle \beta, Q \rangle$ has cyclic centre $\langle t_0 \rangle$. Let S be a Sylow 2-subgroup of $PSp_4(3)$ containing $\langle \beta, Q \rangle$. Since

$$C(t_0) \cap S = \langle \beta, Q \rangle$$

it follows $Z(S) \subseteq \langle \beta, Q\rangle$ and so $Z(S) = \langle t_0 \rangle$. But this gives $S = \langle \beta, Q\rangle$. Hence we have shown that $\langle \beta, Q\rangle$ is a Sylow 2-subgroup of $PSp_4(3)$ and since $Z(\langle \beta, Q\rangle)$ has only one non-trivial element it follows that the structure of $H_0 = C(t_0)$ is uniquely determined. Also we know that $PSp_4(3)$ is a simple group and this shows that $PSp_4(3)$ is a finite group of even order satisfying conditions (a) and (b).

A previous remark shows that $C(\beta) \cap H_0 = \langle t_0, \beta\rangle \times L$, where $\langle t_0, \beta\rangle$ is a four-group and $L \cong A_4 \cong LF(2, 3)$.

We have $S_1 = \langle \alpha_1, \beta_1, \sigma_1 | \alpha_1^2 = \beta_1^2 = t_0, \; t_0^2 = \sigma_1^3 = 1, \; \beta_1^{-1}\alpha_1 \beta_1 = \alpha_1^{-1}, \sigma_1^{-1}\alpha_1 \sigma_1 = \beta_1, \; \sigma_1^{-1}\beta_1 \sigma_1 = \alpha_1 \cdot \beta_1 \rangle$ because $S_1 \cong SL(2, 3)$ and $SL(2, 3)$ is an extension of the quaternion group by an automorphism of order 3. Put $\alpha_2 = \beta \cdot \alpha_1 \cdot \beta$, $\beta_2 = \beta \cdot \beta_1 \cdot \beta$, $\sigma_2 = \beta \cdot \sigma_1 \cdot \beta$. Then $S_2 = \langle \alpha_2, \beta_2, \sigma_2 \rangle$. We may also put $L = \langle \sigma_1 \cdot \sigma_2, \alpha_1 \cdot \alpha_2 \rangle$ because if we put $\rho = \sigma_1 \sigma_2$, $\tau_1 = \alpha_1 \alpha_2$, $\rho^{-1}\tau_1 \rho = \tau_2$, then $\langle \tau_1, \tau_2 \rangle$ is a four-group normalized by ρ, $\langle \rho, \tau_1 \rangle \subseteq C(\beta) \cap K$, and $\langle \rho, \tau_1 \rangle \cap \langle t_0, \beta \rangle = 1$. Every element of H_0 can be written uniquely in the form $\alpha_1^i \beta_1^j \sigma_1^k \tau_1^l \tau_2^m \rho^n \beta^p$, where $i = 0, 1, 2, 3; j = 0, 1; k = 0, 1, 2; l = 0, 1; m = 0, 1; n = 0, 1, 2; p = 0, 1$.

We shall now take a closer look at H_0. In particular we want to determine the conjugate classes of elements of H_0. Obviously $\langle \sigma_1, \sigma_2 \rangle$ is a Sylow 3-subgroup of H_0. This is an elementary abelian group of order 9 and so two non-trivial elements of $\langle \sigma_1, \sigma_2 \rangle$ are conjugate in H_0 if and only if they are conjugate in $N_{H_0}(\langle \sigma_1, \sigma_2 \rangle)$. We want to determine this normalizer. Suppose that

$$x_1 \cdot x_2 \in N_{H_0}(\langle \sigma_1, \sigma_2 \rangle)$$

where $x_i \in S_i$ $(i = 1, 2)$. Then

$$x_2^{-1} \cdot x_1^{-1} \cdot \sigma_1 \cdot x_1 x_2 = x_1^{-1} \sigma_1 x_1 \in S_1 \cap \langle \sigma_1, \sigma_2 \rangle = \langle \sigma_1 \rangle.$$

But $N_{S_1}(\langle \sigma_1 \rangle) = \langle t_0 \rangle \cdot \langle \sigma_1 \rangle$ and so $x_1 \in \langle t_0, \sigma_1 \rangle$. Considering $x_2^{-1} \cdot x_1^{-1} \cdot \sigma_2 \cdot x_1 x_2$ we see that $x_2 \in \langle t_0, \sigma_2 \rangle$. This gives

$$N_K(\langle \sigma_1, \sigma_2 \rangle) = C_K(\langle \sigma_1, \sigma_2 \rangle) = \langle t_0 \rangle \times \langle \sigma_1, \sigma_2 \rangle.$$

Since β normalizes but does not centralize $\langle \sigma_1, \sigma_2 \rangle$ it follows that

$$C_{H_0}(\langle \sigma_1, \sigma_2 \rangle) = \langle t_0 \rangle \times \langle \sigma_1, \sigma_2 \rangle$$

and $N_{H_0}(\langle \sigma_1, \sigma_2 \rangle) = \langle t_0, \beta \rangle \cdot \langle \sigma_1, \sigma_2 \rangle$.

Hence the representatives of conjugate classes of elements of order 3 in H_0 are σ_1, σ_1^{-1}, $\sigma_1 \cdot \sigma_2$, $\sigma_1^{-1} \cdot \sigma_2^{-1}$, and $\sigma_1^{-1} \cdot \sigma_2$. In particular, H_0 has only one real class consisting of elements of order 3. We shall determine the centralizers in H_0 of these representatives. Suppose that $x \in H_0 \backslash K$ and $x \in C_{H_0}(\sigma_1)$. Then $x = \beta \cdot x'$ with $x' \in K$ and so $x^{-1} \sigma_1 x = x'^{-1} \beta^{-1} \sigma_1 \beta x' = x'^{-1} \sigma_2 x' \in S_2$ since $S_2 \lhd K$. But $S_1 \cap S_2 = \langle t_0 \rangle$ and so $x'^{-1} \sigma_2 x' \neq \sigma_1$, a contradiction. Hence $C_{H_0}(\sigma_1) \subseteq K$. We have $C_K(\sigma_1) \supseteq S_2$ and so

$$C_K(\sigma_1) = S_2 \cdot C_{S_1}(\sigma_1) = S_2 \cdot \langle \sigma_1, t_0 \rangle = \langle \sigma_1, \sigma_2 \rangle \cdot Q_2.$$

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Similarly $C_{H_0}(\sigma_1^{-1}) = \langle \sigma_1, \sigma_2 \rangle \cdot Q_2$. We see that a Sylow 2-subgroup of $C_{H_0}(\sigma_1)$ and $C_{H_0}(\sigma_1^{-1})$ is a quaternion group of order 8. Since β centralizes $\sigma_1 \cdot \sigma_2$, it follows that $C_{H_0}(\sigma_1 \sigma_2) = \langle \beta \rangle \cdot C_K(\sigma_1 \cdot \sigma_2)$. Suppose that $x_1 \cdot x_2 \in C(\sigma_1 \cdot \sigma_2)$, where $x_i \in S_i$ $(i = 1, 2)$. Then

$$\sigma_1^{-1} \cdot x_1^{-1} \sigma_1 x_1 = \sigma_2 \cdot x_2^{-1} \sigma_2^{-1} x_2 \in S_1 \cap S_2 = \langle t_0 \rangle.$$

The case $\sigma_1^{-1} \cdot x_1^{-1} \cdot \sigma_1 \cdot x_1 = t_0$ cannot occur because $\sigma_1 \cdot t_0$ is of order 6 and $x_1^{-1} \cdot \sigma_1 \cdot x_1$ is of order 3. Hence $x_1^{-1} \sigma_1 x_1 = \sigma_1$, $x_1 \in C_{S_1}(\sigma_1) = \langle \sigma_1, t_0 \rangle$. Similarly we get $x_2 \in C_{S_2}(\sigma_2) = \langle \sigma_2, t_0 \rangle$ and so $C_K(\sigma_1 \cdot \sigma_2) = \langle \sigma_1, \sigma_2 \rangle \times \langle t_0 \rangle$. We see that a Sylow 2-subgroup of $C_{H_0}(\sigma_1 \cdot \sigma_2)$ and $C_{H_0}(\sigma_1^{-1} \cdot \sigma_2^{-1})$ is elementary abelian of order 4.

We shall now determine the "generalized centralizer" of $\sigma_1^{-1} \cdot \sigma_2$ in H_0 (i.e. the set of all x in H_0 such that $x^{-1} \cdot \sigma_1^{-1} \sigma_2 \cdot x = (\sigma_1^{-1} \sigma_2)^{\pm 1}$). The generalized centralizer $C_{H_0}{}^*(\sigma_1^{-1} \cdot \sigma_2)$ contains β since β inverts $\sigma_1^{-1} \sigma_2$. Hence

$$C_{H_0}{}^*(\sigma_1^{-1} \cdot \sigma_2) = \langle \beta \rangle \cdot C_K{}^*(\sigma_1^{-1} \cdot \sigma_2).$$

Let $x_1 \cdot x_2 \in C_K{}^*(\sigma_1^{-1} \cdot \sigma_2)$, where $x_i \in S_i$ $(i = 1, 2)$. Then

$$\sigma_1 \cdot x_1^{-1} \sigma_1^{-1} x_1 = \sigma_2 \cdot x_2^{-1} \sigma_2^{-1} x_2 \in \langle t_0 \rangle$$

or $\sigma_1^{-1} \cdot x_1^{-1} \sigma_1^{-1} x_1 = \sigma_2^{-1} \cdot x_2^{-1} \sigma_2^{-1} x_2 \in \langle t_0 \rangle$. However, the second case cannot happen because

$$C_{S_i}{}^*(\sigma_i) = C_{S_i}(\sigma_i) \qquad (i = 1, 2).$$

The first case gives $x_i \in \langle t_0, \sigma_i \rangle$ $(i = 1, 2)$, $C_{H_0}{}^*(\sigma_1^{-1} \cdot \sigma_2) = \langle \beta, t_0 \rangle \cdot \langle \sigma_1, \sigma_2 \rangle$. We have proved that a Sylow 2-subgroup of the centralizer in H_0 of a real element of order 3 in H_0 has order 2.

