5

Lagrangian and gauge invariance

5.1 Introduction

After Einstein's identification of the invariance group of space and time in 1905, symmetry principles received an enthusiastic welcome in physics, with the hope that these principles could express the simplicity of nature in its deepest level. Since 1927 [99,100], it has been recognized that Quantum ElectroDynamics (QED) has a *local symmetry* under the transformations in which the electron field has a phase change that can vary point to point in space–time, and the electromagnetic vector potential undergoes a corresponding transformation. This kind of transformation is called a $U(1)$ gauge symmetry due to the fact that a simple phase change can be thought as a multiplication by a 1×1 unitary matrix. Largely motivated by the challenge of giving a field-theoretical framework to the concept of isospin invariance, Yang and Mills [101] in 1954 extended the idea of QED to the $SU(2)$ group of symmetry. However, it appears here that the symmetry would have to be approximate because gauge invariance requires massless vector bosons like the photon, and it seems obvious that strong interactions of pions were not mediated by massless but by the massive ρ mesons. In 1961, there was the idea of dynamic breaking, i.e., the Hamiltonian and commutation relations of a quantum theory could possess an exact symmetry and the symmetry of the Hamiltonian might not turn to be a symmetry of the vacuum. This way of breaking the symmetry would necessarily imply the existence of massless-spin zero particle, the Nambu–Goldstone boson [17] discussed previously. Later on, Higgs and others [102] showed that if the broken symmetry is a local gauge symmetry, as in the case of QED, the Nambu–Goldstone bosons could formally exist, but can be eliminated by a gauge transformation, so that they are not physical particles. These Nambu–Goldstone bosons appear as helicity states of massive vector particles. These ideas were the starting point for building the $SU(2)_L \times U(1)$ electroweak theory by Weinberg and Salam [61] as an improvement of the model proposed earlier by Glashow [61]. The spontaneous breaking of the electroweak group into $U(1)$ via a non-vanishing expectation value of the Higgs scalar field gives masses to the $W^\pm$ and Z^0 but leaves the photon massless. At present time, one even expects that nature has a richer symmetry (*supersymmetry*), which treats in the same manner the fermions and the bosons. However, we do not have yet any *direct evidence* of a such symmetry. In the following, we shall restrict ourselves to the discussion of the symmetry of

https://doi.org/10.1017/9781009290296.011 Published online by Cambridge University Press

QED and QCD described respectively by a $U(1)$ Abelian and $SU(3)_c$ non-Abelian gauge groups.

5.2 The notion of gauge invariance

In quantum mechanics, a multiplication of a state vector by a constant phase factor, $e^{i\alpha}$, does not induce any observable consequences. Now if you take a wave function with two very distant peaks, and multiply one by a phase factor, then you have to multiply the other by the same phase. This is the *local gauge invariance*, i.e., independence under a space–time-dependent phase factor $\exp(i\alpha(x))$, postulated by Weyl [100]. However, this requirement does not even hold for free non-relativistic particles. Indeed, if $\psi(\vec{x},t)$ is the solution of the Schrödinger equation:

$$-\frac{\hbar^2}{2m}\vec{\nabla}^2\psi(\vec{x},t) = i\hbar\partial_t\psi(\vec{x},t)\,, \tag{5.1}$$

the quantity $\exp(i\alpha(x))\,\psi(\vec{x},t)$ is not, in general, a solution of it. Then, gauge invariance necessarily implies that a particle should interact with fields. If indeed, we consider the Schrödinger equation in a magnetic field with a vector potential $\vec{A}(\vec{x},t)$, then the equation becomes:

$$\frac{\hbar^2}{2m}\left(i\vec{\nabla} - \frac{e}{c}\vec{A}(\vec{x},t)\right)^2\psi(\vec{x},t) = i\hbar\partial_t\psi(\vec{x},t)\,. \tag{5.2}$$

In this case, where the vector potential changes.[1]

$$\vec{A}(\vec{x},t) \to \vec{A}(\vec{x},t) - \frac{c}{e}\vec{\nabla}\alpha(\vec{x})\,, \tag{5.3}$$

one can see that both $\psi(\vec{x},t)$ and $\exp(i\alpha(x))\,\psi(\vec{x},t)$ are solutions of the Schrödinger equation. From this example, we learn that a local gauge invariance of the wave function necessary needs a coupling of the particle to a vector field. Such an invariance will be satisfied by the gauge theory Lagrangian that we shall discuss below.

