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A New Sufficient Condition for a Graph To Be (g, f, n)-Critical

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Abstract. Let G be a graph of order p, let a, b, and n be nonnegative integers with $1 \le a < b$, and let g and f be two integer-valued functions defined on $V(G)$ such that $a \le g(x) < f(x) \le b$ for all $x \in V(G)$. A (g, f)-factor of graph G is a spanning subgraph F of G such that $g(x) \le d_F(x) \le f(x)$ for each $x \in V(F)$. Then a graph G is called (g, f, n)-critical if after deleting any n vertices of G the remaining graph of G has a (g, f)-factor. The binding number bind(G) of G is the minimum value of $|N_G(X)|/|X|$ taken over all non-empty subsets X of $V(G)$ such that $N_G(X) \ne V(G)$. In this paper, it is proved that G is a (g, f, n)-critical graph if

$$\operatorname{bind}(G) > \frac{(a+b-1)(p-1)}{(a+1)p-(a+b)-bn+2} \quad \text{and} \quad p \ge \frac{(a+b-1)(a+b-2)}{a+1} + \frac{bn}{a}.$$

Furthermore, it is shown that this result is best possible in some sense.

The graphs considered in this paper are finite undirected simple graphs. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. For any vertex x of G, we denote by $d_G(x)$ the degree of x in G, by $\delta(G)$ the minimum vertex degree of G and by $N_G(x)$ the set of vertices adjacent to x in G. For any $S \subseteq V(G)$, we define $N_G(S) = \bigcup_{x\in S} N_G(x)$, we denote by $G[S]$ the subgraph of G induced by S, and by $G - S$ the subgraph obtained from G by deleting vertices in S together with the edges incident to vertices in S. A subset S of $V(G)$ is *independent* if no two vertices of S are adjacent. The *binding number* bind(G) of G is the minimum value of $|N_G(X)|/|X|$ taken over all non-empty subsets X of $V(G)$ such that $N_G(X) \ne V(G)$ (see [13]).

Let g and f be two nonnegative integer-valued functions defined on $V(G)$ such that $g(x) \le f(x)$ for each $x \in V(G)$. A (g, f)-*factor* of graph G is defined as a spanning subgraph F of G such that $g(x) \le d_F(x) \le f(x)$ for each $x \in V(G)$ (where, of course, d_F denotes the degree in F). If $g(x) = a$ and $f(x) = b$ for all $x \in V(G)$, then a (g, f)-factor is called an $[a, b]$-factor. If $g(x) = f(x) = k$ for all $x \in V(G)$, then a (g, f)-factor is called a k-factor. A graph G is called (g, f, n)-*critical* if after deleting any n vertices of G the remaining graph of G has a (g, f)-factor. If G is (g, f, n)-critical, then we also say that G is a (g, f, n)-critical graph. If $g(x) = a$ and $f(x) = b$ for all $x \in V(G)$, then a (g, f, n)-critical graph is an (a, b, n)-critical graph. If $a = b = k$, then an (a, b, n)-critical graph is simply called a (k, n)-critical graph. In particular, a $(1, n)$-critical graph is simply called an n-critical graph. The other notations and definitions not given in this paper can be found in [1].

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Many authors have investigated (g, f)-factors [3,9,16] and $[a, b]$-factors [6,12]. O. Favaron [4] studied the properties of n-critical graphs. G. Liu and Q. Yu [11] studied the characterization of (k, n)-critical graphs. G. Liu and J. Wang [10] gave the characterization of (a, b, n)-critical graphs with $a < b$. S. Zhou [14] gave two sufficient conditions for graphs to be (a, b, n)-critical. J. Li [7] gave two sufficient conditions for graphs to be (a, b, n)-critical. S. Zhou [15] obtained a sufficient condition for graphs to be (g, f, n)-critical. The characterization of (g, f, n)-critical graphs was given by J. Li and H. Matsuda [8]. In this paper, some binding number conditions for graphs to be (g, f, n)-critical are given. The main results will be given in the following section.

The following results on binding number conditions for graphs to have $[a, b]$-factors and k-factors are known. Katerinis and Woodall proved the following result for the existence of k-factors [5].