Now $\alpha_1 \cdot \alpha_2$ is an element of order 2 and we show easily that

$$\tilde{Q} = C_{H_0}(\alpha_1 \cdot \alpha_2) = \langle \alpha_1, \alpha_2, \beta_1 \cdot \beta_2, \beta \rangle,$$

which is a non-abelian group of order 32. We want to study the structure of $\tilde{Q}$. Since

$$\beta^{-1} \alpha_1 \beta \alpha_1^{-1} = \alpha_1^{-1} \alpha_2 = t_0 \alpha_1 \cdot \alpha_2$$

and

$$(\beta_1 \beta_2)^{-1} \cdot \alpha_1 \cdot \beta_1 \beta_2 \cdot \alpha_1^{-1} = t_0,$$

it follows that the four-group $\langle t_0, \alpha_1 \alpha_2 \rangle$ is contained in the centre and in the commutator group of $\tilde{Q}$. Since $\tilde{Q}/\langle t_0, \alpha_1 \cdot \alpha_2 \rangle$ is abelian, it follows that the commutator group $(\tilde{Q})'$ of $\tilde{Q}$ is equal to $\langle t_0, \alpha_1 \cdot \alpha_2 \rangle$. $\tilde{Q}$ is of class 2. The centre $Z(\tilde{Q})$ is obviously contained in $\langle \alpha_1, \alpha_2, \beta_1 \cdot \beta_2 \rangle$ and $Z(\langle \alpha_1, \alpha_2, \beta_1 \cdot \beta_2 \rangle)$ is contained in $\langle \alpha_1, \alpha_2 \rangle$. However, $\alpha_1 \notin Z(\tilde{Q})$ and so $Z(\tilde{Q}) = \langle t_0, \alpha_1 \alpha_2 \rangle$. We want to study the Sylow 2-subgroup $\langle Q, \beta \rangle$ of H_0. Since

$$\beta^{-1} \beta_1 \beta \beta_1^{-1} = \beta_1 \beta_2 \cdot t_0,$$

it follows that the commutator group $\langle Q, \beta \rangle'$ of $\langle Q, \beta \rangle$ is the elementary abelian group $\langle t_0, \alpha_1 \cdot \alpha_2, \beta_1 \cdot \beta_2 \rangle$ of order 8.

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The non-central involutions of K are conjugate in K to $\alpha_1 \cdot \alpha_2$. All elements of order 4 of K are conjugate to α_1 in H_0 and $C_{H_0}(\alpha_1) = \langle \alpha_1 \rangle \cdot S_2$. It is now easy to determine the centralizers in H_0 of elements $\sigma_1 \cdot t_0$ (order 6), $\sigma_1^{-1} \cdot t_0$ (order 6), $\sigma_1 \cdot \alpha_2$ (order 12), $\sigma_1^{-1} \cdot \alpha_2$ (order 12), $\sigma_1 \cdot \sigma_2 \cdot t_0$ (order 6), $\sigma_1^{-1} \cdot \sigma_2^{-1} \cdot t_0$ (order 6) and $\sigma_1^{-1} \cdot \sigma_2 \cdot t_0$ (order 6). The fact that all these elements are non-conjugate in H_0 follows easily from the fact that σ_1, σ_1^{-1}, $\sigma_1 \sigma_2$, $\sigma_1^{-1}\sigma_2^{-1}$, and $\sigma_1^{-1}\sigma_2$ are non-conjugate in H_0. If, for instance, there exists $z \in H_0$ such that $z^{-1} \cdot \sigma_1 t_0 \cdot z = \sigma_1^{-1} \cdot t_0$, then $z^{-1}\sigma_1 z = \sigma_1^{-1}$, a contradiction. Finally

$$C_{H_0}(\sigma_1 \cdot t_0) = C_{H_0}(\sigma_1), \qquad C_{H_0}(\sigma_1^{-1} t_0) = C_{H_0}(\sigma_1^{-1}),$$

etc., and

$$\begin{aligned} C_{H_0}(\sigma_1 \cdot \alpha_2) &= C_{H_0}(\sigma_1) \cap C_{H_0}(\alpha_2) \\ &= \langle Q_2, \sigma_1, \sigma_2 \rangle \cap \langle \alpha_2 \rangle \cdot S_1 = \langle \alpha_2, \sigma_1 \rangle = C_{H_0}(\sigma_1^{-1} \cdot \alpha_2). \end{aligned}$$

We have determined all conjugate classes of H_0 contained in K. It remains to determine the conjugate classes in $H_0 \backslash K$. We have $C_{H_0}(\beta) = \langle \beta, t_0 \rangle \times L$ and $C_{H_0}(t_0 \beta) = \langle \beta, t_0 \rangle \times L$. We compute that the 12 conjugates of β in H_0 are β, $t_0 \tau_1 \beta$, $t_0 \tau_2 \beta$, $t_0 \tau_1 \tau_2 \beta$, $\sigma_1 \rho\beta$, $\sigma_1^{-1}\rho^{-1}\beta$, $\alpha_1 \sigma_1 \tau_1 \rho\beta$, $\alpha_1^{-1}\sigma_1^{-1}\tau_1 \rho^{-1}\beta$, $\beta_1 \sigma_1 \tau_2 \rho\beta$, $t_0 \beta_1 \sigma_1^{-1}\tau_2 \rho^{-1}\beta$, $\alpha_1 \beta_1 \sigma_1 \tau_1 \tau_2 \rho\beta$, and $\alpha_1^{-1}\beta_1 \sigma_1^{-1}\tau_1 \tau_2 \rho^{-1}\beta$. This is obtained by conjugating β with 1, α_1, β_1, $\beta_1 \alpha_1$, σ_1, σ_1^{-1}, $\alpha_1 \sigma_1$, $\alpha_1 \sigma_1^{-1}$, $\beta_1 \sigma_1$, $\beta_1 \sigma_1^{-1}$, $\beta_1 \alpha_1 \sigma_1$, and $\beta_1 \alpha_1 \sigma_1^{-1}$, respectively. It follows in particular that β and $t_0 \beta$ are not conjugate in H_0. Since ρ and ρ^{-1} are not conjugate in H_0, it follows that $\rho\beta$ and $\rho^{-1}\beta$ are not conjugate in H_0. We have

$$C_{H_0}(\rho\beta) = C_{H_0}(\rho^{-1}\beta) = C_{H_0}(\beta) \cap C_{H_0}(\rho) = \langle t_0, \beta \rangle \times \langle \rho \rangle.$$

We have another two non-conjugate elements of order 6 contained in $H_0 \backslash K$: $t_0 \rho\beta$ and $t_0 \rho^{-1}\beta$ with the same centralizers. Finally $\alpha_1 \beta$ is an element of order 4 contained in $H_0 \backslash K$. $(\alpha_1 \beta)^2 = \tau_1 = \alpha_1 \alpha_2$ and so

$$C_{H_0}(\alpha_1 \beta) \subseteq C_{H_0}(\alpha_1 \alpha_2) = \tilde{Q}.$$

We have to determine $X = C_{\tilde{Q}}(\alpha_1 \beta)$. Obviously $X \supseteq \langle t_0, \alpha_1 \alpha_2 \rangle = Z(\tilde{Q}) = (\tilde{Q})'$ and $X \supseteq \langle \alpha_1 \cdot \beta \rangle$. Hence

$$X \supseteq \langle t_0, \alpha_1 \alpha_2, \alpha_1 \cdot \beta \rangle = \langle t_0 \rangle \times \langle \alpha_1 \beta \rangle,$$

which is an abelian normal subgroup (of order 8) of $\tilde{Q}$. We have four different conjugates of $\alpha_1 \cdot \beta$ in $\tilde{Q}$:

$$\alpha_1 \beta, \quad \beta \cdot \alpha_1 \beta \cdot \beta = \alpha_2 \beta, \quad \beta_1 \beta_2 \cdot \alpha_1 \beta \cdot \beta_1 \beta_2 = \alpha_1^{-1}\beta, \quad \beta \cdot \beta_1 \beta_2 \cdot \alpha_1 \beta\beta\beta_1 \beta_2 = \alpha_2^{-1}\beta$$

and so $X = \langle t_0 \rangle \times \langle \alpha_1 \beta \rangle$.

We have proved that $C_{H_0}(\alpha_1 \beta) = \langle t_0 \rangle \times \langle \alpha_1 \beta \rangle$. Summing up the orders of all conjugate classes of H_0 found so far, we get 576. Hence we have determined all conjugate classes of H_0.

3. The conjugacy classes of involutions and the structures of their centralizers. Let G be a finite group of even order with the properties (a) and

(b) of the theorem. Since $H = C_G(t)$ is isomorphic to H_0, we shall identify H and H_0. We have then $t = t_0$.

LEMMA 1. *The Sylow 2-subgroup $\langle Q, \beta\rangle$ of H is a Sylow 2-subgroup of G.*

Proof. This is obvious since the centre $Z(\langle Q, \beta\rangle) = \langle t\rangle$ is cyclic.

LEMMA 2. *The group G has precisely two conjugate classes of involutions $\mathfrak{K}_1$ and $\mathfrak{K}_2$ with the representatives t and $t\beta$, respectively: $\mathfrak{K}_1 \cap H$ is the union of two conjugate classes of H with the representatives t and β. $\mathfrak{K}_2 \cap H$ is the union of two conjugate classes of H with the representatives $t\beta$ and $\alpha_1\alpha_2$. Let $S = \langle t, \beta, \alpha_1\alpha_2, \beta_1\beta_2\rangle$. Then $C_G(S) = S$ and $N_G(S)/S \cong A_5$.*

Proof. By way of contradiction, suppose that t is conjugate in G to $\alpha_1\cdot\alpha_2$. The group $S = \langle t, \beta, \tau_1, \tau_2\rangle$ is elementary abelian of order 16, where $\tau_1 = \alpha_1\alpha_2$, $\tau_2 = \beta_1\beta_2$. $S \subseteq C(t) = H$ and S contains the commutator group

$$\langle Q, \beta\rangle' = \langle t, \tau_1, \tau_2\rangle$$

of $\langle Q, \beta\rangle$ and so $S \lhd \langle Q, \beta\rangle$. Also S is normalized by $\rho = \sigma_1\cdot\sigma_2$ and so $S \lhd \langle Q, \beta, \rho\rangle = \tilde{\tilde{H}}$. We have $N_G(S) \cap C(t) = \tilde{\tilde{H}}$, since σ_1 does not normalize S. ρ normalizes $\langle Q, \beta\rangle$ and $C(\rho) \cap \langle Q, \beta\rangle = \langle t, \beta\rangle$. Hence ρ does not fix any non-trivial element of $\langle Q, \beta\rangle/S$ and so $\tilde{\tilde{H}}/S \cong A_4$. Now, since $\tau_1 = \alpha_1\alpha_2$ is conjugate in G to t, it follows that $C_G(\tau_1) \cong H$. We know that $C(\tau_1) \cap H = \tilde{Q}$ is a non-abelian group of order 32 and the centre $Z(\tilde{Q}) = \langle t, \tau_1\rangle$ has order 4. Let T be a Sylow 2-subgroup of $C(\tau_1)$ containing $\tilde{Q}$. Then $[T : \tilde{Q}] = 2$. Suppose that S is not normal in T. Then there exists an element $x \in T\backslash\tilde{Q}$ such that $x^{-1}Sx \subseteq \tilde{Q}$ and $x^{-1}Sx \neq S$. It follows that $\tilde{Q} = S\cdot x^{-1}Sx$ and $D = S \cap x^{-1}Sx$ must have order 8 since $|\tilde{Q}| = 32$. But then (since S and $x^{-1}Sx$ are abelian) $C_G(D) \supseteq \langle S, x^{-1}Sx\rangle = \tilde{Q}$, which is a contradiction, since $|Z(\tilde{Q})| = 4$.