5.3 The QED Lagrangian as a prototype

The previous discussion can be illustrated in field theory by the simple Lagrangian of QED. In so doing, one can consider the Lagrangian describing a free Dirac electron field having a mass m:

$$\mathcal{L}_{\text{free}} = \bar{\psi}(x)(i\partial_\mu\gamma^\mu - m)\psi(x)\,. \tag{5.4}$$

Under a $U(1)$ global phase transformation, one has:

$$\psi(x) \to \exp(-i\theta\mathbf{1})\,\psi(x)\,, \tag{5.5}$$

[1] Fortunately, this gauge transformation does not influence the magnetic field.

https://doi.org/10.1017/9781009290296.011 Published online by Cambridge University Press

where θ is an arbitrary constant. Now, if one considers the case where θ depends one the space–time coordinate, one can notice that the Lagrangian is no longer invariant under the phase transformation as the derivative of the field has induced an extra-term:

$$\partial^\mu \psi(x) \to \exp(-i\theta\mathbf{1})\,(\partial^\mu - i\partial^\mu\theta)\,\psi(x)\,, \tag{5.6}$$

which means that, for the theory to be consistent, the same phase convention should be taken at all space–time points. However, this is not natural. *Gauge symmetry* requires that the $U(1)$ phase invariance should hold *locally*. This can be achieved, like for the case of quantum mechanics above, by adding a new spin-1 contribution which can cancel the previous extra term:

$$A_\mu(x) \to A_\mu(x) - \frac{1}{e}\partial_\mu\theta(x)\,, \tag{5.7}$$

and by defining the covariant derivative:

$$D_\mu\psi(x) \equiv (\partial_\mu + ieA_\mu(x))\psi(x)\,, \tag{5.8}$$

which transforms like the field itself:

$$\partial^\mu\psi(x) \to \exp(i\theta\mathbf{1})\,D^\mu\psi(x)\,. \tag{5.9}$$

Therefore, the Lagrangian:

$$\mathcal{L} = \bar\psi(x)(iD_\mu\gamma^\mu - m)\psi(x) = \mathcal{L}_{\rm free} - eA_\mu(x)\bar\psi(x)\gamma^\mu\psi(x)\,, \tag{5.10}$$

is invariant under the *local* $U(1)$ transformation. As in the case of quantum mechanics, the gauge principle necessary needs a coupling of the electron field to the vector field, which is given by the second term of the Lagrangian. A complete QED Lagrangian can be achieved by adding the kinetic term of the electromagnetic field and a gauge term:

$$\mathcal{L}_\gamma = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2\alpha_G}\partial^\mu A_\mu\partial_\mu A^\mu\,, \tag{5.11}$$

which expresses that A_μ can propagate. Here, $F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu$ is the electromagnetic field strength; α_G is the gauge parameter which is 0(resp. 1) in the Landau(resp. Feynman) gauge. On the other hand, a possible $m^2 A^\mu A_\mu$ mass term violates gauge invariance, which then implies that the photon is massless. We have then shown that, with the alone gauge principle, one can rederive the QED Lagrangian, which leads to a very impressive quantum

https://doi.org/10.1017/9781009290296.011 Published online by Cambridge University Press

field theory which applications have been tested to a very high degree of accuracy (see next section).

5.4 The QCD Lagrangian

The case of QCD is very similar to the one of QED though more involved due to the non-Abelian structure of its $SU(3)_c$ gauge group. The QCD Lagrangian density reads:

$$\mathcal{L}_{\rm QCD}(x) = -\frac{1}{4}G_a^{\mu\nu}G^a_{\mu\nu} + i\sum_{j=1}^{n}\bar{\psi}^\alpha_j\gamma^\mu(D_\mu)_{\alpha\beta}\psi^\beta_j - \sum_{j=1}^{n} m_j\bar{\psi}^\alpha_j\psi_{j,\alpha}$$
$$-\frac{1}{2\alpha_G}\partial^\mu A^a_\mu\partial_\mu A^\mu_a - \partial_\mu\bar{\varphi}_a D^\mu\varphi^a\ , \tag{5.12}$$