Theorem 1 *Let $k \geq 2$ be an integer and let G be a graph having $p \geq 4k - 6$ vertices and binding number* $\operatorname{bind}(G)$ *such that kp is even and*

$$\operatorname{bind}(G) > \frac{(2k-1)(p-1)}{k(p-2)+3}.$$

Then G has a k-factor.

C. Chen gave the following result for the existence of $[a, b]$-factors [2].

Theorem 2 *Let G be a graph of order n, $1 \leq a < b$. If*

$$\operatorname{bind}(G) > \frac{(a+b-1)(n-1)}{bn-2b+3} \quad \text{and} \quad n \geq \frac{(a+b-1)(a+b-2)}{b},$$

then G has an $[a, b]$-factor.

Now we state our main results.

Theorem 3 *Let G be a graph of order p, and let a, b, and n be nonnegative integers such that $1 \leq a < b$, and let g and f be two integer-valued functions defined on $V(G)$ such that $a \leq g(x) < f(x) \leq b$ for each $x \in V(G)$. If*

$$\operatorname{bind}(G) > \frac{(a+b-1)(p-1)}{(a+1)p-(a+b)-bn+2} \quad \text{and} \quad p \geq \frac{(a+b-1)(a+b-2)}{a+1} + \frac{bn}{a},$$

then G is a (g, f, n)-critical graph.

In Theorem 3 if $n = 0$, then we get the following corollary.

Corollary 4 *Let G be a graph of order p, and let a, b be nonnegative integers such that $1 \leq a < b$, and let g and f be two integer-valued functions defined on $V(G)$ such that $a \leq g(x) < f(x) \leq b$ for each $x \in V(G)$. If*

$$\operatorname{bind}(G) > \frac{(a+b-1)(p-1)}{(a+1)p-(a+b)+2} \quad \text{and} \quad p \geq \frac{(a+b-1)(a+b-2)}{a+1},$$

then G has a (g, f)-factor.

In Theorem 3, if $g(x) = a$ and $f(x) = b$, then we obtain the following corollary.

Corollary 5 *Let G be a graph of order p, and let a, b, and n be nonnegative integers such that $1 \le a < b$. If*

$$\mathrm{bind}(G) > \frac{(a+b-1)(p-1)}{(a+1)p-(a+b)-bn+2} \quad \textit{and} \quad p \ge \frac{(a+b-1)(a+b-2)}{a+1} + \frac{bn}{a},$$

then G is an (a, b, n)-critical graph.

The proof of Theorem 3 relies heavily on the following theorem.

Theorem 6 [8] *Let G be a graph, $n \ge 0$ an integer, and let g and f be two integer-valued functions defined on $V(G)$ such that $g(x) < f(x)$ for each $x \in V(G)$. Then G is a (g, f, n)-critical graph if and only if*

$$\delta_G(S,T) = f(S) + d_{G-S}(T) - g(T) \ge \max\{f(N) : N \subseteq S, |N| = n\}$$

for all disjoint subsets S and T of $V(G)$ with $|S| \ge n$.

Proof of Theorem 3 Suppose a graph G satisfies the condition of the theorem, but it is not a (g, f, n)-critical graph. Then by Theorem 6, there exist disjoint subsets S and T of $V(G)$ with $|S| \ge n$ such that

$$(1) \qquad \delta_G(S,T) = f(S) + d_{G-S}(T) - g(T) \le \max\{f(N) : N \subseteq S, |N| = n\} - 1.$$

We choose subsets S and T such that $|T|$ is minimum and S and T satisfy (1).

Claim 1 $d_{G-S}(x) \le g(x) - 1 \le b - 2$ for each $x \in T$.

Proof Suppose that there exists a vertex $x \in T$ such that $d_{G-S}(x) \ge g(x)$. Then the subsets S and $T - \{x\}$ satisfy (1), which contradicts the choice of T. ■

If $T = \varnothing$, then by (1)

$$f(S) - 1 \ge \max\{f(N) : N \subseteq S, |N| = n\} - 1 \ge \delta_G(S,T) = f(S),$$

a contradiction. Hence, $T \ne \varnothing$. Let $h = \min\{d_{G-S}(x) : x \in T\}$.