It follows that S is normal in T and so $N_G(S) \nsubseteq H$. On the other hand

$$C_G(S) \subseteq C_G(t) \cap C_G(\tau_1) = \tilde{Q}$$

and so $C_G(S) = S$ since $\tilde{Q}$ is non-abelian. We have proved that $\mathfrak{S} = N_G(S)/S$ is isomorphic to a subgroup of $GL(4, 2) \cong A_8$. Obviously $\mathfrak{V} = \langle Q, \beta\rangle/S$ is a Sylow 2-subgroup (elementary abelian of order 4) of $\mathfrak{S}$ and $\mathfrak{A} = \tilde{\tilde{H}}/S$ is a subgroup of $\mathfrak{S}$ isomorphic to A_4. Hence, in particular, all involutions of $\mathfrak{S}$ are conjugate in $\mathfrak{S}$. However, $\mathfrak{V}_1 = T/S$ and $\mathfrak{V} = \langle Q, \beta\rangle/S$ are two different Sylow 2-subgroups of $\mathfrak{S}$ with the intersection $\mathfrak{D} = \mathfrak{V} \cap \mathfrak{V}_1 = \tilde{Q}/S$ of order 2. This means that Sylow 2-subgroups of $\mathfrak{S}$ are not independent.

Now the order of A_8 is $2^6\cdot 3^2\cdot 5\cdot 7$ and the centralizer of any involution in A_8 has order $2^6\cdot 3$ or $2^5\cdot 3$. Since $C_{\mathfrak{S}}(\mathfrak{D}) \supseteq \langle\mathfrak{V}, \mathfrak{V}_1\rangle$, we get $C_{\mathfrak{S}}(\mathfrak{D}) \supset \mathfrak{V}$. By the above remark about A_8, $C_{\mathfrak{S}}(\mathfrak{D}) = \mathfrak{V}\cdot\mathfrak{U}$, where $|\mathfrak{U}| = 3$ and $\mathfrak{U} \lhd \mathfrak{U}\cdot\mathfrak{V}$. Since $\mathfrak{V}$ and $\mathfrak{V}_1$ are contained in $C_{\mathfrak{S}}(\mathfrak{D})$, it follows that $\mathfrak{U}\cdot\mathfrak{V}$ is not a direct product of $\mathfrak{U}$ and $\mathfrak{V}$.

Suppose at first that $\mathfrak{M} = O(\mathfrak{S}) \neq \langle 1\rangle$. Here $O(\mathfrak{S})$ denotes the maximal normal odd-order subgroup of $\mathfrak{S}$. Considering the action of the four-group $\mathfrak{V}$ on $\mathfrak{M}$ we see that the order of $\mathfrak{M}$ is either 3^3 or 3. However, the first case cannot occur since 3^3 does not divide $|A_8|$. It follows that $|\mathfrak{M}| = 3$, $\mathfrak{B}$ centralizes $\mathfrak{M}$, $\mathfrak{B}\cdot\mathfrak{M} = \mathfrak{B} \times \mathfrak{M} = \mathfrak{B} \times \mathfrak{U}$, a contradiction. Hence $O(\mathfrak{S}) = \langle 1 \rangle$. Using a result of Gorenstein and Walter **(3)** we see that $\mathfrak{S}$ is isomorphic to A_7 or to some $\mathrm{LF}(2, q)$ with $q \equiv \pm 3 \pmod 8$. However, the first case cannot occur since a Sylow 2-subgroup of A_7 has order 8. From the order of A_8 follows that $q = 3$ or 5. But both $\mathrm{LF}(2, 3) \cong A_4$ and $\mathrm{LF}(2, 5) \cong A_5$ have independent Sylow 2-subgroups, a contradiction.

We have proved that t cannot be conjugate to $\alpha_1 \cdot \alpha_2$ in G. Suppose now that G is 2-normal. Since $\langle t \rangle$ is the centre of the Sylow 2-subgroup $\langle Q, \beta \rangle$ of G, it follows by the Hall–Grün theorem **(4)** that the greatest factor group of G which is a 2-group is isomorphic to that of $C_G(t) = H$, i.e. is isomorphic to H/K, which is of order 2. But this contradicts our condition (a).

It follows that G is not 2-normal. This means that there exists an element z in G such that $t \in \langle Q, \beta \rangle \cap z^{-1}\cdot\langle Q, \beta \rangle z$ but $\langle t \rangle$ is not the centre of $z^{-1}\langle Q, \beta \rangle z$. The centre of $z^{-1}\langle Q, \beta \rangle z$ is $\langle z^{-1}tz \rangle$ and so $t \neq z^{-1}tz$. On the other hand, because $z^{-1}tz$ is contained in the centre of $z^{-1}\langle Q, \beta \rangle z$ and also $t \in z^{-1}\cdot\langle Q, \beta \rangle \cdot z$, it follows that t and $z^{-1}tz$ commute. Hence $\tau = z^{-1}tz \in C_G(t) = H$. In other words t is conjugate in G to an involution τ in H and $t \neq \tau$. Since t cannot be conjugate in G to $\alpha_1 \cdot \alpha_2$, it follows that t must be conjugate in G to β or $t\beta$. Interchanging β and $t\beta$, if necessary, we may assume that t is conjugate in G to β.

We are now planning to determine the structure of $N_G(S)$, where $S = \langle t, \beta, \tau_1, \tau_2 \rangle$, $\tau_1 = \alpha_1 \alpha_2$, and $\tau_2 = \beta_1 \beta_2$. Again $S \lhd \langle Q, \beta, \rho \rangle$, where $\rho = \sigma_1 \sigma_2$ and $\rho^{-1}\tau_1 \rho = \tau_2$, $\rho^{-1}\tau_2 \rho = \tau_1 \tau_2$, $\rho t = t\rho$, $\rho\beta = \beta\rho$. Also

$$N_G(S) \cap C_G(t) = \langle Q, \beta, \rho \rangle = \tilde{\tilde{H}}$$

and $\tilde{\tilde{H}}/S \cong A_4$. Now, since β is conjugate in G to t, we have $C_G(\beta) \cong H = C_G(t)$. We know that $C(\beta) \cap C(t) = S \cdot \langle \rho \rangle = D$. Let T be a Sylow 2-subgroup of $C(\beta)$ containing S. Since D is 2-closed, $T \cap C(t) = S$ and $[T : S] = 4$. In particular $N_G(S) \not\subseteq H$ and $\mathfrak{S} = N(S)/S$ is not 2-closed since $(N(S) \cap T)/S$ is a non-trivial 2-subgroup of $\mathfrak{S}$ which is not contained in $\mathfrak{B} = \langle Q, \beta \rangle / S$. Here $\mathfrak{B}$ is a Sylow 2-subgroup of $\mathfrak{S}$ and $\mathfrak{B}$ is elementary abelian of order 4. All involutions are conjugate in $\mathfrak{S}$ since $\tilde{\tilde{H}}/S$ is a subgroup of $\mathfrak{S}$. Obviously $C_G(S) = S$ and so $\mathfrak{S}$ is isomorphic to a subgroup of $\mathrm{GL}(4, 2) \cong A_8$. We want to determine $N_{\mathfrak{S}}(\mathfrak{B})$. We have $N_G(\langle Q, \beta \rangle) \subseteq C_G(t) = H$ and so

$$N_G(\langle Q, \beta \rangle) = \tilde{\tilde{H}}.$$

It follows that $N_{\mathfrak{S}}(\mathfrak{B}) = \tilde{\tilde{H}}/S \cong A_4$.

Suppose at first that $O(\mathfrak{S}) = \mathfrak{M} \neq \langle 1 \rangle$. Then considering the action of $\mathfrak{B}$ on $\mathfrak{M}$ and using the fact that all involutions are conjugate in $\mathfrak{S}$ and also the fact that the centralizer of any involution in A_8 has order $3 \cdot 32$ or $3 \cdot 64$, it follows that either $|\mathfrak{M}| = 27$ or $|\mathfrak{M}| = 3$ and $\mathfrak{B} \cdot \mathfrak{M} = \mathfrak{B} \times \mathfrak{M}$. However, the first case is not possible because 27 does not divide the order of A_8. The second case is also not possible because $N_{\mathfrak{S}}(\mathfrak{B}) \cong A_4$. We have proved that $O(\mathfrak{S}) = \langle 1 \rangle$ and $\mathfrak{S}$ has no subgroups of index 2. If d is an involution in $\mathfrak{B}$, then again by

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the structure of A_8 we have either $C_{\mathfrak{S}}(d) = \mathfrak{U}\cdot\mathfrak{V}$ with $\mathfrak{U} \lhd \mathfrak{U}\cdot\mathfrak{V}$ and $|\mathfrak{U}| = 3$ or $C_{\mathfrak{S}}(d) = \mathfrak{V}$. In the first case by a result of Gorenstein and Walter **(3)** we have $\mathfrak{S} \cong \mathrm{LF}(2, q)$ with $q \pm 1 = 12 = 3\cdot 4 = |C_{\mathfrak{S}}(d)|$. Hence $q = 11$ or $q = 13$, which contradicts the order of A_8. Hence the second case must be involved and so $\mathfrak{S} \cong A_5$. Let μ be an element of order 5 contained in $N_G(S)$. Since $C_G(S) = S$, it follows that μ acts fixed-point-free on S. Now we take a closer look at the elements of S. Let $\mathfrak{K}_1$ be the conjugate class in G with the representative t. Then

$$\mathfrak{K}_1 \cap S \supseteq \{t, \beta, t\tau_1\beta, t\tau_2\beta, t\tau_1\tau_2\beta\}.$$

The six involutions $\tau_1, \tau_2, \tau_1\tau_2, t\tau_1, t\tau_2, t\tau_1\tau_2$ are conjugate in G to τ_1 and the four involutions $t\beta, \tau_1\beta, \tau_2\beta, \tau_1\tau_2\beta$ are conjugate in G to $t\beta$. Since t is not conjugate in G to τ_1, it follows that τ_1 must be conjugate (in $N(S)$) to $t\beta$ and t is not conjugate in G to $t\beta$. Lemma 2 is completely proved.

