# 14
# Minimizers and Quasiminimizers

## 14.1 Quasiminimizers

In addition to currents and varifolds, there are several other ways to model minimal surfaces and related objects, see [139, 161]. Quasiminimizers provide a very natural and general setting for many variational problems. Let $E \subset \mathbb{R}^n$ be closed and unbounded such that for a fixed positive integer $m$, $0 < \mathcal{H}^m(E \cap B(x,r)) < \infty$ for $x \in E, r > 0$. We say that $E$ is an *m-quasiminimizer* if for some $M < \infty$,

$$\mathcal{H}^m(E \cap W) \leq M\mathcal{H}^m(f(E \cap W))$$

for all Lipschitz mappings $f\colon \mathbb{R}^n \to \mathbb{R}^n$ such that $W = \{x\colon f(x) \neq x\}$ is bounded. If this holds with $M = 1$, then $E$ minimizes $m$-dimensional Hausdorff measure. The setting in the papers quoted below is more general. In particular, there is also a local, often very useful, version, but we skip it here. The quasiminimizers were introduced by Almgren in [9] under the name restricted sets. He proved that they are AD-$m$-regular and $m$-rectifiable. David and Semmes investigated them in [150]. They re-proved Almgren's results and went further. The following is a special case of their results:

**Theorem 14.1** *If $E \subset \mathbb{R}^n$ is a closed m-quasiminimizer, then E is AD-m-regular, uniformly m-rectifiable and it contains big pieces of Lipschitz graphs (recall Section 5.2).*

Both Almgren's and David–Semmes's proofs use Lipschitz projections into $k$-dimensional cubical skeleta like in the Federer–Fleming proof of the deformation theorem of currents. First this gives AD-regularity. Then, by David and Semmes, via many complicated constructions, the big pieces of the Lipschitz graphs condition are verified.

The codimension 1 case was studied by different methods in [149] and [264].

All these papers contain many interesting results on and connections with various geometric variational problems.

There is much later work along these lines, see David's long paper [139] for a very general setting, for discussion and references. It seems to give the most general rectifiability results. In particular, there he used sliding conditions; the deformations were required to preserve given boundary pieces but were allowed to slide along them.

When minimizing Hausdorff measure the existence of minimizers is often a difficult question, both for the lack of lower semicontinuity and compactness. De Lellis, Ghiraldin and Maggi [162] established a general result to deal with this. For this they used Preiss's Theorem 4.11.

## 14.2 Mumford–Shah Functional

Let $\Omega \subset \mathbb{R}^n$ be a domain and $g$ a bounded measurable function in $\Omega$. The *Mumford–Shah functional* $J$ is then defined by

$$J(u, K) = \int_{\Omega \setminus K} (u - g)^2 + \mathcal{H}^{n-1}(K) + \int_{\Omega \setminus K} |\nabla u|^2$$

for

$$(u, K) \in \mathcal{A}(\Omega) := \{(u, K) \colon K \subset \Omega \text{ relatively closed and } u \in W^{1,2}_{loc}(\Omega \setminus K)\}.$$

We assume that there are $(u, K) \in \mathcal{A}(\Omega)$ with $J(u, K) < \infty$, which is always true if $\Omega \subset \mathbb{R}^n$ is bounded. For many aspects of the Mumford–Shah functional, including applications to image segmentation and conjectures and results on minimizers, see the books [15] and [138]. Here we restrict the discussion to things related to rectifiability.

A minimizer for $J$ is a pair $(u, K) \in \mathcal{A}(\Omega)$ which gives the smallest value for $J$. Minimizers always exist, although it is far from obvious since Hausdorff measure is not lower semicontinuous. One way to prove the existence is to first minimize

$$\int_{\Omega} (u - g)^2 + \mathcal{H}^{n-1}(S_u) + \int_{\Omega} |\nabla u|^2$$

for $u \in SBV(\Omega)$, recall Section 12.3. Minimizers for this exist by the compactness properties of SBV. To get from this a minimizer for $J$, the problem that $S_u$ need not be closed has to be dealt with. Here one cannot use the full $BV(\Omega)$, since it would give 0 for the infimum. Anyway, now $S_u$ is $(n-1)$-rectifiable by Theorem 12.13. This approach is discussed in [15]. In [138], a different approach without SBV is explained.