According to Claim 1, we have $0 \le h \le b - 2$. We shall consider various cases according to the value of h and derive contradictions.

Case 1. $h = 0$.

At first, we prove the following claim.

Claim 2 $\dfrac{(a+1)p-(a+b)-bn+2}{p-1} > 1.$

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Proof Since
$$p \geq \frac{(a+b-1)(a+b-2)}{a+1} + \frac{bn}{a},$$
we have

$$\begin{aligned}(a+1)p-(a+b)-bn+2-(p-1) &= ap-(a+b)-bn+3\\ &\geq a\Big(\frac{(a+b-1)(a+b-2)}{a+1}+\frac{bn}{a}\Big)\\ &\quad -(a+b)-bn+3\\ &= \frac{a(a+b-1)(a+b-2)}{a+1}-(a+b)+3\\ &\geq (a+b-2)-(a+b)+3>0\end{aligned}$$

Thus, we have
$$\frac{(a+1)p-(a+b)-bn+2}{p-1} > 1. \qquad \blacksquare$$

Let $m = |\{x : x \in T, d_{G-S}(x) = 0\}|$, and let $Y = V(G) \setminus S$. Then $N_G(Y) \neq V(G)$ since $h = 0$. In view of the definition of the binding number bind(G), we get that
$$|N_G(Y)| \geq \text{bind}(G)|Y|.$$

Thus, we have $p - m \geq |N_G(Y)| \geq \text{bind}(G)|Y| = \text{bind}(G)(p - |S|)$, that is,

$$|S| \geq p - \frac{p-m}{\text{bind}(G)}. \tag{2}$$

Using $|S| + |T| \leq p$ and (1) and (2) and Claim 2, we obtain

$$\begin{aligned} bn-1 &\geq \max\{f(N) : N \subseteq S, |N| = n\} - 1\\ &\geq \delta_G(S,T) = f(S) + d_{G-S}(T) - g(T)\\ &\geq (a+1)|S| + |T| - m - (b-1)|T|\\ &= (a+1)|S| - (b-2)|T| - m\\ &\geq (a+1)|S| - (b-2)(p-|S|) - m\\ &= (a+b-1)|S| - (b-2)p - m\\ &\geq (a+b-1)\Big(p - \frac{p-m}{\text{bind}(G)}\Big) - (b-2)p - m\\ &= (a+1)p - (a+b-1)\frac{p-m}{\text{bind}(G)} - m\\ &> (a+1)p - (a+b-1)\frac{(p-m)((a+1)p-(a+b)-bn+2)}{(a+b-1)(p-1)} - m \end{aligned}$$

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$$= (a+1)p - \frac{(p-m)((a+1)p-(a+b)-bn+2)}{p-1} - m$$
$$\geq (a+1)p - \frac{(p-1)((a+1)p-(a+b)-bn+2)}{p-1} - 1$$
$$= bn + (a+b) - 3$$
$$\geq bn,$$

which is a contradiction.

Case 2. $1 \leq h \leq b-2$. Let x_1 be a vertex in T such that $d_{G-S}(x_1) = h$, and let $Y = (V(G) \setminus S) \setminus N_{G-S}(x_1)$. Then $x_1 \in Y \setminus N_G(Y)$, so $Y \neq \varnothing$ and $N_G(Y) \neq V(G)$. In view of the definition of the binding number $\operatorname{bind}(G)$, we obtain

$$\frac{|N_G(Y)|}{|Y|} \geq \operatorname{bind}(G).$$

Thus, we get that $p - 1 \geq |N_G(Y)| \geq \operatorname{bind}(G)|Y| = \operatorname{bind}(G)(p - h - |S|)$, that is,