LEMMA 3. *The group G is not an N-group in the sense of J. G. Thompson* **(7)**.

LEMMA 4. *We have the following two possibilities for the structure of* $C_G(t\beta)$:
(i) $C_G(t\beta)$ *is isomorphic to the centralizer of an involution in* A_8 *which does not lie in the centre of any Sylow 2-subgroup of* A_8.
(ii) $C_G(t\beta)$ *is the non-splitting central extension of* $\langle t\beta\rangle$ *by* S_6.

Proof. Again put $S = \langle t, \beta, \tau_1, \tau_2\rangle$, where $\tau_1 = \alpha_1\alpha_2$, $\tau_2 = \beta_1\beta_2$. Obviously $\tilde{Q} = C(\tau_1) \cap C(t)$ is contained in $N(S)$ and $\tilde{Q}$ is a Sylow 2-subgroup of $C(\tau_1)$. Namely, τ_1 is not conjugate to t in G and so τ_1 does not lie in the centre of any Sylow 2-subgroup of G. We have $\rho^{-1}\tau_1\rho = \tau_2$, $\rho^{-1}\tau_2\rho = \tau_1\tau_2$, where $\rho = \sigma_1\sigma_2 \in N(S)$ and so $|C(x) \cap N(S)|$ is divisible by 32 for any $x \in \{\tau_1, \tau_2, \tau_1\cdot\tau_2, t\tau_1, t\tau_2, t\tau_1\tau_2\}$. Also we know that $\langle Q, \beta\rangle \subseteq N(S)$ (since S contains the commutator group of $\langle Q, \beta\rangle$) and $t\beta, \tau_1\beta, \tau_2\beta, \tau_1\tau_2\beta$ are all conjugate in $\langle Q, \beta\rangle \subseteq N(S)$. It follows that $t\beta$ is conjugate in $N(S)$ to an element of $\{\tau_1, \tau_2, \tau_1\tau_2, t\tau_1, t\tau_2, t\tau_1\tau_2\}$ and so $Y = C(t\beta) \cap N(S) = \tilde{\tilde{Q}}\cdot\langle\rho\rangle$, where $[\tilde{\tilde{Q}} : S] = 2$, $\tilde{\tilde{Q}} \cong \tilde{Q}$, and $C_G(t\beta) \cap C_G(t) = S\cdot\langle\rho\rangle$. By the structure of $A_5 \cong N(S)/S$, Y is not 2-closed. Y is also not 3-closed since ρ does not act trivially on S.

$$N(\langle\rho\rangle) \cap \langle\rho\rangle\cdot S = \langle\rho\rangle \times C_S(\rho) = \langle\rho\rangle \times \langle t, \beta\rangle.$$

Since Y/S is non-abelian of order 6, $N_Y(\langle\rho\rangle) \neq C_Y(\rho)$. Hence

$$Y = N_Y(\langle\rho\rangle)\cdot S, \qquad N_Y(\langle\rho\rangle) \cap S = \langle t, \beta\rangle,$$

ρ is real in Y, and $\langle t, \beta\rangle$ is normal in Y. However, $C_G(\langle t, \beta\rangle = S\cdot\langle\rho\rangle$ and so $N_G(\langle t, \beta\rangle) = Y$ because t and $t\beta$ are not conjugate in G. $S\cdot\langle\rho\rangle$ is a normal subgroup of index 2 in Y. Let B be a Sylow 2-subgroup of $N_Y(\langle\rho\rangle)$. Then there exists an element z of 2-power order in B such that $z^{-1}tz = \beta$. Hence B is the dihedral group of order 8 and so we may choose z to be an involution. The group $\langle\rho\rangle\cdot\langle\tau_1, \tau_2\rangle$ is isomorphic to A_4. On the other hand $S\cdot\langle\rho\rangle$ has the normal subgroup $\langle\rho\rangle\cdot\langle\tau_1, \tau_2\rangle$ of index 4 which is the smallest normal subgroup

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of $S\cdot\langle\rho\rangle$ with 2-factor group. Hence $\langle\rho\rangle\cdot\langle\tau_1, \tau_2\rangle$ is characteristic in $S\cdot\langle\rho\rangle$ and so $\langle\rho\rangle\cdot\langle\tau_1, \tau_2\rangle$ is normal in Y. But $\langle\tau_1, \tau_2\rangle$ is characteristic in $\langle\rho\rangle\cdot\langle\tau_1, \tau_2\rangle$ and so $\langle\tau_1, \tau_2\rangle$ is normal in Y. Also the involution z normalizes $\langle\rho\rangle$ and because $C_Y(\rho) = \langle\rho\rangle \times \langle t, \beta\rangle$ and $z \notin \langle t, \beta\rangle$ ($\langle z, t, \beta\rangle$ being dihedral of order 8), we have $z\rho z = \rho^{-1}$. We also have $\langle z, S\rangle = \tilde{\tilde{Q}}$ and this is isomorphic to $\tilde{Q}$. It follows that the centre of $\tilde{\tilde{Q}}$ has order 4 and so $|C_S(z)| = 4$. On the other hand, $C_S(z) \supseteq \langle t\beta\rangle$ and so $|C(z) \cap \langle\tau_1, \tau_2\rangle| = 2$ (using the fact that $\langle\tau_1, \tau_2\rangle \lhd Y$). We may put $z^{-1}\cdot\tau_1\tau_2\cdot z = \tau_1\tau_2$ and $z^{-1}\tau_1 z = \tau_2\cdot$ $\langle z, \tau_1, \tau_2\rangle$ is the dihedral group of order 8. The structure of Y is completely determined.

We see that $Y/\langle t\beta\rangle$ is the direct product of $\langle t, \beta\rangle/\langle t\cdot\beta\rangle$ and $\langle z, \rho, \tau_1, \tau_2\rangle\cdot\langle t\beta\rangle/\langle t\beta\rangle$, which is isomorphic to $\langle z, \rho, \tau_1, \tau_2\rangle$ and this is isomorphic to S_4. Also $N(\langle t, \beta\rangle) \cap C_G(t\beta) = Y$ and so $C_G(t\beta)/\langle t\beta\rangle$ satisfies the condition (1) of Proposition 1, because Y contains a Sylow 2-subgroup of $C_G(t\beta)$. Now, $\tilde{Q}/\langle\tau_1\rangle$ is a Sylow 2-subgroup of $C_G(\tau_1)/\langle\tau_1\rangle$ and $\langle t, \tau_1\rangle/\langle\tau_1\rangle$ is the commutator group of $\tilde{Q}/\langle\tau_1\rangle$. On the other hand, $N_G(\langle t, \tau_1\rangle)$ is contained in $C_G(t) = H$ because t is not conjugate in G to either $\tau_1 = \alpha_1\alpha_2$ or $t\tau_1 = \alpha_1{}^{-1}\alpha_2$. It follows that

$$N_G(\langle t, \tau_1\rangle) \cap C_G(\tau_1) \subseteq C_G(t) \cap C_G(\tau_1) = \tilde{Q}.$$

Since τ_1 is conjugate in G to $t\beta$, it follows that the centralizer in $C_G(t\beta)/\langle t\cdot\beta\rangle$ of the commutator group of $\tilde{\tilde{Q}}/\langle t\beta\rangle$ is equal to $\tilde{\tilde{Q}}/\langle t\cdot\beta\rangle$. This shows that the condition (2) of Proposition 1 is also satisfied.

Applying the Proposition 1 on the group $C_G(t\beta)/\langle t\beta\rangle$ (and using the fact that since τ_1 is a square of $\alpha_1\beta$ we have that $\langle\tau_1\rangle$ does not split in $\tilde{Q}$ and consequently $\langle t\beta\rangle$ does not split in $\tilde{\tilde{Q}}$) we get that either

$$C_G(t\beta) = Y = C_G(t\beta) \cap N_G(S) \quad \text{or} \quad C_G(t\beta)$$

is the non-splitting central extension of $\langle t\beta\rangle$ by S_6 (symmetric group in six letters).

It remains to show that Y is isomorphic to the centralizer of an involution in A_8 which does not lie in the centre of any Sylow 2-subgroup of A_8. We establish the isomorphism from Y onto $C(\mu)$ in the notation of Wong **(9)**, by mapping the generators $\rho, \tau_1, \tau_2, t, \beta, z$ of Y onto the generators $\nu,^{-1}\, \tau\lambda$, $\pi\mu\cdot\tau\lambda$, λ, $\lambda\mu$, μ' (in this order) of $C(\mu)$ and then verifying that the same relations are satisfied by both systems of generators. The lemma is proved.