For a minimizer $(u, K)$, $u$ is in $C^1(\Omega \setminus K)$, which follows from the fact that it solves the PDE $\Delta u = u - g$. For $K$ there are conjectures which are only partially solved. David and Semmes proved the following in [148], see also [138]:

**Theorem 14.2** *If $(u, K)$ is a minimizer for $J$ and $B(x, 2r) \subset \Omega$, then $K \cap B(x, r)$ is contained in an AD-$(n-1)$-regular uniformly $(n-1)$-rectifiable set.*

The key to the proof is that the failure of the Poincaré inequality in the complement of an AD-$(n-1)$-regular set $E$ at most scales implies uniform rectifiability of $E$. This is understandable because the validity of the Poincaré inequality requires that $E$ does not separate the space too much. More precisely: $E$ is uniformly $(n-1)$-rectifiable if there exists a positive number $c$ such that for all $M \geq 1$ the set $F(E, c, M)$ of pairs $(x, r), x \in E, 0 < r < d(E)$, satisfying the following condition, is a Carleson set: for all balls $B(x_i, r_i) \subset B(x, r) \setminus E, i = 1, 2$, with $r_i > cr$ and for all $f \in W^{1,1}(B(x, Mr) \setminus E)$,

$$\left| r_1^{-n} \int_{B(x_1,r_1)} f - r_2^{-n} \int_{B(x_2,r_2)} f \right| \leq M r^{1-n} \int_{B(Mx,r)\setminus E} |\nabla f|. \tag{14.1}$$

David and Semmes proved this by showing that this condition implies the local symmetry of Theorem 5.9. Another proof is described in [138]. The converse is false; an example is a coordinate hyperplane with the balls of radius $1/10$ centred in the integer lattice removed.

For slight simplicity, assume $\Omega = \mathbb{R}^n$. To prove that for a minimizer $(u, K)$ the set $F(K, c, M)$ is a Carleson set, one applies (14.1) with $u = f$ and constructs a competitor to get for some $p < 2$,

$$\omega_p(x, Mr) = r^{p/2-n} \int_{B(x,Mr)\setminus K} |\nabla u|^p > \varepsilon(M) > 0.$$

As $r^{1-n} \int_{B(x,r)\setminus K} |\nabla u|^2$ is bounded, it is not very difficult to prove that the set of $(x, r)$ such that $\omega_p(x, r) > \varepsilon$ satisfies a Carleson condition, from which it follows that $F(K, c, M)$ is a Carleson set.

Theorem 14.2 holds for a much larger class of quasiminimizers.

## 14.3 Some Free Boundary Problems

In [141], David, Engelstein and Toro studied the following two-phase free boundary problem. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain and $q_+$ and $q_-$ bounded continuous functions on $\Omega$. Let

$$J(u) = \int_\Omega \left( |\nabla u(x)|^2 + q_+(x)^2 \chi_{\{u>0\}}(x) + q_-(x)^2 \chi_{\{u<0\}}(x) \right) dx.$$

Among other things they proved that if $u$ is an almost minimizer (we omit the definition) for $J$, then, under slight extra conditions, the sets $\Omega \cap \partial\{x \in \Omega\colon u(x) > 0\}$ and $\Omega \cap \partial\{x \in \Omega\colon u(x) < 0\}$ are locally AD-$(n-1)$-regular and uniformly $(n-1)$-rectifiable. The proof is a complicated mixture of potential theory and geometric measure theory. In particular, proving the AD-regularity is quite demanding and achieved with estimates for the harmonic measure.

We shall return to the corresponding one-phase problem in Section 15.6.

Rigot [390] proved the uniform rectifiability of sets almost minimizing perimeter, recall Section 12.1. Let $g(0,\infty) \to (0,\infty)$ with $g(x) = o(x^{(n-1)/n})$.

**Theorem 14.3** *Let $E \subset \mathbb{R}^n$ be Lebesgue measurable. If*

$$P(E) \le P(F) + g(\mathcal{L}^n((E \setminus F) \cup (F \setminus E))$$

*whenever $F \subset \mathbb{R}^n$ is Lebesgue measurable and $F = E$ outside some compact set, then $E$ is equivalent to $E'$ for which $\partial E'$ is AD-$(n-1)$-regular and uniformly $(n-1)$-rectifiable.*

She proved this by showing that $\partial E'$ is a Semmes surface, recall Section 8.7.