$$|S| \geq p - h - \frac{p-1}{\operatorname{bind}(G)}. \tag{3}$$

By $|S| + |T| \leq p$ and (1) and (3), we have

$$\begin{aligned} bn - 1 &\geq \max\{f(N) : N \subseteq S, |N| = n\} - 1 \\ &\geq \delta_G(S,T) = f(S) + d_{G-S}(T) - g(T) \\ &\geq (a+1)|S| + d_{G-S}(T) - (b-1)|T| \\ &\geq (a+1)|S| + h|T| - (b-1)|T| \\ &= (a+1)|S| - (b-h-1)|T| \\ &\geq (a+1)|S| - (b-h-1)(p-|S|) \\ &= (a+b-h)|S| - (b-h-1)p \\ &\geq (a+b-h)\Big(p - h - \frac{p-1}{\operatorname{bind}(G)}\Big) - (b-h-1)p \\ &> (a+b-h)\Big(p - h - \frac{(a+1)p-(a+b)-bn+2}{a+b-1}\Big) \\ &\quad - (b-h-1)p. \end{aligned} \tag{4}$$

Let $f(h) = (a+b-h)(p-h-\frac{(a+1)p-(a+b)-bn+2}{a+b-1}) - (b-h-1)p$. In fact, the function $f(h)$ attains its minimum value at $h = 1$, since $1 \leq h \leq b-2$ is an integer. Then we get $f(h) \geq f(1)$. Combining this with (4), we obtain

$$bn - 1 > f(1) = (a+b-1)\Big(p - 1 - \frac{(a+1)p-(a+b)-bn+2}{a+b-1}\Big) - (b-2)p$$

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$$= (a+b-1)(p-1) - ((a+1)p - (a+b) - bn + 2) - (b-2)p$$
$$= bn - 1,$$

which is a contradiction.

From the argument above, we deduce the contradictions, so the hypothesis cannot hold. Hence, G is a (g, f, n)-critical graph. ■

Remark Let us show that the condition $\operatorname{bind}(G) > \frac{(a+b-1)(p-1)}{(a+1)p-(a+b)-bn+2}$ in Theorem 3 cannot be replaced by $\operatorname{bind}(G) \geq \frac{(a+b-1)(p-1)}{(a+1)p-(a+b)-bn+2}$. Let $a \geq 2, b = a+1, n \geq 0$ be three integers such that $a+b+n$ is odd, and let $p = \frac{(a+b-1)(a+b-2)+(a+b-2)+(a+2b-1)n}{b}$ be an integer, and let $l = \frac{a+b+n-1}{2}$ and

$$m = p - 2l = p - (a+b+n-1) = \frac{(a+b-1)(a-2)+(a+b-2)+(a+b-1)n}{b}.$$

Clearly, m is an integer. Let $H = K_m \vee lK_2$. Let $X = V(lK_2)$, for any $x \in X$, then $|N_H(X \setminus x)| = p-1$. By the definition of $\operatorname{bind}(H)$,

$$\operatorname{bind}(H) = \frac{|N_H(X \setminus x)|}{|X \setminus x|} = \frac{p-1}{2l-1} = \frac{p-1}{a+b+n-2}$$
$$= \frac{(a+b-1)(p-1)}{bp-(a+b)-bn+2} = \frac{(a+b-1)(p-1)}{(a+1)p-(a+b)-bn+2}.$$

Let $S = V(K_m) \subseteq V(H)$, $T = V(lK_2) \subseteq V(H)$, then $|S| = m \geq n$, $|T| = 2l$. Since $a \leq g(x) < f(x) \leq b$ and $b = a+1$, then we have $g(x) = a$ and $f(x) = b = a+1$. Thus, we get

$$\delta_H(S,T) = f(S) + d_{H-S}(T) - g(T) = (a+1)|S| + |T| - (b-1)|T|$$
$$= (a+1)|S| - (b-2)|T| = b|S| - (a-1)|T|$$
$$= b\frac{(a+b-1)(a-2)+(a+b-2)+(a+b-1)n}{b}$$
$$- (a-1)(a+b+n-1)$$
$$= bn - 1 < bn = \max\{f(N) : N \subseteq S, |N| = n\}.$$

By Theorem 6, H is not a (g, f, n)-critical graph. In the above sense, the result of Theorem 3 is best possible.

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