LEMMA 5. *The case* (ii) *of Lemma* 4 *cannot happen.*

Proof. Suppose that we have case (ii) of Lemma 4. There are precisely three conjugate classes of involutions in S_6. Note that the centre Z of a Sylow 2-subgroup of S_6 is elementary of order 4, and that the three involutions in Z are not conjugate in S_6. Hence $C_G(t\beta)$ has precisely three conjugate classes of subgroups of order 4 containing $\langle t\beta\rangle$. Since $t\beta$ is conjugate in G to $\alpha_1\alpha_2 = \tau_1$, we may consider $C_G(\tau_1)$. We want to find explicitly the three subgroups non-conjugate in $C_G(\tau_1)$ which are of order 4 and contain $\langle\tau_1\rangle$. They are $\langle t, \tau_1\rangle$, $\langle\alpha_1\beta\rangle$, and $\langle\beta\tau_2, \tau_1\rangle$, where $\tau_2 = \beta_1\beta_2$. Clearly $\langle\alpha_1\beta\rangle$, being cyclic of order 4, cannot be conjugate to any of the four-groups $\langle t, \tau_1\rangle$ and $\langle \beta\tau_2, \tau_1\rangle$. On the other hand $\langle t, \tau_1\rangle/\langle\tau_1\rangle$ is the commutator group of $\tilde{Q}/\langle\tau_1\rangle$ and $\langle\beta\tau_2, \tau_1\rangle/\langle\tau_1\rangle$ is the subgroup of order 2 contained in the centre of $\tilde{Q}/\langle\tau_1\rangle$ and is different from $\langle t, \tau_1\rangle/\langle\tau_1\rangle$. Hence the four-groups $\langle t, \tau_1\rangle$ and $\langle\beta\tau_2, \tau_1\rangle$ cannot be conjugate in $C_G(\tau_1)$. The four-group $\langle\beta\tau_2, \tau_1\rangle$ is normal in $\tilde{Q}$ but is not contained in the centre of $\tilde{Q}$ and so $\beta\tau_2$ and $\beta\tau_1\tau_2$ are conjugate in $C_G(\tau_1)$. Since

$$N(\langle t, \tau_1\rangle) \cap C(\tau_1) = \tilde{Q},$$

it follows that

$$C(t) \cap C(\tau_1) = C(t\tau_1) \cap C(\tau_1) = \tilde{Q}.$$

Using the structure of S_6, it follows that $N(\langle\beta\tau_2, \tau_1\rangle) \cap C(\tau_1) = \tilde{Q}\cdot X$, where $X \subseteq C(\tau_1)$ is a subgroup of order 3 and so

$$C(\beta\tau_2) \cap C(\tau_1) = X\cdot\langle t, \tau_1, \tau_2, \beta\rangle.$$

Let $\mathfrak{R}_1$ and $\mathfrak{R}_2$ have the same meaning as in Lemma 2. Then $t \in \mathfrak{R}_1$, $t\tau_1 \in \mathfrak{R}_2$, $\beta\tau_2 \in \mathfrak{R}_2$, and $\beta\tau_1\tau_2 \in \mathfrak{R}_2$.

Now let x be an involution in $C(\tau_1)$. Suppose $x \neq \tau_1$ and consider the four-group $\langle x, \tau_1\rangle$. Because S_6 has precisely three conjugate classes of involutions, it follows that every group of order 4 in $C(\tau_1)$ which contains τ_1 must be conjugate in $C(\tau_1)$ to one of the following groups (of order 4): $\langle t, \tau_1\rangle$, $\langle\alpha_1\beta\rangle$, and $\langle\beta\tau_2, \tau_1\rangle$. Since $\langle x, \tau_1\rangle$ is a four-group, $\langle x, \tau_1\rangle$ is conjugate in $C(\tau_1)$ to (only one of) $\langle t, \tau_1\rangle$ or $\langle\beta\tau_2, \tau_1\rangle$. The involutions t and $t\tau_1$ cannot be conjugate in $C(\tau_1)$ because $t \in \mathfrak{R}_1$ and $t\tau_1 \in \mathfrak{R}_2$. However, $\beta\tau_2$ and $\beta\tau_1\tau_2$ are elements of $\mathfrak{R}_2$ and are conjugate in $C(\tau_1)$. It follows that x must be conjugate in $C(\tau_1)$ to one of the involutions t, $t\tau_1$, and $\beta\tau_2$. In particular, we have proved that $C(\tau_1)$ has precisely four conjugate classes of involutions and only one of them (with the representative t) lies in $\mathfrak{R}_1$ and $C(t) \cap C(\tau_1)$ is a 2-group.

Consider now $C_G(t\beta)$. We have $\beta \in C_G(t\beta)$, $\beta \in \mathfrak{R}_1$, and $C(\beta) \cap C(t\beta)$ contains $\langle t, \beta\rangle \times \langle\tau_1, \tau_2, \rho\rangle$, where $\rho = \sigma_1\sigma_2$ and so $C(\beta) \cap C(t\beta)$ is not a 2-group. This is a contradiction. The lemma is proved.

Let us find some conjugate classes in $C_G(t\beta)$. First of all we have one conjugate class of involutions consisting of one single involution $t\beta \in \mathfrak{R}_2$. ($\mathfrak{R}_1$, $\mathfrak{R}_2$ have the same meaning as in Lemma 2). The conjugate class of $t \in \mathfrak{R}_1$ consists of two elements and

$$C(t) \cap C(t\beta) = \langle t, \beta\rangle \times \langle\rho\rangle\cdot\langle\tau_1, \tau_2\rangle.$$

The conjugate class of $\tau_1 \in \mathfrak{R}_2$ consists of three elements and

$$C(\tau_1\tau_2) \cap C(t\beta) = (\langle t, \beta\rangle \times \langle\tau_1, \tau_2\rangle)\cdot\langle z\rangle,$$

where $\tau_1\tau_2$ is conjugate to τ_1 in $C(t\beta)$. The conjugate class of $t\beta\tau_1 \in \mathfrak{R}_1$ consists of three involutions. The conjugate class of $t\tau_1 \in \mathfrak{R}_2$ obviously consists of six elements and

$$C(t\tau_1) \cap C(t\beta) = \langle t, \beta\rangle \times \langle\tau_1, \tau_2\rangle.$$

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Finally the conjugate class of the involution z consists of 12 involutions, namely,

$$C(\rho) \cap C(t\beta) = \langle \rho \rangle \times \langle t, \beta \rangle.$$

On the other hand z inverts ρ and so $C(t\beta)$ has precisely one conjugate class of elements of order 3 consisting of eight elements. We have

$$C(z) \cap C(t\beta) = \langle z \rangle \times (C(z) \cap ((\langle t, \beta \rangle \times \langle \tau_1, \tau_2 \rangle)\langle \rho \rangle)).$$

Suppose that z fixes an element x in

$$W = ((\langle t, \beta \rangle \times \langle \tau_1, \tau_2 \rangle)\langle \rho \rangle$$

which is not a 2-element. Then x fixes an element of order 3 lying in W and so a conjugate of z under an element of W fixes ρ, a contradiction. Hence

$$C_W(z) = C(z) \cap (\langle t, \beta \rangle \times \langle \tau_1, \tau_2 \rangle) = \langle t\beta, \tau_1 \tau_2 \rangle$$

because a Sylow 2-subgroup of $C(t\beta)$ is isomorphic to $\tilde{Q}$ and $|Z(\tilde{Q})| = 4$. It follows that

$$C(z) \cap C(t\beta) = \langle z \rangle \times \langle t\beta, \tau_1 \tau_2 \rangle.$$

There are three conjugacy classes of elements of order 6 (with the representatives $\rho t\beta$, ρt, and $\rho\beta$) with eight elements in each class and

$$C(\rho t\beta) \cap C(t\beta) = C(\rho t) \cap C(t\beta) = C(\rho\beta) \cap C(t\beta) = \langle \rho \rangle \times \langle t, \beta \rangle.$$

We are able to show that we have found all conjugate classes of involutions in $C_G(t\beta)$. Namely, any involution of $C_G(t\beta)$ is conjugate to z or to an involution in $\langle t, \beta \rangle \times \langle \tau_1, \tau_2 \rangle$ or to $z \cdot x$, where

$$x \in C(z) \cap (\langle t, \beta \rangle \times \langle \tau_1, \tau_2 \rangle) = \langle t\beta, \tau_1 \tau_2 \rangle \qquad (x \neq 1).$$

But $\langle z, t, \beta \rangle$ is dihedral with the centre $\langle t\beta \rangle$ and so in this group z is conjugate to $z \cdot t\beta$. Similarly, working in the dihedral group $\langle z, \tau_1, \tau_2 \rangle$, we see that z is conjugate to $z \cdot \tau_1 \tau_2$. Hence z is also conjugate to $z \cdot t\beta \cdot \tau_1 \tau_2$.

Since $S = \langle t, \beta, \tau_1, \tau_2 \rangle$ contains the commutator group of $\langle Q, \beta \rangle$, it follows that $\tilde{Q}$ is contained in $N(S)$. But also $\langle z, t, \beta, \tau_1, \tau_2 \rangle$ is contained in $N(S)$. We now use the fact that $N(S)/S \cong A_5$ and that all involutions in A_5 are conjugate. Hence there exists an element $y \in N(S)$ such that $z' = y^{-1}zy \in \tilde{Q} \backslash S$. The involution $\alpha_1 \beta_1 \beta_2$ is contained in $\tilde{Q} \backslash S$ and $C(\alpha_1 \beta_1 \beta_2) \cap S = Z(\tilde{Q})$ has order 4. Hence the conjugate class of $\alpha_1 \beta_1 \beta_2$ in $\tilde{Q}$ has order 4. On the other hand, we have either $z' = \alpha_1 \beta_1 \beta_2$ or $z' = \alpha_1 \beta_1 \beta_2 x$, where $x \neq 1$ and $x \in S \cap C(\alpha_1 \beta_1 \beta_2) = Z(\tilde{Q})$. Hence there are only four involutions in $\tilde{Q} \backslash S$ and so z' is conjugate to $\alpha_1 \beta_1 \beta_2$. This gives $z \in \mathfrak{K}_2$.

4. The simplicity of *G*.

We are now in the position to prove

LEMMA 6. *G is a simple group.*

Proof. Suppose at first that $O(G) \neq 1$. Act on $O(G)$ by the four-group $\langle t, \beta \rangle$. We know that $C_G(x)$ does not have a non-trivial normal odd-order subgroup for any involution $x \in \langle t, \beta \rangle$. Hence $\langle t, \beta \rangle$ acts fixed-point-free on $O(G)$, a contradiction. We have proved that G has no non-trivial odd-order normal subgroups.

Suppose now that G has a proper normal subgroup N with odd-order factor-group G/N. Then $\langle Q, \beta \rangle$ (being a Sylow 2-subgroup of G) is contained in N. The Frattini argument gives $G = N \cdot N(\langle Q, \beta \rangle)$ and the fact that $\langle t \rangle$ is the centre of $\langle Q, \beta \rangle$ gives

$$N_G(\langle Q, \beta \rangle) \subseteq C_G(t) = H.$$

Hence

$$N_G(\langle Q, \beta \rangle) = \langle Q, \beta \rangle \cdot \langle \rho \rangle,$$

where $\rho = \sigma_1 \sigma_2$ and

$$N_G(\langle Q, \beta \rangle) \cap N = \langle Q, \beta \rangle.$$

On the other hand, ρ is contained in $C_G(t\beta)$ and $t\beta \in N$. This is a contradiction because $C_G(t\beta)$ does not have proper normal subgroups with an odd-order factor-group. Hence G has no proper normal subgroups with odd-order factor-group.