where $G^a_{\mu\nu} \equiv \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g\ f_{abc}A^b_\mu A^c_\nu$ $(a \equiv 1, 2, \ldots, 8)$ are the Yang–Mills field strengths constructed from the gluon fields $A^a_\mu(x)$ [101]. ψ_j is the field of the quark flavour j while $\varphi^a(x)$ are eight anti-commuting scalar fields in the $\underline{8}$ of $SU(3)$. $(D_\mu)_{\alpha\beta} \equiv \delta_{\alpha\beta}\partial_\mu - ig\sum_a \frac{1}{2}\lambda^a_{\alpha\beta}A^a_\mu$ are the covariant derivatives acting on the quark colour component $\alpha, \beta \equiv$ red, blue and yellow; $\lambda^a_{\alpha\beta}$ are the eight 3×3 colour matrices and f_{abc} are *real* structure constants which close the $SU(3)$ Lie algebra:[2]

$$[T_a, T_b] = i\ f_{abc}\ T_c\ , \tag{5.13}$$

where $(T^a)_{\alpha\beta} = \frac{1}{2}\lambda^a_{\alpha\beta}$ in the fundamental colour $\underline{3}$ representation, while $(T_a)_{bc} = -i\ f_{abc}$ in the adjoint $\underline{8}$ representation of gluon basis. The last two terms in the Lagrangian are respectively the gauge-fixing term necessary for a covariant quantization in the gluon sector [$\alpha_G = 1(0)$ in the Feynman (Landau) gauge] and the Faddeev–Popov ghost term [97] necessary to eliminate unphysical particles from the theory.

One can rewrite the above Lagrangian in a more explicit form:

$$\mathcal{L}_{\rm QCD} = \mathcal{L}_{\rm free} + \mathcal{L}^{qg}_{\rm I} + \mathcal{L}^{gg}_{\rm I} + \mathcal{L}^{FPg}_{\rm I}, \tag{5.14}$$

where:

$$\mathcal{L}_{\rm free} = \mathcal{L}^g_{\rm free} + \mathcal{L}^q_{\rm free} + \mathcal{L}^{FP}_{\rm free}\ , \tag{5.15}$$

is the free-field Lagrangian containing the kinetic terms of the different fields, with:

$$\mathcal{L}^g_{\rm free} = -\frac{1}{4}\left(\partial^\mu A^\nu_a - \partial^\nu A^\mu_a\right)\left(\partial^\mu A^\nu_a - \partial^\nu A^\mu_a\right) - \frac{1}{2\alpha_G}\partial^\mu A^a_\mu\partial_\mu A^\mu_a$$
$$\mathcal{L}^q_{\rm free} = i\sum_{j=1}^{n}\bar{\psi}^\alpha_j\gamma^\mu(\partial_\mu)_{\alpha\beta}\psi^\beta_j - \sum_{j=1}^{n} m_j\bar{\psi}^\alpha_j\psi_{j,\alpha}$$
$$\mathcal{L}^{FP}_{\rm free} = -\partial_\mu\bar{\varphi}_a\partial^\mu\varphi^a\ . \tag{5.16}$$

[2] More general and useful properties of the λ matrices are given in Appendix B.

https://doi.org/10.1017/9781009290296.011 Published online by Cambridge University Press

The interaction Lagrangian of the gluon fields respectively with the quarks, gluons and Faddeev–Popov ghosts reads:

$$\begin{aligned}\mathcal{L}_{\mathrm{I}}^{qg} &= gA_a^\mu \sum_{j=1}^{n} \bar{\psi}_j^\alpha \gamma^\mu \left(\frac{\lambda_a}{2}\right)_{\alpha\beta} \psi_j^\beta \,, \\ \mathcal{L}_{\mathrm{I}}^{gg} &= -\frac{g}{2} f^{abc} \left(\partial^\mu A_a^\nu - \partial^\nu A_a^\mu\right) A_{b,\mu} A_{c,\nu} - \frac{g^2}{4} f^{abc} f_{ade} A_b^\mu A_c^\nu A_\mu^d A_\nu^e \,, \\ \mathcal{L}_{\mathrm{I}}^{FPg} &= g f_{abc} (\partial_\mu \bar{\varphi}^a) \varphi^b A_\mu^c \,. \end{aligned} \tag{5.17}$$