Suppose now that G has a proper non-trivial normal subgroup M. Then both numbers $|M|$ and $[G : M]$ are even. Denote by $\mathfrak{K}_1$ and $\mathfrak{K}_2$ the conjugate classes of involutions in G with the representatives t and $t\beta$, respectively. Suppose that $\mathfrak{K}_1 \cap M \neq \emptyset$. Then $\mathfrak{K}_1 \subseteq M$. In particular, t and β are contained in M. Hence $t\beta \in M$ and so $\mathfrak{K}_2 \cap M \neq \emptyset$, $\mathfrak{K}_2 \subseteq M$. All involutions of G are contained in M. It follows that $\langle Q, \beta \rangle \subseteq M$ (because $\langle Q, \beta \rangle$ is generated by its involutions), a contradiction. This gives $\mathfrak{K}_1 \cap M = \emptyset$. It follows that $\mathfrak{K}_2 \subseteq M$. This gives $Q \subseteq M$, $t \in M$, a contradiction. The proof of Lemma 6 is complete.

5. The 3-structure of G**.** We want to determine the structure of a Sylow 3-normalizer in G. Put $T = \langle \sigma_1, \sigma_2 \rangle \subseteq C_G(t) = H$. We know that

$$C_H(T) = \langle t \rangle \times T$$

and $N_H(T) = \langle t, \beta \rangle \cdot T$. Consider now $N_G(T)$. We have $C_G(T) \lhd N_G(T)$ and $\langle t \rangle$ is a Sylow 2-subgroup of $C_G(T)$. It follows that $C_G(T)$ has the normal 2-complement $M \supseteq T$. Since M char $C_G(T)$, it follows $M \lhd N_G(T)$. By a Frattini argument $N(T) = \langle t, \beta \rangle M$. We know that $C_M(t) = T$, $\langle t, \beta \rangle$ centralizes $\langle \sigma_1 \sigma_2 \rangle$ and $C_M(\langle t, \beta \rangle) = \langle \rho \rangle$. Also by the structure of $C_G(t\beta)$ we have $C_M(t\beta) = \langle \rho \rangle$. By way of contradiction, suppose that $C_M(\beta) = \langle \rho \rangle$. Then $|M| = |T|$ and so $M = T$, $N_G(T) = T \cdot \langle t, \beta \rangle$, T is an elementary abelian Sylow 3-subgroup of G and $\langle \rho \rangle$ is contained in the centre of $N_G(T)$. This contradicts the simplicity of G. Hence $C_M(\beta) = T_1$ is an elementary abelian group of order 9 and $T \cap T_1 = \langle \rho \rangle$. We get $|M| = 27$, $M = T \cdot T_1$, M is abelian, and so M is elementary of order 27. We have $T = \langle \rho, \zeta \rangle$, $\zeta = \sigma_1 \sigma_2^{-1}$, $T_1 = \langle \rho, \zeta_1 \rangle$, ζ is inverted by β and $t\beta$, and ζ_1 is inverted by t and $t\beta$. The structure of $N_G(T)$ is determined.

By way of contradiction, suppose that $N_G(M) = N_G(T)$. Then $N_G(T)$ is a Sylow 3-normalizer and (by a theorem of Burnside) T and T_1, being conjugate in G, must be conjugate in $N_G(T)$, a contradiction. Hence $N_G(M) \supset N_G(T)$. Obviously $O(N_G(M)) = M$. Also all involutions in $N_G(M)$ are not conjugate in $N_G(M)$ and so $\langle t, \beta\rangle$ is not a Sylow 2-subgroup of $N_G(M)$.

Let us determine the structure of a Sylow 2-subgroup U $(\supset\langle t, \beta\rangle)$ of $N_G(M)$. We have

$$C(t) \cap U = C(\beta) \cap U = \langle t, \beta\rangle.$$

In particular U is non-abelian and $Z(U) = \langle t\beta\rangle$. Also considering

$$C(t\beta) \cap N_G(M)$$

we see that $\langle\rho\rangle$ is normalized by U and $U\cdot\langle\rho\rangle \subseteq C_G(t\beta)$. By the structure of $C_G(t\beta)$, we know that U is a dihedral group of order 8, the involution $z \in U\backslash\langle t, \beta\rangle$ inverts ρ, $\langle t, \beta\rangle$ centralizes ρ, and z is conjugate to $t\beta$ in G.

Suppose that $N_G(M)$ has a normal 2-complement. It follows that $N(M) = M\cdot U$ and so M is a Sylow 3-subgroup of G. Since T and T_1 are conjugate in G, they must be conjugate in $N_G(M)$. It follows that $z^{-1}Tz = T_1$ and so since z inverts ρ we may choose $\zeta_1 = z^{-1}\zeta z$. We know that z is conjugate in G to $t\beta$ and so $C_M(t\beta) = \langle\rho\rangle$ should be conjugate in $N(M)$ to $C_M(z) = \langle\zeta\zeta_1\rangle$, which is a contradiction.

Suppose now that $N_G(M)$ does not have a normal 2-complement. We see that $N(M)$ has a normal subgroup L of index 2 which does not have a normal subgroup of index 2 and a Sylow 2-subgroup of L is a four-group. We have $M \subseteq L$, $M = O(L)$, $[U : (U \cap L)] = 2$, and $U \cap L$ is a four-group. Because $Z(U) = \langle t\beta\rangle$, $t\beta \in U \cap L$. All involutions in L must be conjugate in L. It follows that $U \cap L = \langle z, t\beta\rangle$ and $t \in U\backslash L$. We want to determine $C_L(t\beta)$. We get $C_M(t\beta) = \langle\rho\rangle$ and so $\langle\rho\rangle$ is normalized by $C_L(t\beta)$. By the structure of $C_G(t\beta)$ we have $C_L(t\beta) = \langle z, t\beta\rangle\langle\rho\rangle$. In particular, $C_L(t\beta)$ has an abelian 2-complement $\langle\rho\rangle$ of order 3 and so by a result of Gorenstein and Walter **(3)** we get $L/M \cong \mathrm{PSL}(2, q)$, q odd.

On the other hand $C_G(M) = M$ and so L/M is isomorphic to a subgroup of $\mathrm{GL}(3, 3)$. It follows that $q = 3$ and so $L/M \cong \mathrm{PSL}(2, 3) \cong A_4$. Since $C_M(t\beta) = \langle\rho\rangle$ and ρ is inverted by z, we get $C_M(\langle t\beta, z\rangle) = \langle 1\rangle$. By the structure of A_4, we have $\langle t\beta, z\rangle\cdot M \lhd L$. There is an element $\mu \in L\backslash\langle t\beta, z\rangle\cdot M$ such that $\langle t\beta, z\rangle\cdot\langle\mu\rangle \cong A_4$ and so we may put $\mu^{-1}\cdot t\beta\cdot\mu = z$, $\mu^{-1}z\mu = t\beta z$. Replacing μ by $\mu\cdot x$ with $x \in \langle t\beta, z\rangle$, if necessary, we have that t normalizes $\langle\mu\rangle$. By the structure of $C_G(t)$ and the fact that $|C_M(t)| = 9$, it follows that $t\mu t = \mu^{-1}$. Hence $\langle t, \mu, t\beta, z\rangle \cong S_4$ and so $N_G(M)$ is a splitting extension of M by S_4. Since $t\beta$ centralizes ρ and z inverts ρ, it follows that $C_M(\langle t\beta, z\rangle) = \langle 1\rangle$. Acting by μ on $\langle t\beta, z\rangle$ and M we see that $M = \langle\rho\rangle \times \langle\rho^{\mu}\rangle \times \langle\rho^{\mu^2}\rangle$ and $C_M(t\beta) = \langle\rho\rangle$, $C_M(z) = \langle\rho^{\mu}\rangle$, $C_M(t\beta z) = \langle\rho^{\mu^2}\rangle$. The action of $\langle t\beta, z, \mu\rangle$ on M is determined. It remains to determine the action of t on M. Representing $\langle t\beta, z, \mu, t\rangle$ on the "vector space" M over GF(3), we get in terms of the "basis" ρ, ρ^{μ}, ρ^{μ^2}:

$$\mu \to \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \qquad z \to \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix},$$

$$t\beta z \to \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

The matrix representing t will be determined by the conditions $t^2 = 1$, $t\mu t = \mu^{-1}$, $tzt = t\beta z$, $t\rho t = \rho$. We get

$$t \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

and so $t\rho t = \rho$, $t\rho^{\mu}t = \rho^{\mu^2}$, $t\rho^{\mu^2}t = \rho^{\mu}$. The structure of $N_G(M)$ is determined. Put $\mathfrak{M} = \langle\mu\rangle \cdot M$. Then $\mathfrak{M}$ is a Sylow 3-subgroup of $N_G(M)$. The centre $Z(\mathfrak{M})$ of $\mathfrak{M}$ is obviously contained in M and so $Z(\mathfrak{M}) = C_M(\mu)$. We find that $Z(\mathfrak{M}) = \langle \rho \cdot \rho^{\mu} \cdot \rho^{\mu^2} \rangle$.

We are going to show that $N_G(\mathfrak{M}) \subseteq N_G(M)$. Let $x \in N_G(\mathfrak{M})$ but $x \notin N_G(M)$. Then $M^x = x^{-1}Mx \subseteq \mathfrak{M}$ and $M^x \neq M$. Because $M \cdot M^x = \mathfrak{M}$ and $[\mathfrak{M} : M] = 3$, we get $|M \cap M^x| = 9$. On the other hand,

$$C_{\mathfrak{M}}(M \cap M^x) \supseteq \langle M, M^x \rangle = \mathfrak{M},$$

which contradicts the fact that $|Z(\mathfrak{M})| = 3$.

We have proved that $N_G(\mathfrak{M}) \subseteq N_G(M)$ and so $\mathfrak{M}$ is a Sylow 3-subgroup of G. We are now able to determine the structure of $N_G(\mathfrak{M})$. Certainly t normalizes $\mathfrak{M}$ because t inverts μ and normalizes M. We have

$$N_G(M) = (\mathfrak{M}\langle t \rangle) \cdot \langle t\beta, z \rangle$$

and so if $N_G(\mathfrak{M}) \supset \mathfrak{M}\langle t \rangle$ we would get that $t\beta$ normalizes $\mathfrak{M}$, which is not the case. We have proved that $\mathfrak{M} \cdot \langle t \rangle$ is a Sylow 3-normalizer in G. We have proved the following result:

LEMMA 7. *A Sylow 3-normalizer in G has order* $2 \cdot 3^4$ *and is given by*

$$\langle \rho, \rho^{\mu}, \rho^{\mu^2}, \mu, t | \rho^3 = \mu^3 = t^2 = 1, [\rho, \rho^{\mu}] = [\rho, \rho^{\mu^2}] = [\rho^{\mu}, \rho^{\mu^2}] = 1,$$
$$t\rho t = \rho, t\rho^{\mu}t = \rho^{\mu^2}, t\rho^{\mu^2}t = \rho^{\mu}, t\mu t = \mu^{-1} \rangle.$$

We shall now study various 3-subgroups of G and their normalizers. The commutator group $\mathfrak{M}'$ of $\mathfrak{M}$ is the set of all $\rho^i(\rho^{\mu})^j(\rho^{\mu^2})^{-i-j}$. It follows that $\mathfrak{M}' = \langle \rho \cdot \rho^{\mu} \cdot \rho^{\mu^2}, \ \rho(\rho^{\mu})^{-1} \rangle$ is elementary of order 9 containing the centre $Z(\mathfrak{M}) = \langle \rho \cdot \rho^{\mu} \cdot \rho^{\mu^2} \rangle$. Hence $[\mathfrak{M}, \mathfrak{M}'] = Z(\mathfrak{M})$ and so $\mathfrak{M}$ is a 3-group of class 3. We also have that $\mathfrak{M}^3$ (the group generated by all third powers of elements of $\mathfrak{M}$) is equal to $Z(\mathfrak{M})$ and so the Frattini subgroup $\phi(\mathfrak{M}) = \mathfrak{M}'$. Hence $\mathfrak{M}$ has precisely four maximal subgroups: M (which is characteristic in $\mathfrak{M}$ and is the unique maximal normal abelian subgroup of $\mathfrak{M}$ of an order $\geqslant 27$), $\langle \mathfrak{M}', \mu \rangle$ (which is characteristic in $\mathfrak{M}$ and is the unique non-abelian maximal subgroup of exponent 3), and M_1 and M_2, which are both non-abelian of exponent 9. We have $M_1{}^t = M_2$.

Put as before $T = C_M(t)$, $T_1 = C_M(\beta)$. Then

$$T \cap T_1 = \langle \rho \rangle = C_M(t\beta), \qquad \langle \rho^\mu \rangle = C_M(z),$$

$$\langle \rho^{\mu^2} \rangle = C_M(t\beta z), \qquad T = \langle \rho, \rho^\mu \rho^{\mu^2} \rangle,$$

where

$$\langle \rho^\mu \rho^{\mu^2} \rangle = \langle \sigma_1 \sigma_2^{-1} \rangle, \qquad T = \langle \sigma_1, \sigma_2 \rangle,$$

and

$$T_1 = \langle \rho, \rho^\mu (\rho^{\mu^2})^{-1} \rangle.$$

We want to determine at first the structure of $N_G(\langle \rho \rangle)$. Since z inverts ρ, we shall determine at first $C_G(\rho)$. We know that

$$C_G(\rho) \cap N(M) = M \cdot \langle t, \beta \rangle.$$

Let U be a Sylow 2-subgroup of $C_G(\rho)$ containing $\langle t, \beta \rangle$. If $U \supset \langle t, \beta \rangle$, then there is an involution x in $\langle t, \beta \rangle$ such that a Sylow 2-subgroup of $C_G(x) \cap C_G(\rho)$ has order $\geqslant 8$, which contradicts the structure of $C_G(t)$ and $C_G(t\beta)$. It follows that $\langle t, \beta \rangle$ is a Sylow 2-subgroup of $C_G(\rho)$. All involutions are not conjugate in $C_G(\rho)$. It follows that $C_G(\rho)$ has a normal 2-complement X containing M. The order of X cannot be greater than 3^3 and so $X = M$. We have proved that $N_G(\langle \rho \rangle) \subseteq N_G(M)$ and so $N_G(\langle \rho \rangle) = M \cdot \langle t, \beta, z \rangle$ is a splitting extension of the elementary group M of order 27 by the dihedral group $\langle t, \beta, z \rangle$ of order 8. The element ρ is real.

We are now going to determine the structure of $N_G(\langle \rho^\mu \cdot \rho^{\mu^2} \rangle)$. Put $\zeta = \rho^\mu \rho^{\mu^2}$. We know that

$$N_G(\langle \zeta \rangle) \cap N_G(M) = M \cdot \langle t, \beta \rangle,$$

where t centralizes ζ and β inverts ζ. Since $\langle \zeta \rangle = \langle \sigma_1 \sigma_2^{-1} \rangle$, it follows by the structure of $C(t)$ that $\langle t \rangle$ is a Sylow 2-subgroup of $C_G(\zeta)$ and so $N_G(\langle \zeta \rangle)$ has a normal 2-complement X_1 ($\supseteq M$) acted upon by the four-group $\langle t, \beta \rangle$ and so $X_1 = M$. We have proved that $N_G(\langle \zeta \rangle) \subseteq N_G(M)$ and so

$$N_G(\langle \rho^\mu \cdot \rho^{\mu^2} \rangle) = M \cdot \langle t, \beta \rangle$$

is a splitting extension of the elementary group M of order 27 by the four-group $\langle t, \beta \rangle$. The element $\rho^\mu \cdot \rho^{\mu^2}$ is real and $C_G(\sigma_1 \sigma_2^{-1}) = M \cdot \langle t \rangle$. In particular, $\sigma_1 \sigma_2^{-1}$ is not conjugate in G to $\sigma_1 \sigma_2$.

We are going to show that μ is conjugate in G to $\sigma_1 \sigma_2^{-1}$. For this purpose we shall determine the structure of $N_G(\langle t\beta, z \rangle)$. By the structure of $C_G(t\beta)$ we have that $C_G(\langle t\beta, z \rangle) = \langle t\beta, z, \tau_1 \tau_2 \rangle$ is elementary of order 8. On the other hand, the non-abelian group $\langle t, \mu \rangle$ of order 6 acts faithfully on $\langle t\beta, z \rangle$ and so

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$N_G(\langle t\beta, z\rangle)$ is a splitting extension of $\langle t\beta, z, \tau_1\tau_2\rangle$ by $\langle t, \mu\rangle$. Let $\mathfrak{K}_1$ and $\mathfrak{K}_2$ have the same meaning as in Lemma 2. Then $t\beta$, z, $t\beta z$, $\tau_1\tau_2$, $z\tau_1\tau_2$, and $t\beta z\tau_1\tau_2$ are in $\mathfrak{K}_2$ and only $t\beta\tau_1\tau_2$ is in $\mathfrak{K}_1$. It follows that $\langle t, \mu\rangle$ centralizes $t\beta\tau_1\tau_2$. Hence μ is real in $C_G(t\beta\tau_1\tau_2)$ and so by the structure of $H = C_G(t)$ we have that μ is conjugate in G to $\sigma_1\sigma_2^{-1}$.

We shall put $\rho\cdot\rho^{\mu}\cdot\rho^{\mu^2} = \lambda$ and we shall determine the structure of $N_G(\langle\lambda\rangle)$. We note that $\langle\lambda\rangle = Z(\mathfrak{M})$ and $\lambda = \sigma_1^{-1}$ or σ_2^{-1}. It follows that λ is not real in G (because t centralizes λ and $\langle t\rangle\mathfrak{M}$ is a Sylow 3-normalizer in G) and by the structure of $C_G(t)$ we have that $C_G(\lambda) \subseteq \langle\lambda\rangle \times S_i$, where

$$S_i = Q_i\langle\sigma_i\rangle \cong \mathrm{SL}(2, 3)$$

and $i = 1$ or 2. Here Q_i is a quaternion group containing t. Also

$$C_G(\lambda) \cap C_G(t) = \langle\lambda\rangle \times S_i.$$

Let U be a Sylow 2-subgroup of $C(\lambda)$ containing Q_i. If $U \supset Q_i$, then $C(t) \cap U \supset Q_i$, which contradicts $C(\lambda) \cap C(t) = \langle\lambda\rangle \times S_i$. Hence the quaternion group Q_i is a Sylow 2-subgroup of $C(\lambda)$. Put $V = O(C_G(\lambda))$. Then $V \supseteq \langle\lambda\rangle$ and by a result of Brauer and Suzuki **(2)** $C(\lambda)/V$ has only one involution $t\cdot V$. Hence $\langle t\rangle V$ is normal in $C(\lambda)$ and $C_V(t) = \langle\lambda\rangle$ because otherwise $\langle\lambda\rangle \times S_i$ would be 3-closed, which is not the case. We get

$$C_G(\lambda) = (C(t) \cap C(\lambda))\cdot V = S_i\langle\lambda\rangle V = S_i\cdot V,$$
$$S_i \cap V = \langle 1\rangle.$$

On the other hand, we know that $\mathfrak{M} \subseteq C_G(\lambda)$ and so $\mathfrak{M}_1 = \mathfrak{M} \cap V$ is a maximal subgroup of $\mathfrak{M}$. Since $\sigma_i \in T \subseteq M$ and $\sigma_i \in S_i$ ($\sigma_i \notin V$), it follows that $\mathfrak{M}_1 \neq M$. Because t acts fixed-point-free on $V/\langle\lambda\rangle$, it follows that $V/\langle\lambda\rangle$ is abelian and so V is nilpotent (of class 2). Hence t normalizes $\mathfrak{M}_1$ and so $\mathfrak{M}_1 = \langle\mathfrak{M}', \mu\rangle$. The fact that μ is conjugate in G to $\sigma_1\sigma_2^{-1}$ and the structure of $C_G(\sigma_1\sigma_2^{-1})$ imply that a Sylow 3-complement of V is $\langle 1\rangle$ and so $V = \langle\mathfrak{M}', \mu\rangle$. It follows that $C_G(\lambda)$ is a splitting extension of the non-abelian group $\langle\mathfrak{M}', \mu\rangle$ of order 27 and exponent 3 by S_i which is isomorphic to $\mathrm{SL}(2, 3)$. The element λ is not real.