The new piece compared with the usual Abelian QED Lagrangian is the the appearance of the gluon self-interaction $\mathcal{L}_{gg}^{I}$, which is a specific feature of the non-Abelian group $SU(3)_c$. Because of this new piece, the Faddeev–Popov ghosts fields are introduced (as mentioned above) for a proper quantization of the theory, which can be done formally using path integral techniques. This method is discussed in details in various textbooks and will be briefly sketched in the next section.

5.5 Local invariance and BRST transformation

$\mathcal{L}_{\mathrm{QCD}}(x)$ is locally invariant under the BRST transformation [103]:

$$\begin{aligned} A_\mu(x) &\to A_\mu(x) + \omega D_\mu \, \varphi \,, \\ \psi_i(x) &\to \exp(-ig\omega \vec{T} \cdot \vec{\varphi}) \, \psi_i \,, \\ \bar{\varphi} &\to \bar{\varphi} + \frac{\omega}{\alpha_G} \partial_\mu A^\mu \,, \\ \phi &\to \phi - \frac{1}{2} g\omega \vec{\varphi} \times \vec{\varphi} \,, \end{aligned} \tag{5.18}$$

where $\omega(x)$ is an arbitrary parameter. In order to see the usefulness of the BRST transformations for generating the Slavnov–Taylor–Ward identities [104,20], let's consider the gluon propagator:

$$i D_{\mu\nu}^{ab}(k) = \int d^4x \; e^{ikx} \langle 0 | \mathcal{T} A_\mu^a(x) A_\nu^b(0) | 0 \rangle \,. \tag{5.19}$$

We shall prove that order by order in perturbation theory, the non-transverse part of the gluon propagator remains the same as for the free propagator:

$$k^\mu k^\nu i D_{\mu\nu}^{ab}(k) = -i\alpha_G \delta^{ab} \,. \tag{5.20}$$

In so doing, we start with the trivial identity:

$$\langle 0 | \partial^\mu A_\mu^a(x) \bar{\varphi}^b(0) | 0 \rangle = 0 \,. \tag{5.21}$$

The BRST invariance implies:

$$\langle 0 | \partial^\mu A_\mu^a(x) \bar{\varphi}^b(0) | 0 \rangle = \langle 0 | \partial^\mu \left(A_\mu^a\right)'(x) (\bar{\varphi}^b)'(0) | 0 \rangle \,, \tag{5.22}$$

https://doi.org/10.1017/9781009290296.011 Published online by Cambridge University Press

where the new fields:

$$A'^a_\mu = A^a_\mu + \omega D_\mu \varphi^a \, ,$$
$$\bar{\varphi}'^a = \bar{\varphi}^a + \frac{\omega}{\alpha_G} \partial^\mu A^a_\mu \, , \tag{5.23}$$

have been introduced. Then, one can deduce:

$$\frac{\omega}{\alpha_G} \langle 0 | \partial^\mu A^a_\mu(x) \partial^\nu A^b_\nu(0) | 0 \rangle = 0 \, . \tag{5.24}$$

By taking its Fourier transform, one obtains the Ward identity written in Eq. (2.39). Using the canonical commutation relation:

$$\left[\Pi^a_\mu(x), A^b_\nu(0) \right] \delta(x_0) = -i g_{\mu\nu} \delta^{ab} \delta^4(x) \, , \tag{5.25}$$

where:

$$\Pi^a_\mu(x) = -G^a_{0\mu}(x) - \frac{1}{\alpha_G} g_{0\mu} \partial^\nu A^a_\nu(x) \, , \tag{5.26}$$

we obtain:

$$\left[A^a_0(x), \partial^\nu A^b_\nu(0) \right] \delta(x_0) = -i \alpha_G \delta^4(x) \, . \tag{5.27}$$

Using Eq. (5.27) into the Ward identity in Eq. (2.39), one can deduce the result in Eq. (5.20).