The centralizer of the element $\mu\cdot\rho$ of order 9 must be contained in $C(\lambda)$, because $(\mu\rho)^3 = \lambda$. We get $C_G(\mu\rho) = \langle\mu\rho\rangle$. Also the generalized centralizer of $\mu\rho$ must be contained in $C(\lambda)$ because λ is not real. The fact that $C(\lambda)/V \cong \mathrm{SL}(2, 3)$ does not contain a non-abelian subgroup of order 6 gives the result that this generalized centralizer is equal to $\langle\mu\rho\rangle$. It follows that $\mu\rho$ is not real and

$$C_G(\mu\rho) = C_G((\mu\rho)^{-1}) = \langle\mu\rho\rangle.$$

We are going to show that we have found all conjugate classes of 3-elements of G. We have to show that every non-trivial 3-element in $\mathfrak{M}$ is conjugate in G to one of

$$\rho,\quad \rho^{\mu}\rho^{\mu^2},\quad \rho\rho^{\mu}\rho^{\mu^2},\quad \rho^{-1}(\rho^{\mu})^{-1}(\rho^{\mu^2})^{-1},\quad \mu\rho,\quad \rho^{-1}\mu^{-1}.$$

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Because $C_G(\rho) = M\cdot\langle t, \beta\rangle$, ρ has (under the conjugation by the elements of $N_G(M)$) 6 conjugates in M. Because $C_G(\rho^\mu\rho^{\mu^2}) = M\cdot\langle t\rangle$, $\rho^\mu\cdot\rho^{\mu^2}$ has (under the conjugation by the elements of $N(M)$) 12 conjugates in M. Because

$$C_{N(M)}(\rho\rho^\mu\rho^{\mu^2}) = C_{N(M)}(\rho^{-1}(\rho^\mu)^{-1}(\rho^{\mu^2})^{-1}) = \mathfrak{M}\cdot\langle t\rangle,$$

$\rho\rho^\mu\rho^{\mu^2}$ has 4 conjugates and $\rho^{-1}(\rho^\mu)^{-1}(\rho^{\mu^2})^{-1}$ has also 4 conjugates in M. Now μ has 18 conjugates in $\langle\mathfrak{M}', \mu\rangle\backslash\mathfrak{M}'$ under the conjugation by the elements of $\mathfrak{M}\cdot\langle t\rangle$ since $|C_{\mathfrak{M}\cdot\langle t\rangle}(\mu)| = 9$. But μ is conjugate in G to $\rho^\mu\rho^{\mu^2}$ and so we have found all conjugate classes of elements of order 3 in G. It remains to determine the conjugate classes in G consisting of elements of order 9. The element $\mu\rho$ (of order 9) has 18 conjugates in $\mathfrak{M}$ under the conjugation by the elements of $\mathfrak{M}\cdot\langle t\rangle$ since $C_G(\mu\rho) = \langle\mu\rho\rangle$ and also $\rho^{-1}\mu^{-1} = (\mu\rho)^{-1}$ has 18 conjugates in $\mathfrak{M}$ and $\mu\rho$ and $(\mu\rho)^{-1}$ are not conjugate in G. We have proved the following result:

LEMMA 8. *The group G has precisely 4 conjugate classes of elements of order 3 with the representatives σ_1 (non-real), σ_1^{-1} (non-real), $\rho = \sigma_1\cdot\sigma_2$ (real), and $\sigma_1\cdot\sigma_2^{-1}$ (real). Also G has precisely 2 conjugate classes of elements of order 9 with the representatives $\mu\rho$ (non-real) and $(\mu\rho)^{-1}$ (non-real). We have*

$$|C_G(\sigma_1)| = |C_G(\sigma_1^{-1})| = 81\cdot 8, \qquad |C_G(\sigma_1\,\sigma_2)| = 27\cdot 4,$$

$$|C_G(\sigma_1\,\sigma_2^{-1}) = 27\cdot 2, \qquad \text{and } |C_G(\mu\rho)| = |C_G(\mu\rho)^{-1}| = 9.$$

6. The identification of G with PSp$_4$(3). We are now in a position to apply the following result of J. G. Thompson **(7)**.

THEOREM A. PSp$_4(3)$ *is the only finite simple group G with the following properties:*

(i) *G contains an elementary subgroup of order 27.*

(ii) *If P is an S_3-subgroup of G and $A \in \mathfrak{S}\mathfrak{C}\mathfrak{N}_3(P)$, then $\mathsf{M}(A)$ is trivial.*

(iii) *The centre of an S_3-subgroup of G is cyclic.*

(iv) *The normalizer of every non-identity 3-subgroup of G is soluble.*

(v) *S_2-subgroups of G contain normal elementary subgroups of order 8.*

(vi) *If T is a S_2-subgroup of G, then $Z(T)$ is cyclic and if $B \in \mathfrak{S}\mathfrak{C}\mathfrak{N}_3(T)$, then $\mathsf{M}(B)$ is trivial.*

(vii) *The centralizer of every involution of G is soluble.*

(viii) *G contains a soluble subgroup S with the following two properties: $(\alpha)S$ contains an elementary subgroup D of order 9 such that, for each $x \in D$, $C_G(x)$ contains an elementary subgroup E_x of order 9 with $[G : N_G(E_x)]$ prime to 3. $(\beta)S$ contains an elementary subgroup L of order 8 such that for each $y \in L$, $C_G(y)$ contains an elementary subgroup E_y of order 4 with $[G : N_G(E_y)]$ prime to 2.*

Here $\mathfrak{S}\mathfrak{C}\mathfrak{N}_3(X)$ denotes the set of self-centralizing normal subgroups (of a group X) which cannot be generated by less than 3 generators and $\mathsf{M}_X(V) = \mathsf{M}(V)$ is the set of subgroups of X which V normalizes and which intersect V in the identity only. Finally an S_p-subgroup of a group X is a Sylow p-subgroup of X.

We are now able to complete the proof of our theorem by showing that our group G satisfies the conditions (i) to (viii) of Theorem A. First of all, by Lemma 6 the group G is simple. Now using Lemma 7, we see that G satisfies the conditions (i) and (iii). Also using Lemma 1 and the assumption (b) of the theorem, we see that the condition (v) is satisfied and that a Sylow 2-subgroup of G has cyclic centre. By Lemmas 2, 4, and 5 we see that the condition (vii) is satisfied. It is not difficult to see that the condition (viii) is satisfied if we take for S the soluble subgroup $H = C_G(t)$, for D the Sylow 3-subgroup $\langle \sigma_1, \sigma_2 \rangle$ of H, and for L the commutator subgroup of the Sylow 2-subgroup $\langle Q, \beta \rangle$ of H. We know that $\langle \sigma_1, \sigma_2 \rangle \subset M$, M is elementary abelian of order 27 containing the commutator group $\mathfrak{M}'$ (which is elementary of order 9) of the Sylow 3-subgroup $\mathfrak{M}$ of G, and so we may put for any $x \in D = \langle \sigma_1, \sigma_2 \rangle$, $E_x = \mathfrak{M}'$. Let $\mathfrak{K}_1$ and $\mathfrak{K}_2$ have the same meaning as in Lemma 2. If $y \in L$ lies also in $\mathfrak{K}_1$, then we can take for E_y any normal four-subgroup of a Sylow 2-subgroup of $C_G(y)$. Such four-subgroups exist because the commutator group of a Sylow 2-subgroup of $C(y)$ is elementary of order 8. If $y \in L$ lies in $\mathfrak{K}_2$, then we may suppose (by conjugating) that $y = \tau_1 = \alpha_1 \alpha_2$. In this case we take $E_y = Z(\tilde{Q})$, which is elementary of order 4 and E_y is normal in $\langle Q, \beta \rangle$ because $\tilde{Q}$ is normal in $\langle Q, \beta \rangle$.

We shall now show that the group G satisfies the condition (ii). Take the Sylow 3-subgroup $\mathfrak{M}$ of G and note that the only element of $\mathfrak{S}\mathfrak{C}\mathfrak{N}_3(\mathfrak{M})$ is the subgroup M. Let $V \neq 1$ be an element of $\mathsf{M}(M)$. Since a Sylow 3-subgroup of G is not abelian, the order $|V|$ is prime to 3. By Lemma 8, V is a 2-group. If M acts faithfully on $V/\phi(V)$, then $|V/\phi(V)| = 2^6$, which is not possible. Hence $M_1 = C_M(V) \neq \langle 1 \rangle$. Using Lemma 8 again, we see that $|V| \leqslant 8$. It is clear that V cannot possess a characteristic subgroup of order 2 because the order of the centralizer of an involution is not divisible by 27. It follows that V must be elementary of order 4. But then $|M_1| = 9$ and $M_1 V = M_1 \times V$, which contradicts the structure of $C(t) = H$. We have proved that the group G satisfies the condition (ii).

We shall now prove that G satisfies the condition (iv). By Lemma 8, the centralizer of any non-trivial 3-subgroup of G is soluble. Also a Sylow 3-normalizer is soluble. It follows that it is enough to show that $N_G(X)$ is soluble, where X is any subgroup of order 27 which does not possess a characteristic subgroup of order 3. This means that it has to be shown only that $N_G(M)$ is soluble. This has been done before.

It remains to be shown that $\mathsf{M}(B)$ is trivial, where B is an element of $\mathfrak{S}\mathfrak{C}\mathfrak{N}_3(\langle Q, \beta \rangle)$. By way of contradiction, suppose that $W \neq \langle 1 \rangle$ and $W \in \mathsf{M}(B)$. Lemma 3.10 of **(6)** shows that $|W|$ is odd. By the structure of centralizers of involutions, W is a 3-group. Obviously, W cannot be a Sylow 3-subgroup of G and also W cannot have a characteristic subgroup of order 3 (Lemma 8). Using the structure of $N_G(M)$, we see that W must be elementary of order 9. A Sylow 2-subgroup of $GL(2, 3)$ is semi-dihedral of order 16 and so B does not act faithfully on W. There is an involution τ contained in $B \cap \mathfrak{K}_1$ which

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centralizes W. This contradicts the structure of $C(t) = H$. The proof of our theorem is completed.

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Monash University,
Melbourne, Australia